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ON RATIONAL AND ADAPTIVE SPECTRAL METHODS

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Abstract: In this work, we give a general overview on pseudo-spectral methods. Three types of its variants which are: the adaptive spectral methods conceived for functions with rapid variation in certain domains, the rational adaptive spectral method using rational interpolants, and the adaptive grid Chebyshev points thanks to conformal maps, another type of approximation by the Chebyshev spectral method constituted by a collection of algorithms coded in an objet-oriented MATLAB software environment are considered.

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1. Introduction

Spectral methods form a classe of provided numerical methods to solve partial differential equations. Among these spectral methods include Galerkin method, Tau method, Tau-collocation method, collocation method, and so many others.

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We will be interested in the two last one which constitute spectral collocation methods often called pseudo-spectral methods.

Spectral methods have known a big success favor partly about books published in 1977 by Gottlieb et al. [19], Gottlieb et al. [20](1984), and the first publishing of the book of Canuto et al. [12], which covers aspects of the application of spectral methods as long on plans theoretical that practice. But practices of spectral methods were evolved with B. Fornberg's works [18]. It explains the methods under a specific angle, like being the limit of finite difference methods and the book [32]. These methods are widely studied in the literature [21, 22, 26]. We can estimate that pseudo-spectral methods knew their real development from 1970s, after Cooley et al [13] proposed in 1965 a first algorithm of the Fast Fourier Transform (FFT).

The concept of pseudo-spectral (PS) methods is to approach unknown solutions of partial differential equations by global interpolants which satisfy collocation equations. The pseudo-spectral Chebyshev method is based on the algebraic polynomial interpolant and the Chebyshev points $x_k = -\cos(k\pi/N)$ where N is the degree of the polynomial interpolant. It is applied to problems with the nonperiodic boundary conditions on bounded intervals.

The pseudo-spectral Fourier method uses the trigonometric interpolating polynomial and the equidistant points $x_k = k\pi/N$ with $k = 0, 1, ..., N-1$ and also applied to problems with periodic boundary conditions on a bounded interval. See their theorems of convergence in [29](1986).

These two types of methods are applied to the resolution of problems with solutions sufficiently smooth. When a real function f can be continued analytically to the closed ellipse E_ρ(ellipse with foci ± 1, and ρ the sum of semi axis lengths), the spectral Chebyshev method using $N+1$ collocation points approaches the derivative of this function on $[-1, 1]$ with an error which decreases at a rate $O\left(\rho^{-N}\right)$ as $N \longrightarrow \infty$ [29]. Such a convergence means that a few degrees of freedom are sufficient to achieve a high degree of accuracy. That is not the case with other methods of weak order, such as the finite difference (FD) or finite element methods (FE) methods, for which a correct accuracy is obtained only at the price of large increase of the number of points (and consequently, of computer time). A comparison between PS Fourier and FD methods is given in the following table 1 extracted from [11]:

But if the function f has singularities in the complex plane close to $[-1, 1]$ so that $\rho \approx 1$, convergence can be too slow for the method to be effective. In this case, it would the function f has to be analytical into a large region nonelliptical region by building a larger ellipse with foci ± 1 with the aid of conformal maps g wherein f is analytical with the aim of improving the convergence rate.

N	FD 2nd order	FD 4th order	PS Fourier
16	$3.4020e-001$	$1.3844e-001$	$4.3179e-003$
32	$9.3589e-002$	$1.5750e-002$	$1.7619e-007$
64	$2.5833e-002$	$1.1318e-003$	$2.3870e-014$
128	$6.5118e-003$	$7.5900e-005$	$7.2054e-014$

Table 1: The table shows the error decay of approximating the first derivative of $e^{\sin x}$ through a second and a fourth order FD method and a Fourier pseudo-spectral method.

This paper is organized as follows: after this introduction in section one, adaptive spectral method as well as some theoretical results are presented in Section 2. In Section 3, we will focus on adaptive rational spectral methods. A view on an other type of approximation by the spectral Chebyshev method constitutes Section 4. Remarks and open questions are expressed in the last section.

