International Journal of Applied Mathematics

Volume 28 No. 6 2015, 637-650

ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)

doi: http://dx.doi.org/10.12732/ijam.v28i6.1

NUMERICAL RANGES OF WEIGHTED COMPOSITION OPERATORS ON $\ell^2(\mathbb{N}) - II$

Mitu Gupta[1], B.S. Komal[2 §]

[1]Department of Mathematics
University of Jammu
Baba Sahib Bhim Rao Ambedkar Road
Jammu, 180006, INDIA

[2]MIET Kot Bhalwal
Jammu, 180001, INDIA

[2]House No 4 Sector 3
Green Avenue, Digiana Ashram
Digiana, Jammu, 180010, INDIA

Abstract: In this paper we obtain some results on the numerical ranges of weighted composition operators on $\ell^2(\mathbb{N})$.

AMS Subject Classification: 47B37, 47A12

Key Words: numerical range, weighted composition operator, orbit, unilateral weighted shift, compact operator

1. Introduction

Let $\mathbb{N}$ denote the set of positive integers and let $\ell^2(\mathbb{N})$ be the Hilbert space of

Received: June 15, 2015

§Correspondence author

square summable sequences of complex numbers. Suppose $\theta : \mathbb{N} \to \mathbb{C}$ and $\phi : \mathbb{N} \to \mathbb{N}$ are two mappings. Then a bounded linear transformation $C_{\theta,\ \phi} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ defined by $C_{\theta,\ \phi}\ (f) = \theta.\ f \circ \phi$ for every $f \in \ell^2(\mathbb{N})$ is known as a weighted composition operator induced by θ and ϕ. It is easy to see that $C_{\theta,\ \phi}$ is a bounded operator if and only if there exists $M > 0$ such that $\sum\limits_{m \in \phi^{-1}(\{n\})} |\theta(m)|^2 \leq M$ for every $n \in \mathbb{N}$ and $\|C_{\theta,\ \phi}\| = \sup\limits_{n \in \phi(\mathbb{N})} \sqrt{\sum\limits_{k \in \phi^{-1}(\{n\})} \left|\theta(k)\right|^2}$.
For $n \in \mathbb{N}$, the orbit of n with respect to ϕ is defined as

$$O_{\phi}(n) = \{m \in \mathbb{N} : \phi^r(m) = \phi^s(n) \text{ for some } r, s \in \mathbb{N}\},$$

whereby the symbol ϕ^r we shall mean $\underbrace{\phi \circ \phi \circ \phi \circ ...\phi}_{r\ times}$. A complex number λ is called an eigenvalue of a bounded linear operator $A : H \to H$ from Hilbert space H into itself if there exists non-zero $f \in H$ such that $Af = \lambda f$. The set of all complex numbers λ such that $A - \lambda I$ is not invertible is called the spectrum of A and it is denoted by $\sigma(A)$. The Banach algebra of all bounded linear operators from H into itself is denoted by B(H). For $G \subset \mathbb{N}$, let $\ell^2(G) = \left\{f \in \ell^2(\mathbb{N}) : f(m) = 0 \text{ for every } m \notin G\right\}$. The symbol $C_{\theta,\ \phi}\big|_{\ell^2(G)}$ denotes the restriction of $C_{\theta,\ \phi}$ to $\ell^2(G)$ and the symbol $\#(G)$ denotes the cardinality of the set G. For $A \in B(H)$, the numerical range of A is defined as $W(A) = \{< Ax, x >: x \in H \text{ and } \|x\| = 1\}$ and numerical radius of A is defined as $w(A) = sup\{|\lambda| : \lambda \in W(A)\}$. So far as we know, very little is known about the numerical ranges of weighted composition operators as well as other bounded linear operators. Numerical ranges of some operators are obtained by Bourdon and Shapiro [1], Gustafson and Rao [3], Komal and Sharma [5], Ridge [6], Tam [8], etc. In the following proposition we list some well known results regarding numerical ranges.