2. Adaptive Spectral Methods

Adaptive spectral methods were initially designed for functions with rapid variation adapting collocation points in form to functions thanks to conformal maps [4, 5, 21, 25]. There are two issues that must be addressed in the systematic application of mappings to enhance the accuracy of the pseudo-spectral Chebyshev method. These are:

- the construction of an appropriate family of mappings,
- the criteria for choosing a particular mapping from this family according to the behavior of the solution to be approximated.

In [4], the chosen transformation is

$$x = 1 + \frac{\pi}{4}\arctan\left(\alpha_1 \tan\left(\frac{\pi}{4}\left(\frac{\alpha_2 - y}{\alpha_2 y - 1} - 1\right)\right)\right), \qquad (1)$$

with $-1 \leq y \leq 1$, $-1 < \alpha_2 < 1$ and $\alpha_1 > 0$. This maps Chebyshev points in the transformed coordinate into points that cluster in physical coordinates.

To describe adaptive spectral methods [4], the following non linear reaction, diffusion, convection equation is considered

$$u_t = u_{xx} + u_x + R(u),\ -1 \leq x \leq 1, \qquad (2)$$

where $R(u)$ is a nonlinear term. Let u be a solution of the equation (2), the chosen functional is

$$I_2(u) = \left(\int_{-1}^{1}\left(\left|\frac{d^2u}{dx^2}\right|^2 + \left|\frac{du}{dx}\right|^2 + |u|^2\right)\frac{1}{\sqrt{1-x^2}}dx\right)^{1/2}, \tag{3}$$

which is an increasing of the error of approximation of u by the spectral Chebyshev method measured in a weighted discrete norm L_2. This enables to find $\underset{\sim}{\alpha} = (\alpha_1, \alpha_2)$ such that

$$\widetilde{I}\left(\underset{\sim}{\alpha}\right) = \left(\int_{-1}^{1}\left(\left|\frac{d^2u}{dy^2}\right|^2 + \left|\frac{du}{dy}\right|^2 + |u|^2\right)\frac{1}{\sqrt{1-y^2}}dy\right)^{1/2}, \tag{4}$$

is minimized.

Kosloff et al [25] have considered the first following order hyperbolic initial boundary value problem

$$\begin{aligned} u_t - u_x &= 0, && -1 < x < 1, \quad t \geq 0, \\ u(x,0) &= u_0(x), && -1 < x < 1, \\ u(1,t) &= s(t) && t > 0, \end{aligned}$$

and, have generalized it in the second order case.The collocation points are found from the *stretching* of regular Chebychev collocation points. Precisely, if x is the physical coordinates and y its transformed ones, then one can write

$$x = g(y;\alpha) = \frac{\arcsin(\alpha y)}{\arcsin(\alpha)}; \quad 0 \leq \alpha \leq 1 \quad x, y \in [-1, 1], \tag{5}$$

where α is a parameter of the *stretching* function $g(y;\alpha)$. The identity is recovered when $\alpha \to 0$. The interest of (5) is to transform the Gauss-Lobatto y_i points into equally spaced x_i points when $\alpha \to 1$. This limit is singular so that the optimal use of the transformation corresponds to values of α closed to 1, with $\alpha = \cos(k\pi/N)$ where N is the polynomial degree and $k \geq 1$. The differentiation of a function u is written by

$$\frac{du}{dx} = \frac{1}{g'(y)}\frac{du}{dy}, \tag{6}$$

where

$$g'(y;\alpha) = \frac{\alpha}{\arcsin\alpha\sqrt{1-(\alpha y)^2}}.$$

Consequently, the first order differentiation operator D is replaced by $\widetilde{D} = AD$ where A is a diagonal matrix with entries $A_{ii} = \dfrac{1}{g'(y_i, \alpha)}$ and $y_i = \cos\left(\frac{i\pi}{N}\right)$. The second order differentiation is

$$\frac{d^2u}{dx^2} = \frac{1}{(g'(y))^2}\frac{d^2u}{dy^2} - \frac{g''(y)}{(g'(y))^3}\frac{du}{dy} \tag{7}$$

and the second order derivative operator is replaced by $\widetilde{D_2} = A^2D_2 + BD$ where $A^2 = A \times A$ and B represents the diagonal matrix with entries $B_{ii} = -\dfrac{g''(y_i;\alpha)}{(g'(y_i;\alpha))^3}$.

Kosloff et al. [25] have verified that the eigenvalues of the matrix $\widetilde{D}$ are insensitive with respect to perturbations. The evaluation of the matrix $\widetilde{D_2}$ for various values of N and α shows that matrices $\widetilde{D_2}$ are nearly well-conditioned. The transformation decreases the spectral radius of the differential operator of $O\left(N^2\right)$ in $O(N)$, increases timestep from $O\left(N^{-2}\right)$ to $O\left(N^{-1}\right)$ for hyperbolic problems and from $O\left(N^{-4}\right)$ to $O\left(N^{-2}\right)$ for those parabolic.

In 1992, several coordinate transformations were compared [5] based on the tangent function judges numerically to be the best:

$$x = f(y, a_1, a_2) = a_1 + a_2 \tan\left[\lambda\left(y - y_0\right)\right] \tag{8}$$

$$\text{with} \quad y_0 = \frac{\kappa - 1}{\kappa + 1}, \kappa = \frac{\arctan\left[\left(1 + a_1\right)/a_2\right]}{\arctan\left[\left(1 - a_1\right)/a_2\right]}, \quad \text{and}$$

$$\lambda = \frac{\arctan\left[\left(1 + a_1\right)/a_2\right]}{1 - y_0}.$$

In the region of rapid variation, characterized by $x = a_1$ and $y = y_0$, one has $f'(y_0, a_1, a_2) \sim \pi a_2/2$ if $a_2 \to 0$. Under these conditions, the inverse $y = g(x, a_1, a_2)$ gives an approximation of a quasi-step function with a near-discontinuity at $x = a_1$. For large values of a_2, the transformation (8) is closed to the identity.

The Chebyshev polynomial approximation is naturally well-adapted to the representation of boundary layers. It may happen, however, that the thickness of the boundary layer is so small that a high-degree polynomial is necessary to capture it accurately. Then, it is advisable to introduce an adapted mapping. The coordinates transformation (8) is singular for boundary layers, namely for $a_1 = \pm 1$.

An alternative mapping has been employed with success for various problems dependent of the one parameter:

$$x = f(y, a) = \frac{4}{\pi} \tan^{-1} \left[a \tan \frac{\pi}{4} (y - 1) \right] + 1. \tag{9}$$

It maps the interval $-1 \leq x \leq 1$ onto $-1 \leq y \leq 1$ with $f'(-1, a) = 1/a$ and $f'(1, a) = a$. Therefore, the coordinates transformation (9) is adapted to the capture of the boundary layer near $x = -1$ where $a > 1$ and near $x = 1$ where $a < 1$.

The effectiveness for transformations (5), (8) and (9) in pseudo-spectral approximations were compared [5], the approach was to build the pseudo-spectral polynomial for the transformed function and then to measure the maximum norm of the error. This maximum norm of the error is then computed by comparing the approximating polynomial and the given function over a large grid of points then they computed the discrete L_2 norm of the error in the new coordinate system. Numerically, the transformation (8) appears more effective than (5) and (9).

3. Rational Adaptive Spectral Methods

Adaptive rational spectral methods are based on rational interpolants and adaptive grid collocation points to solve problems whose solutions present singularities in the complex plane. We notice that it is more accurately to approach functions with localized regions of rapid variation in some domains by rational functions than by polynomials with the same number of degrees of freedom.