Proposition 1.1. *Let $A \in B(H)$. Then:*

(a) *W(A) lies in the closed disc of radius $\|A\|$ centered at the origin.*

(b) *W(A) is always convex.*

(c) *$W(\alpha A + \beta I) = \alpha W(A) + \beta$ where α and β are complex numbers.*

(d) *W(A) is invariant under a unitary operator.*

(e) *Numerical range of the unilateral shift is the open unit disc centered at origin.*

(f) *The closure of the numerical range of a normal operator is the convex hull of its spectrum.*

(g) $W(A^*) = \overline{W(A)} = \{\overline{\lambda} : \lambda \in W(A)\}$, *where* A^* *denotes the adjoint of operator* A *and* $\overline{\lambda}$ *is the conjugate of complex number* λ.

(h) $w(A) \leq \|A\|$.

In this paper we obtain some results on the numerical ranges of weighted composition operators. We show that the shape of the numerical range of a weighted composition operator is influenced by the inducing functions.

2. Numerical Ranges of Weighted Composition Operators Induced by Fixed Point Mappings

In this section we compute the numerical range of a weighted composition operator $C_{\theta,\ \phi}$ when each $n \in \phi(\mathbb{N})$ is a fixed point of ϕ. We first state and prove the following lemma.

Lemma 2.1. *Suppose* $C_{\theta,\ \phi} \in B(\ell^2(\mathbb{N}))$. *Let* $\phi : \mathbb{N} \to \mathbb{N}$ *be such that* $\#(\phi^{-1}(\{n_1\})) \geq 2$ *for some fixed point* n_1 *of* ϕ. *Then* $W(C_{\theta,\ \phi}|_{\ell^2(E_{n_1})})$ *is closed elliptical disc with focii at 0 and* $\theta(n_1)$, *where* $E_{n_1} = \phi^{-1}(\{n_1\})$.

Proof. Suppose n_1 is a fixed point of ϕ and $\#(\phi^{-1}(\{n_1\})) \geq 2$. Let $\phi^{-1}(\{n_1\}) = \{n_1, n_2, \ldots.., n_k\}$ for $k \geq 2$. Take $g = \|\theta\|_2\, e_{n_1}$, where $\|\theta\|_2 = \sqrt{\sum_{i=1}^{k} \left|\theta(n_i)\right|^2}$ and $h = \frac{\theta.\chi_{E_{n_1}}}{\|\theta\|_2}$, where $\chi_{E_{n_1}}$ is the characteristic function of E_{n_1}. Then

$$
\begin{aligned}
C_{\theta,\ \phi}|_{\ell^2(E_{n_1})} f &= \theta(n_1) f(n_1) e_{n_1} + \theta(n_2) f(n_1) e_{n_2} + \ldots. + \theta(n_k) f(n_1) e_{n_k} \\
&= f(n_1)\ \theta.\chi_{E_{n_1}} \\
&= < f, g > h \text{ for every } f \in \ell^2(E_{n_1}),
\end{aligned}
$$

where $\theta.\chi_{E_{n_1}}$ is the pointwise multiplication of functions and $e_n(m) = \delta_{nm} =$ the Kronecker delta.

Hence by Proposition 2.5 of Bourdon and Shapiro [1] $W(C_{\theta,\ \phi}|_{\ell^2(E_{n_1})})$ is closed elliptical disc with focii at 0 and $\langle h, g\rangle = \theta(n_1)$. This completes the proof. □

If each point of $\phi(\mathbb{N})$ is a fixed point of ϕ, then we can partition $\mathbb{N}$ into two disjoints sets E_1 and E_2, where

$$E_1 = \Big\{n \in \mathbb{N} : n \text{ is a fixed point of } \phi \text{ having only one element in its preimage}\Big\}$$

and

$$E_2 = \Big\{n \in \mathbb{N} : n \text{ is a fixed point of } \phi \text{ having more than one element in its preimage}\Big\}.$$

For $n \in E_2$, write $F_n = \phi^{-1}(\{n\})$.

Theorem 2.2. *Suppose $C_{\theta,\ \phi} \in B(\ell^2(\mathbb{N}))$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be such that each point of $\phi(\mathbb{N})$ is a fixed point of ϕ. Then*

$$W(C_{\theta,\ \phi}) = Co\left((\bigcup_{n\in E_2} D_n) \cup \{\theta(m) : m \in E_1\} \right),$$

where D_n is closed elliptical disc with focii at 0 and $\theta(n)$.