3.1. Linear Rational Interpolants

Berrut et al. [7] have developed spectral methods based on rational interpolants by using a barycentric formulation. This form r that interpolates a function f at $N + 1$ distinct points x_0, x_1,...,x_N on the interval $[-1, 1]$ is

$$r(x) = \frac{\displaystyle\sum_{k=0}^{N} \frac{\omega_k}{x - x_k} f(x_k)}{\displaystyle\sum_{k=0}^{N} \frac{\omega_k}{x - x_k}}, \tag{10}$$

where ω_0, ω_1,..., ω_n are nonzero numbers called barycentric weights. At the Chebyshev points, one has $\omega_0 = c/2$, $\omega_k = c(-1)^k$, for $k = 1, 2, ..., N - 1$ and

$\omega_N = (-1)^N c/2$ for a same constant c [10]. An other method for calculating barycentric weights is proposed in [6].

The derivatives of r can serve to determine the nth order differentiation matrix $\left(D^{(n)}\right)$ associated with r represented by (10) at the points x_j :

$$r^{(n)}(x_j) = \sum_{k=0}^{N} \frac{d^n}{dx^n} \left(\frac{\dfrac{\omega_k}{x - x_k} f(x_k)}{\displaystyle\sum_{l=0}^{N} \frac{\omega_l}{x - x_l}} \right)_{x=x_j} \equiv \sum_{k=0}^{N} D_{jk}^{(n)} f(x_k), \quad j = 0, ..., N. \tag{11}$$

This expression enable to compute the differentiation matrix $D^{(n)}$ for various values of n if f is known or unknown at Chebyshev points. The entries of the first and second order differentiation matrices are [27]:

$$D_{jk}^{(1)} = \begin{cases} \dfrac{\omega_k}{\omega_j (x_j - x_k)}, & \text{if } j \neq k; \quad j,k = 0,1,..N, \\ -\displaystyle\sum_{i=0\ i\neq k}^{n} D_{ji}^{(1)}, & \text{if } j = k, \end{cases} \tag{12}$$

and

$$D_{jk}^{(2)} = \begin{cases} 2D_{jk}^{(1)} \left(D_{jj}^{(1)} - \dfrac{1}{x_j - x_k} \right), & \text{if } j \neq k; \quad j,k = 0,1,...,N \\ -\displaystyle\sum_{i=0\ i\neq k}^{n} D_{ji}^{(2)}, & \text{if } j = k. \end{cases} \tag{13}$$

The entries of the nth order differentiation matrices of the algebraic rational interpolants r in their form (10) are [30]:

$$D_{jk}^{(n)} = \begin{cases} \dfrac{n}{(x_j - x_k)} \left(\dfrac{\omega_k}{\omega_j} D_{jj}^{(n-1)} - D_{jk}^{(n-1)} \right), & \text{if } j \neq k; \\ -\displaystyle\sum_{l=0\ i\neq k}^{n} D_{jl}^{(n)}, & \text{if } j = k, \end{cases} \tag{14}$$

where $D^{(0)}$ is the identity matrix.

Berrut et al. [8] have considered the interpolation problem of a continuous function at $N+1$ distinct points x_0, x_1,...,x_N in the interval $[-1,1]$ to solve the problem with boundary conditions. Let $\mathcal{P}_m$ and $\mathcal{R}_{mn}$ be respectively the linear space of all polynomial degree $\leq m$ and the set of all rational functions with the numerator of degree $\leq m$ and the denominator of degree $\leq n$.