Proof. It can be easily seen that $\mathbb{N} = \left(\bigcup_{n\in E_2} F_n\right) \cup E_1$. Therefore

$$\ell^2(\mathbb{N}) = \left(\sum_{n\in E_2} \oplus\ \ell^2(F_n)\right) \ \oplus\ \left(\ell^2(E_1)\right),$$

$$\begin{aligned} C_{\theta,\ \phi} &= \left(\sum_{n\in E_2} \oplus\, C_{\theta,\ \phi}|\ell^2(F_n)\right) \oplus \left(C_{\theta,\ \phi}|\ell^2(E_1)\right) \\ &= \left(\sum_{n\in E_2} \oplus\, C_{\theta,\ \phi}|\ell^2(F_n)\right) \oplus \left(M_\theta|\ell^2(E_1)\right). \end{aligned}$$

Since the closure of the numerical range of a normal operator is the convex hull of its spectrum and since $\sigma(M_\theta) = \overline{ran\theta}$, it follows by Lemma 2.1 that

$$\begin{aligned} W(C_{\theta,\ \phi}) &= Co\left(\left(\bigcup_{n\in E_2} D_n\right) \cup \left\{\overline{ran\theta}|_{E_1}\right\}\right) \\ &= Co\left(\left(\bigcup_{n\in E_2} D_n\right) \cup \left\{\overline{\theta(m)} : m \in E_1\right\}\right). \end{aligned}$$

This proves the theorem. □

Example 2.3. Let $p \in \mathbb{N}$. Define a relation $\sim$ on $\mathbb{N}$ as follows: $m \sim n$ if $m \equiv n\ (modp)$. This relation will divide $\mathbb{N}$ into p distinct equivalence classes namely $\bar{1}, \bar{2}, ..., \bar{p}$. Define $\phi : \mathbb{N} \to \mathbb{N}$ by $\phi(n) = p$ if $n \in \bar{p}$ (p is the smallest integer in the equivalence class $\bar{p}$). Let $\theta : \mathbb{N} \to \mathbb{C}$ be defined by $\theta(n) = \frac{e^{2i\pi/p}}{n}$ whenever $n \in \bar{p}$. Then $C_{\theta,\ \phi}$ is a bounded operator and each point of $\phi(\mathbb{N})$ is a fixed point of ϕ. Let $Q_m = \phi^{-1}(\{m\})$, where m is a fixed point of ϕ and $1 \le m \le p$. Hence in view of Theorem 2.1, $W(C_{\theta,\ \phi}) = Co\left(\bigcup_{m=1}^{p} Q_m\right)$, where Q_m is closed elliptical disc with focii at 0 and $\theta(m) = \frac{e^{2i\pi/m}}{m}$.

Example 2.4. Let $\phi : \mathbb{N} \to \mathbb{N}$ be defined by

$$\phi(n) = \begin{cases} n, & \text{if n is odd number} \\ n-1, & \text{if n is an even number.} \end{cases}$$

For $n \in \mathbb{N}$, let $\theta : \mathbb{N} \to \mathbb{C}$ be a bounded function. Then $C_{\theta,\ \phi}$ is a bounded operator.

Let $E_{2n-1} = \{2n-1,\ 2n\}$. Then $\mathbb{N} = \bigcup_{n=1}^{\infty} E_{2n-1}$, $\ell^2(\mathbb{N}) = \sum_{n=1}^{\infty} \oplus\ \ell^2(E_{2n-1})$ and $C_{\theta,\ \phi} = \sum_{n=1}^{\infty} C_{\theta,\ \phi}\left|_{\ell^2(E_{2n-1})}\right.$.

For $n = 1$, the matrix of $C_{\theta,\ \phi}\left|_{\ell^2(E_{2n-1})}\right. = \begin{bmatrix} \theta(1) & 0 \\ \theta(2) & 0 \end{bmatrix}$. The eigenvalues of $C_{\theta,\ \phi}\big|_{\ell^2(E_{2n-1})}$ are $\theta(1)$ and 0 and the corresponding unit eigenvectors are

$$f = \left(\frac{\theta(1)}{\sqrt{\left|\theta(1)\right|^2 + \left|\theta(2)\right|^2}}, \frac{\theta(2)}{\sqrt{\left|\theta(1)\right|^2 + \left|\theta(2)\right|^2}} \right), \text{ and } g = (0, 1).$$

Now let $r = |\langle f, g\rangle| = \frac{\left|\theta(2)\right|}{\sqrt{\left|\theta(1)\right|^2 + \left|\theta(2)\right|^2}}$, $\quad \sqrt{1 - r^2} = \frac{\left|\theta(1)\right|}{\sqrt{\left|\theta(1)\right|^2 + \left|\theta(2)\right|^2}}$.