Let P $(P \leq N)$ be the number of the poles z_i, $i = 1, .., N$, connected to polynomial of denominator of the set $\mathcal{R}_{mn}$. If the same rational interpolant $r \in \mathcal{R}_{N,P}$ exists with the poles z_i, so the denominator takes at the points x_k the following values:

$$d_k := a \prod_{i}^{P} (x_k - z_i), \tag{15}$$

where a is an arbitrary nonzero complex constant, and the values of numerator to the same points $f_k d_k$. By using (10), we get:

$$r(x) := \frac{\displaystyle\sum_{k=0}^{N} \frac{\omega_k \prod_i^P (x_k - z_i)}{x - x_k} f(x_k)}{\displaystyle\sum_{k}^{N} \frac{\omega_k \prod_i^P (x_k - z_i)}{x - x_k}}. \tag{16}$$

The expression (16) is affected to weights $v_k := w_k \prod_i^P (x_k - z_i)$ and exist for all rational interpolants r in $\mathcal{R}_{N,N}$ [9] and can be written as follows

$$r(x) = \frac{\displaystyle\sum_{j=0}^{N} \frac{\beta_j}{x - x_j} f(x_j)}{\displaystyle\sum_{j=0}^{N} \frac{\beta_j}{x - x_j}}, \tag{17}$$

the numbers β_j are arbitrary by reason of one per node. This form is employed for the classical problem of rational interpolation whose poles are described previously.

As example, we consider the problem of finding the solution of the following differential equation

$$\begin{cases} u''(x) + p(x)u'(x) + q(x)u(x) = h(x), \quad x \in]-1, 1[, \\ u(-1) = u_l, \quad u(1) = u_r, \end{cases} \tag{18}$$

where functions p, q, and h are chosen such that the problem is well-posed, and u_l and u_l are given real numbers. By reason of presence in the problem of non-homogeneous boundary values, the restriction is made at the node containing the ends -1 and 1 for β fixed $\beta = [\beta_0, \beta_1, ...\beta_N]^T$. The functions

$$L_j^{(\beta)}(x) := \frac{\dfrac{\beta_j}{x - x_j}}{\displaystyle\sum_{k=0}^{N} \frac{\beta_j}{x - x_k}}, j = 0, 1, ..., N \tag{19}$$

form a basis in the space $\mathcal{R}_N^{(\beta)}$ of all interpolants and satisfy the following Lagrange property $L_j^{(\beta)}(x_j) = \delta_{ij}$. The solution of the equation (18) as interpolant by linear rational collocation method at the points x_j is expressed by

$$\widetilde{u}(x) = \sum_{j=0}^{N} u_j L_j^{(\beta)}(x), \tag{20}$$

then we have the equation

$$\sum_{j=0}^{N} u_j L_j^{(\beta)''}(x) + p(x) \sum_{j=0}^{N} u_j L_j^{(\beta)'}(x) + q(x) \sum_{j=0}^{N} u_j L_j^{(\beta)}(x) = h(x) \tag{21}$$

which provide a system of equations for unknown values u_1,..., u_{N-1} of $\widetilde{u}$ at the Chebyshev points x_1,..., x_{N-1} (u_0 and u_N being known from boundary conditions) by setting

$$\sum_{j=0}^{N} u_j L_j^{(\beta)''}(x_i) + p(x_i) \sum_{j=0}^{N} u_j L_j^{(\beta)'}(x_i) + q(x_i) \sum_{j=0}^{N} u_j L_j^{(\beta)}(x_i) = h(x_i), \tag{22}$$

for $i = 1, ..., N-1$. This system of linear equations can be written as

$$\mathbf{Au} = \mathbf{h}, \tag{23}$$

where we have set

$$\mathbf{A} = \mathbf{D}^{(2)} + \mathbf{PD}^{(1)} + \mathbf{Q}; \quad \mathbf{u} = [u_1, u_2, ..., u_{N-1}]^T, \tag{24}$$

with

$$\left\{\begin{array}{lcl}\mathbf{D}^{(1)} & = & \left(D_{ij}^{(1)}\right), \quad D_{ij}^{(1)} = L_j^{(\beta)'}(x_i), \\ \mathbf{D}^{(2)} & = & \left(D_{ij}^{(2)}\right), \quad \left(D_{ij}^{(2)}\right) = L_j^{(\beta)''}(x_i), \\ \mathbf{P} & = & diag\,(p(x_i))\,, \quad \mathbf{Q} = diag\,(q(x_i))\,, \\ \mathbf{h} & = & \Big[h(x_i) - u_r\left(L_0^{(\beta)''}(x_i) + p(x_i)L_0^{(\beta)'}(x_i)\right) \\ & - & u_l\left(L_N^{(\beta)''}(x_i) + p(x_i)L_N^{(\beta)'}(x_i)\right)\Big]^T, \\ & & \text{for} \quad i,j = 1,...,N-1.\end{array}\right. \tag{25}$$