Hence by Remark 2.4 of Bourdon and Shapiro [1], $W(C_{\theta,\ \phi}\big|_{\ell^2(E_1)})$ is a closed elliptical disc with foci at 0 and $\theta(1)$, major axis $= \sqrt{\left|\theta(1)\right|^2 + \left|\theta(2)\right|^2}$ and minor axis $= \left|\theta(2)\right|$. In a similar manner we can show that $W\left(C_{\theta,\ \phi}\big|_{\ell^2(E_{2n-1})}\right)$

is a closed elliptical disc D_{2n-1} with foci at 0 and $\theta(2n-1)$, major axis = $\sqrt{|\theta(2n-1)|^2 + |\theta(2n)|^2}$ and minor axis = $|\theta(2n)|$.

Hence $W(C_{\theta,\phi}) = \bigcup_{n=1}^{\infty} W\left(C_{\theta,\ \phi}|_{\ell^2(E_{2n-1})}\right).$

$$= Co\left(\bigcup_{n=1}^{\infty} D_{2n-1}\right).$$

In particular, if we take

$$\theta(n) = \begin{cases} 5 & \text{if } n = 1 \\ 12 & \text{if } n = 2 \\ 9 & \text{if } n = 3 \\ 40 & \text{if } n = 4 \\ 13 & \text{if } n = 5 \\ 84 & \text{if } n = 6 \\ 17 & \text{if } n = 7 \\ 144 & \text{if } n = 8 \\ \frac{500}{n(n+1)} & \text{if } n \geq 9 \end{cases},$$

then the shape of the numerical range is as shown in Figure 1.

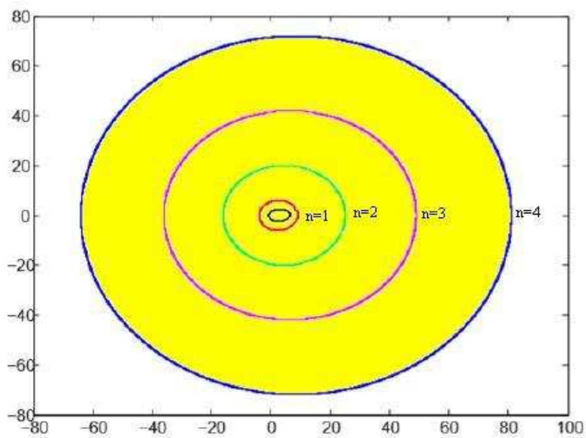

Figure 1

3. Numerical Ranges of Weighted Composition Operators Induced by Non-Fixed Point Mappings

In this section we obtain the numerical ranges of weighted composition operators when ϕ and ϕ^2 do not have any fixed point.

Lemma 3.1. *Suppose $C_{\theta,\,\phi} \in B(\ell^2(\mathbb{N}))$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be such that $\#(\phi^{-1}(\{n\})) \geq 2$ for some $n \in \mathbb{N}$ and n is neither a fixed point of ϕ nor of ϕ^2. Then*

$$W\left(C_{\theta,\phi}\big|_{\ell^2(E_n)}\right) = \left\{\alpha \in \mathbb{C} : |\alpha| \leq \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\},$$

where $E_n = \phi^{-1}(\{n\}) \cup (\{n\})$.

Proof. Suppose $\phi^{-1}(\{n\}) = \{n_1, n_2, \ldots.., n_k\}$. Then for $f \in \ell^2(E_n)$ with $\|f\| = 1$, we have

$$\begin{aligned} |\langle C_{\theta,\,\phi} f, f\rangle| &= \left|\sum_{k\in\phi^{-1}(\{n\})} \theta(k) f(\phi(k))\overline{f(k)}\right| \\ &\leq |f(n)| \sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|\,|f(k)|\,. \end{aligned}$$

Write $|f(n)| = t_n$ and $|f(k)| = t_k$ for each $k \in \phi^{-1}(\{n\})$.