The advantage of the barycentric representation is the simplicity of the formula for the entries of the matrices $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ as analogues of (12) and (13). The reason of the efficiency of the method in the polynomial case based on the spectral exponential convergence of $\widetilde{u}$ toward u, provided that all functions resulting in the problem are analytic within ellipses containing the interval $[-1,1]$ in their interior.

3.2. Adaptive Rational Spectral Methods

Tee et al. [31] have presented the spectral collocation method that uses adaptively transformed Chebyshev nodes. This method is based on the exponential convergence of rational approximations which interpolates at transformed Chebyshev nodes of the form

$$r_N = \frac{\displaystyle\sum_{k=0}^{N} \frac{(-1)^k}{x - x_k} f(x_k)}{\displaystyle\sum_{k=0}^{N} \frac{(-1)^k}{x - x_k}}, \quad x_k = g\left(-\cos\left(\frac{k\pi}{N}\right)\right). \tag{26}$$

Here, the interpolation points are the images of the Chebyshev points by a conformal map g defined of $[-1,1]$ on itself and such as $f \circ g$ is analytical on an appropriate neighborhood of the complex plane.

The expression (26) has been exploited to prove theorems on the exponential convergence of rational functions [2] and its derivatives [30].

The authors [31] have considered solutions with a relevant front, and have constructed g using two singularities of the solution, $\delta \pm \epsilon\imath$, that are symmetric with respect to the real line. The proposed conformal map is:

$$g(z) = \delta + \epsilon \sinh\left(\left(\sinh^{-1}\left(\frac{1-\delta}{\epsilon}\right) + \sinh^{-1}\left(\frac{1+\delta}{\epsilon}\right)\right)\frac{z-1}{2} + \sinh^{-1}\left(\frac{1-\delta}{\epsilon}\right)\right), \quad (27)$$

where $\sinh^{-1}$ is the inverse of *sine* hyperbolic. Once the grid adapted as $x_k = g\left(-\cos\left(\frac{k\pi}{N}\right)\right)$, the rational interpolant is used with differentiation matrices given by the expression (14). By using the map (27), they have proposed for the approximate location of singularities in the complex plane and conformal mapping in order to transform the Chebyshev grid into one that adaptively clusters points near steep gradients of the solution (singular lines).

Two evolution problems were to be solved via the algorithm [31]: Franck-Kamenetskii or Gelfand called *blow up* (reaction-diffusion) and viscous Burgers equations [31]. They are discretized temporally using the adaptive Runge-Kutta 5.4 method [14] and its variants for nonstiff time-dependent and stiff problems [22, 23]. Solutions are determined by the Chebyshev and adaptive rational spectral methods for various values of N. In both examples, the sharp features of the solution evolve in time and the explicit time marching allows of the singularities. Numerically, the singularity location requires the definition of a local approximation of the solution, which is used to approximate the poles using the Chebyshev Padé process. This is done by selecting an interval around the previously computed poles $\delta \pm \epsilon\imath$, such as $I = [\delta - \xi, \delta - \xi]$ with $\xi = \min(10\epsilon, 1 - |\delta|)$. The solution is interpolated at the $N + 1$ Chebyshev points in the interval I, the singularities domain provides the new poles. The new value of ϵ is multiplied by a factor equal to 0.75. After the above approach, the minimum limit value of ϵ is $\epsilon = \max(\epsilon_0, \epsilon^*, \epsilon_{\min})$, where ϵ_0 is the value provided by singularities of the domain due to Chebyshev Padé approximation, ϵ^* is a value that progressively decreases in the process, and $\epsilon_{\min}$ is a minimum limit value due to the explosive growth of the condition number of the differentiation matrices for small values of ϵ.