By the method of Lagrange's multiplier, we maximize

$$t_n \sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|\, t_k$$

subject to the condition $\displaystyle\sum_{k\in\phi^{-1}(\{n\})} t_k^2 + t_n^2 = 1.$

Let

$$F = t_n \sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|\, t_k - \lambda\left(\sum_{k\in\phi^{-1}(\{n\})} t_k^2 + t_n^2 - 1\right) = 0. \tag{1}$$

Then

$$\frac{\partial F}{\partial t_n} = \sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|\, t_k - 2\lambda t_n = 0 \tag{2}$$

and

$$\frac{\partial F}{\partial t_k} = t_n\,|\theta(k)|-2\lambda t_k = 0 \text{ for all } k \in \phi^{-1}(\{n\}). \tag{3}$$

Multiplying equation (3.2) by t_n, equation (3.3) by t_k and adding, we find that

$$\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|\, t_k t_n = \lambda. \tag{4}$$

Substituting the value of t_k from (3.3) in $t_n^2 + \sum_{k\in\phi^{-1}(\{n\})} t_k^2 = 1$, we obtain

$$t_n^2 = \frac{4\lambda^2}{4\lambda^2 + \sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}. \tag{5}$$

Solving (3.4) and (3.5) for λ, we get $\lambda = \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}$.

Hence

$$W\left(C_{\theta,\phi}\big|_{\ell^2(E_n)}\right) \subset \left\{\alpha \in \mathbb{C} : |\alpha| \le \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\}.$$

To prove the converse part, let

$$f(k) = \begin{cases} \frac{1}{\sqrt{2}}\frac{\theta(k)}{2\lambda}e^{-i\beta}, & \text{for } k \in \phi^{-1}(\{n\}),\ 0 \le \beta \le 2\pi, \\ \frac{1}{\sqrt{2}}, & \text{for } k = n, \\ 0, & \text{for } k \notin E_n. \end{cases}$$

Then $\|f\| = 1$ and

$$\begin{aligned} \left\langle C_{\theta,\phi}\big|_{\ell^2(E_n)} f, f\right\rangle &= \langle C_{\theta,\phi} f, f\rangle = f(n) \sum_{k\in\phi^{-1}(\{n\})} \theta(k)\overline{f(k)} \\ &= \frac{1}{\sqrt{2}} \sum_{k\in\phi^{-1}(\{n\})} \frac{1}{\sqrt{2}} \frac{\overline{\theta(k)}\theta(k)e^{i\beta}}{2\lambda} \\ &= \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\; e^{i\beta}, \quad 0 \le \beta \le 2\pi. \end{aligned}$$

Since numerical range is convex, we conclude that

$$\left\{\alpha \in \mathbb{C} : |\alpha| \leq \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\} \subset W\left(C_{\theta,\phi}|_{\ell^2(E_n)}\right).$$

Hence

$$W\left(C_{\theta,\phi}|_{\ell^2(E_n)}\right) = \left\{\alpha \in \mathbb{C} : |\alpha| \leq \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\}.$$

□

Theorem 3.2. *Let $C_{\theta,\ \phi} \in B(\ell^2(\mathbb{N}))$. Suppose $\#(\phi^{-1}(\{n\})) \geq 2$ for all $n \in \phi(\mathbb{N})$ and each point of $\phi(\mathbb{N})$ is neither a fixed point of ϕ nor of ϕ^2. Then*

$$\left\{\alpha \in \mathbb{C} : \left|\alpha\right| \leq \sup_{n\in\phi(\mathbb{N})} \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\} \subset W(C_{\theta,\ \phi}).$$

Proof. Take $n \in \phi(\mathbb{N})$. In view of Lemma 3.1, we obtain

$$\left\{\alpha \in \mathbb{C} : \left|\alpha\right| \leq \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\} = W\left(C_{\theta,\ \phi}|_{\ell^2(E_n)}\right) \subset W(C_{\theta,\ \phi}),$$

where E_n is as described in Lemma 3.1. Since the numerical range of an operator is convex, so

$$Co\left(\bigcup_{n\in\mathbb{N}}\left\{\alpha \in \mathbb{C} : \left|\alpha\right| \leq \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\}\right) \subset W(C_{\theta,\ \phi})$$

and hence

$$\left\{\alpha \in \mathbb{C} : \left|\alpha\right| \leq \sup_{n\in\phi(\mathbb{N})} \frac{1}{2}\sqrt{\sum_{k\in\phi^{-1}(\{n\})} |\theta(k)|^2}\right\} \subset W(C_{\theta,\ \phi}).$$

□

Lemma 3.3. *Suppose $C_{\theta,\ \phi} \in B(\ell^2(\mathbb{N}))$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be such that $\#(\phi^{-1}(\{n\})) \geq 2$ for some $n \in \phi(N)$. Suppose n is not a fixed point of ϕ but a fixed point of ϕ^2. Then,*

$$W\left(C_{\theta,\ \phi}\big|_{\ell^2(E_n)}\right) \subset \left\{\alpha \in \mathbb{C} : |\alpha| \leq \frac{1}{2}\sqrt{\sum_{k\in E_n} \left|\theta^2(k)\right| + 2\left|\theta(n)\right|\left|\theta(\phi(n))\right|}\right\},$$

where $E_n = \phi^{-1}(\{n\})$.

Proof. Let $\phi^{-1}(\{n\}) = \{n_1, n_2, ..., n_k\}$. Then $\phi(n) \in \phi^{-1}(\{n\})$ and so $\phi(n) = n_j$ for some $1 \leq j \leq k$. For $f \in \ell^2(E_n)$, consider

$$\begin{aligned} |\langle C_{\theta,\ \phi} f, f\rangle| &= \left|\sum_{i=1}^{k} \theta(n_i) f(\phi(n_i))\overline{f(n_i)}\right| \\ &\leq \sum_{i\neq j, i=1}^{k} |\theta(n_i)|\,|f(n)|\,|f(n_i)| + |\theta(n_j)|\,|f(n)|\,|f(n_j)| \\ &\qquad + |\theta(n)|\,|f(n_j)|\,|f(n)|. \end{aligned} \tag{6}$$

Write $|f(n)| = t_n$, $\quad |f(n_i)| = t_{n_i}$, $\quad |f(n_j)| = t_{n_j}$. This yields that

$$\left|\left\langle C_{\theta,\ \phi}\big|_{\ell^2(E_n)} f, f\right\rangle\right| \leq \sum_{i\neq j, i=1}^{k} |\theta(n_i)|\, t_n t_{n_i} + |\theta(n_j)|\, t_n t_{n_j} + |\theta(n)|\, t_{n_j} t_n.$$

By the method of Lagrange's multiplier, we maximize

$$\sum_{i\neq j,\ i=1}^{k} |\theta(n_i)|\, t_n t_{n_i} + |\theta(n_j)|\, t_n t_{n_j} + |\theta(n)|\, t_{n_j} t_n$$

subject to the condition $\sum_{i=1}^{k} t_{n_i}^2 + t_n^2 = 1$.

Proceeding in a similar manner as in Lemma 3.1, we obtain

$$\lambda \;=\; \frac{1}{2}\sqrt{\sum_{k\in E_n} \left|\theta^2(k)\right| + \left|\theta^2(n)\right| + 2\left|\theta(n)\right|\left|\theta(\phi(n))\right|}.$$

Therefore,

$$W\left(C_{\theta,\phi}\big|_{\ell^2(E_n)}\right) \subset \left\{\alpha \in \mathbb{C} : |\alpha| \leq \frac{1}{2}\right.$$
$$\left.\sqrt{\sum_{k\in E_n} \left|\theta^2(k)\right| + \left|\theta^2(n)\right| + 2\left|\theta(n)\right|\left|\theta(\phi(n))\right|}\right\}.$$

□

Theorem 3.4. *Suppose $C_{\theta,\ \phi} \in B(\ell^2(\mathbb{N}))$. Let $\phi : \mathbb{N} \to \mathbb{N}$ be such that $\#\left(\phi^{-1}(\{n\})\right) \geq 2$ for each $n \in \phi(\mathbb{N})$. Suppose each point of $\phi(\mathbb{N})$ is not a fixed point of ϕ but a fixed point of ϕ^2. Then,*

$$W(C_{\theta,\ \phi}) \subset \left\{\alpha \in \mathbb{C} : \left|\alpha\right| \leq \sup_n \frac{1}{2}\sqrt{\sum_{k\in E_n} \left|\theta(k)\right|^2 + 2\left|\theta(n)\right|\left|\theta(\phi(n))\right|}\right\},$$

where $E_n = \phi^{-1}(\{n\})$.