W. Tee [30] have developed these adaptive rational spectral methods to solve the periodic and nonperiodic differential equations.

For nonperiodic problems, the transformation can be done into two stages. The first one appeared in the article of Szegő [28] attributed to Schwarz, which maps the interior of the ellipse E_ρ to the unit disk by the following transformation:

$$h_1(z) = \sqrt[4]{m} sn\left(\frac{2K(m)}{\pi}\arcsin(z) \mid m\right) \tag{28}$$

where $sn(\cdot \mid m)$ is the Jacobi elliptic *sine* function defined in [1]

$$sn(z \mid m) = \sin(\phi) \tag{29}$$

and

$$z = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2(\theta)}}, \tag{30}$$

with $m \in]0, 1[$ is an elliptic parameter. The conformal map (28) maps the interval $[-1, 1]$ into the interval $[-\sqrt[4]{m}, \sqrt[4]{m}]$, and the elliptic parameter m satisfies

$$\rho = \exp\left(\frac{\pi K'(m)}{4K(m)}\right), \tag{31}$$

where $K(m)$ and $K'(m)$ are the complete elliptic integrals of the first kind and second kind respectively [1]. They are implemented in MATLAB routines `ellipj`, `ellipke` for real and `ellipjc` for complex arguments [15].

The second stage is the mapping of the unit disk, the analytical region of f into a slit $S_{\delta+\imath\epsilon}$ in which f is analytic and defined by

$$S_{\delta+\imath\epsilon} = \mathbb{C}\backslash\left\{[\delta - \imath\epsilon, \delta - \imath\infty] \bigcup [\delta + \imath\epsilon, \delta + \imath\infty]\right\}, \tag{32}$$

where $\epsilon > 0$. The use of the Schwarz-Christoffel formula [17], the expression (28) becomes

$$h_2(z) = A - C\left(\frac{1 - \cos(\theta)}{2(z-1)} + \frac{1 + \cos(\theta)}{2(z+1)}\right), \tag{33}$$

where the four unknowns A, C, θ and m can be determined under the following conditions:

$$h_2\left(-\sqrt[4]{m}\right) = -1, \quad h_2\left(\sqrt[4]{m}\right) = 1, \tag{34}$$

$$\Re\left(h_2(z_1)\right) = \delta, \quad \text{and} \quad \Im\left(h_2(z_1)\right) = \epsilon \tag{35}$$

with $z_1 = \exp(\imath\theta)$.

Thus, the conformal map $g = h_2 \circ h_1$ maps the interior of the ellipse E_ρ into a plane slit $S_{\delta+\imath\epsilon}$ (for illustration, see [24]).

In the periodic case, the problem is to map an infinite horizontal slit

$$S_{\delta+\imath\epsilon} = \mathbb{C}\backslash \bigcup_{j=-\infty}^{\infty} [(\delta + 2j\pi) \pm \imath\epsilon, (\delta + 2j\pi) \pm \imath\infty] \tag{36}$$

that is symmetric around the real axis of the region of analyticity of the function f thanks to the transformation

$$g(z) = \delta + 2 \arcsin\left(\sqrt{\frac{(m-1)sn^2(\frac{K}{\pi}(z-\delta)|m)}{msn^2(\frac{K}{\pi}(z-\delta)|m)-1}}\right), \qquad (37)$$

where m is the elliptic parameter given by $m = sech^2(\frac{\epsilon}{2})$.

4. Chebfun System

`Chebfun` is a software system written in object-oriented **MATLAB**. It was implemented in a 2004 paper [3] for the smooth functions on the interval $[-1,1]$ with the aim of building some links between the discrete and continuous linear algebra in particular to extend the **MATLAB** operations made on vectors and matrix in the functions and the operators.