Proof. By an application of Lemma 3.3,

$$W(C_{\theta,\ \phi}\big|_{\ell^2(E_n)}) \subset \left\{\alpha \in \mathbb{C} : \left|\alpha\right| \leq \frac{1}{2}\sqrt{\sum_{k\in E_n} \left|\theta(k)\right|^2 + 2\left|\theta(n)\right|\left|\theta(\phi(n))\right|}\right\}$$

and hence

$$W(C_{\theta,\ \phi}) \subset \left\{\alpha \in \mathbb{C} : \left|\alpha\right| \leq \sup_n \frac{1}{2}\sqrt{\sum_{k\in E_n} \left|\theta(k)\right|^2 + 2\left|\theta(n)\right|\left|\theta(\phi(n))\right|}\right\}.$$

□

Theorem 3.5. *Let $C_{\theta,\ \phi} \in B(\ell^2(N))$. Suppose ϕ^k has no fixed point for any $k \in \mathbb{N}$ and $\lim_{n\to\infty} \theta(n) = \|\theta\|_\infty$. Then,*

$$\left\{\lambda \in \mathbb{C} : \left|\lambda\right| < \|\theta\|_\infty\right\} \subset W(C_{\theta,\ \phi}) \subset \left\{\lambda \in \mathbb{C} : \left|\lambda\right|\right.$$
$$\left. < \sup_{n\in\mathbb{N}} \sqrt{\sum_{m\in\phi^{-1}(\{n\})} \left|\theta(m)\right|}\right\}.$$

The equality holds if ϕ is an injection.

Proof. We shall divide the proof in two steps:

Step I : Suppose there exists $n_1 \in \mathbb{N}$ s.t $\phi^{-1}(\{n_1\})$ is empty. Write $\phi^k(n_1) = n_{k+1}$. Let $E_{n_1} = \{n_k : k \in \mathbb{N}\}$. Then E_{n_1} is an infinite subset of $\mathbb{N}$. It is easy to see that $C^*_{\theta,\ \phi}\ell^2(E_{n_1}) \subset \ell^2(E_{n_1})$. Define $S : \ell^2(E_{n_1}) \to \ell^2(\mathbb{N})$ by $S(e_{n_k}) = e_k$ for every $k \in \mathbb{N}$. Clearly S is a unitary operator and $C^*_{\theta,\ \phi}\big|_{\ell^2(E_{n_1})} = S^{-1}US$, where U is the unilateral shift operator with weights $\left\{\overline{\theta(n)} : n \in E_1\right\}$. Hence by Theorem 1(i) of Tam [8],

$$W\left(C^*_{\theta,\ \phi}\big|_{\ell^2(E_{n_1})}\right) = W(U) = \left\{\lambda \in \mathbb{C} : |\lambda| < \sup_{n \in E_{n_1}} |\theta(n)|\right\}$$

or that

$$W\left(C_{\theta,\ \phi}\big|_{\ell^2(E_{n_1})}\right) = \left\{\lambda \in \mathbb{C} : |\lambda| < \sup_{n \in E_{n_1}} |\theta(n)|\right\}.$$

Consequently,

$$\left\{\lambda \in \mathbb{C} : |\lambda| < \sup_{n \in E_{n_1}} |\theta(n)|\right\} \subset W\left(C_{\theta,\ \phi}\right).$$

It follows that if $E_{n_1} = \mathbb{N}$, then

$$\left\{\lambda \in \mathbb{C} : |\lambda| < \|\theta\|_\infty\right\} \subset W(C_{\theta,\ \phi}) \subset \left\{\lambda \in \mathbb{C} : |\lambda| < \sqrt{\sup_{n\in\mathbb{N}} \sum_{m \in \phi^{-1}(\{n\})} |\theta(m)|^2}\right\}. \quad (7)$$

Step II: If $\phi^{-1}(\{n\})$ is nonempty for every $n \in \mathbb{N}$, then for $n_0 \in \mathbb{N}$, write $\phi^k(n_0) = n_k$ and let $n_{-k} \in \mathbb{N}$ to be s.t. $\phi^k(n_{-k}) = n_0$.

Let $F_{n_0} = \{n_k : k \in \mathbb{Z}\}$. Define $S : \ell^2(F_{n_0}) \to \ell^2(\mathbb{Z})$ by $S(e_{n_k}) = e_k$. Then $C^*_{\theta,\ \phi}\big|_{\ell^2(F_{n_0})} = S^{-1}WS$, where W is the bilateral weighted shift with shifts $\left\{\overline{\theta(n)} : n \in F_{n_0}\right\}$. Hence by Theorem 1(ii) of Tam [8],

$$W\left(C^*_{\theta,\ \phi}\big|_{\ell^2(F_{n_0})}\right) = \left\{\lambda \in \mathbb{C} : |\lambda| < \sup_{n \in F_{n_0}} |\theta(n)|\right\}$$

or that

$$W\left(C_{\theta,\ \phi}\big|_{\ell^2(F_{n_0})}\right) = \{\lambda \in \mathbb{C} : |\lambda| < \sup_{n\in F_{n_0}} |\theta(n)|\}.$$

As $n \to \infty$, $|\theta(n)| \to \|\theta\|_\infty$ we can conclude that

$$\{\lambda \in \mathbb{C} : |\lambda| < \|\theta\|_\infty\} \subset W(C_{\theta,\ \phi}) \subset \left\{\lambda \in \mathbb{C} : |\lambda| < \sqrt{\sup_{n\in\phi(\mathbb{N})} \sum_{m\in\phi^{-1}(\{n\})} |\theta(m)|^2}\right\}.$$

Let $G = \left\{O_\phi(n) : n \in \mathbb{N} \text{ and } \phi^{-1}(\{m\}) \text{ is empty for some } m \in O_\phi(n)\right\}$ and $H = N - G$.

Clearly, $\ell^2(G) = \sum_{m\in G \text{ and } \phi^{-1}(\{m\})=\phi} \oplus\, \ell^2(E_m)$, $\ell^2(H) = \sum_{p\in H} \oplus\, \ell^2(F_p)$ and $\ell^2(\mathbb{N}) = \ell^2(G) \oplus \ell^2(H)$.

From the first part of the proof we can conclude that

$$\{\lambda \in \mathbb{C} : |\lambda| < \|\theta\|_\infty\} \subset W(C_{\theta,\ \phi}) \subset \left\{\lambda \in \mathbb{C} : |\lambda| < \sqrt{\sup_{n\in\mathbb{N}} \sum_{m\in\phi^{-1}(\{n\})} |\theta(m)|^2}\right\}. \quad (8)$$

If ϕ is injective, then

$$\|C_{\theta,\ \phi}\| = \|\theta\|_\infty$$

and hence

$$W(C_{\theta,\ \phi}) \subset \{\lambda \in \mathbb{C} : |\lambda| < \|\theta\|_\infty\}.$$

Thus equality holds in view of (3.7) and (3.8). □

References

[1] P.S. Bourdon and J.H. Shapiro, When is zero in the numerical range of a composition operator, *Integ. Equ. Oper. Theory*, **44** (2002), 410-441.

[2] J.B. Conway, A course in functional analysis, *Trans. Amer. Math. Soc.*, **239** (1978), 1-31.

[3] K.E. Gustafson and D.K.M. Rao, *Numerical Range: The Field of Values of Linear Operators and Matrices*, Springer-Verlag, Heidelberg - Berlin - New York (1997).

[4] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, New York (1991).

[5] B.S. Komal and S. Sharma, Numerical ranges of composition operators on l^2, *Hokkaido Math. J.*, **35** (2006), 1-12.

[6] W. Ridge, Numerical ranges of a weighted shift with periodic weights, *Proc. Amer. Math. Soc.*, **55**, No 1 (1976), 107-110.

[7] J.H. Shapiro, *Composition Operator and Classisical Function Theory*, Springer-Verlag, New York - Heidelberg - Berlin (1993).

[8] T.Y. Tam, On a conjecture of Ridge, *Proc. Amer. Math. Soc.*, **125** (1997), 3581-3592.