The basis of `Chebfun` is Chebyshev polynomials [33, 34]. `Chebfun` computes Chebyshev coefficients by examining them in the machine precision of a function `g` and represents it to the Chebyshev points by using barycentric interpolation [10]. For example, we can obtain the Chebyshev coefficients a_k in `Chebfun` corresponding to $\exp(x)$:

```
>>g=chebfun(@(x) exp(x));
>>[length(g)]
ans =
    15
>>Coef=chebpoly(g);% compute the Chebyshev coefficients
```

1.266065877752008	0.000542926311914	0.000000000550590
1.130318207984970	0.000044977322954	0.000000000024980
0.271495339534077	0.000003198436463	0.000000000001039
0.044336849848664	0.000000199212481	0.000000000000040
0.005474240442094	0.000000011036772	0.000000000000001

Note that the last coefficient is about the level of machine precision fixed approximatively at 10^{-15}. The indication `length` means that 15 is the number of Chebyshev points necessary to represent this function to the machine precision. Then, the function g is approximated by the polynomial of degree 14.

The commands like `diff`, `sum` and `norm` of MATLAB allow to compute the derivative, the definite integral and the norm respectively. Also, there was an implementation of the continuous and piecewise functions, the notions of linear algebra including functions such as QR decomposition of a matrix A which spells $A = QR$ where Q is an orthogonal matrix and R an upper triangular matrix and, the singular value decomposition process SVD of a matrix A which factorizes like $A = USV^T$ where U in is the same size that A, S size $n \times n$ and diagonal with the entries non negative, V size $n \times n$ and orthogonal.

`Chebfun` have the capability to solve the nonlinear partial differential and ordinary differential equations [16]. If p is a polynomial of degree n, it is determined by its values on the $(n + 1)$ point Chebyshev grid in $[-1, 1]$. The derivative $p^{'}$, a polynomial of degree $n - 1$ is determined by its values on the same grid. The spectral differentiation matrix associated with this grid is the $(n+1)\times(n+1)$ matrix that represents the linear map from the vector of values of p on the grid to the vector of values of p'. For example, we can calculate the values of p' on the same grid by the command `diff(p)`. But we can also obtain on the differentiation matrix explicitly with the commands:

```
>>[d,x]=domain([-1 1]);%domain of computation
>>D=diff(d); %Chebyshev differentiation operator
>>D15=D(15); %Discrete matrix of size 15x15
```

Generally, `D=diff(d,n)` for $n \geq 2$ definite the nth order of Chebyshev differentiation matrices, `eye` identity matrix. If `D` is applied to an integer argument, the matrix of that dimension is produced. Thus, the differential operator corresponding to the map

$$L : u \longrightarrow u'' + u' + 100u \quad \text{on} \quad [-1, 1]$$

is

`L(5)= diff(d,2)+diff(d)+100*eye.`

Here is the 5×5 realization of this operator

$$\begin{pmatrix} 111.5000 & -21.6569 & 16.0000 & -10.3431 & 4.5000 \\ 7.5355 & 86.7071 & 7.4142 & -2.7071 & 1.0503 \\ -0.5000 & 2.5858 & 94.0000 & 5.4142 & -1.5000 \\ 0.4645 & -1.2929 & 4.5858 & 85.2929 & 10.9497 \\ 5.5000 & -12.6863 & 20.0000 & -35.3137 & 122.5000 \end{pmatrix}$$

If we impose boundary conditions (bc), the standard way of doing this is to modify one or more initial or final rows of the matrix, one row for each boundary condition [32, 34].

5. Concluding Remarks

Common features to adaptive rational spectral methods and `Chebfun` are the barycentric formulation of algebraic interpolants. Most of these methods are not extended to the resolution of differential equations in the domains to complex geometries and in higher dimensions.

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