WEIGHTED GAUSSIAN QUADRATURES FOR ORTHOGONAL WAVELET FUNCTIONS IN THE INTERVAL

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Abstract: We study the numerical evaluation of integrals involving scaling functions from the Cohen-Daubechies-Vial (CDV) family of compactly supported orthogonal wavelets on the interval. The computation of the wavelet coefficients is performed by a weighted Gaussian quadrature, in conjunction with the Chebyshev and modified Chebyshev algorithms. We validate the proposed quadratures with the numerical approximation of a Fredholm integral equation of second kind by the Galerkin method with CDV scaling functions as basis functions.

1. Introduction

The solid and elegant theoretical background and the design of efficient algorithms for a wide range of applications in sciences and engineering have turned wavelet theory into one of the pillars of computational mathematics. Among the several contributions to this field, we point out the classical families of wavelet functions proposed by I. Daubechies [4, 5], which preserve the orthogonality and the compactness of support from the Haar system but provide increasing regularity (see, e.g., [4, p. 226]).

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Unlike the Haar system, it is not trivial to construct a basis for square-integrable functions in an interval (say $L^2([0,1])$) from the Daubechies family. One of the most successful constructions was proposed by Cohen, Daubechies, and Vial (CDV, [2]), which we consider in this work. The motivation for revisiting the CDV wavelet basis is the increasing popularity of Wavelet-Galerkin methods for integral equations on bounded intervals [8, 14, 17].

Another interesting property of the Haar system, which is not preserved on compactly-supported orthogonal wavelets with higher regularity, is that a closed formula is available for the scaling function (namely, $\phi^{Haar}(x) = \chi_{[0,1]}(x)$, where $\chi_I(\cdot)$ is the characteristic function over the interval I). The Daubechies and CDV scaling functions are indirectly given by scaling (or refinement) equations in the form

$$\phi(x) = \sum_{n \in \mathbb{Z}} c_n \phi(2x - n).$$

How to compute an integral that involves a function defined by a scaling equation? This is a crucial task in wavelet analysis. Even though most integrals may be avoided by means of recursive pyramid algorithms, they must be performed in at least in one refinement level.

Several authors have considered this problem ([3, 16], among others). In particular, Barinka et al. [1] proposed a modified Gaussian quadrature which is well suited to inner products of smooth functions and (non necessarily smooth) wavelets. The Gaussian quadrature is modified in the sense that the integration points and weights are found with respect to a weighting function chosen as the scaling or the wavelet function. Barinka et al. employed this approach to derive quadrature rules for cardinal B-splines. Later on, Maleknejad et al. [10] employed the same approach on the Daubechies' family of classical, extremal phase wavelets [4].

In this work, we follow the methodology proposed by Barinka et al. to derive weighted Gaussian quadratures to integrals involving CDV scaling functions. Since the CDV functions were developed from the Daubechies' least-asymmetric (i.e., symmlet) family [5], it is natural to consider [10] as a starting point. The algorithm proposed therein is improved by avoiding the evaluation of determinants with the aid of the Chebyshev algorithm [7] and by a more convenient selection of the support of the interior scaling functions.

The paper is organized as follows: Section 2 reviews the work in [1] in the context of the Daubechies' family of wavelets and proposes improvements over the approach in [10]. In Section 3, we consider the CDV basis. For this purpose, we also need the modified Chebyshev algorithm [6, 15]. Section 4 illustrates the effectiveness of the proposed quadrature in the numerical approximation

of a homogeneous Fredholm integral equation of second kind by the Galerkin method with the CDV basis.

2. Gaussian Quadratures for Daubechies' Functions

Let us consider a function $\phi \in L^2(\mathbb{R})$, which we refer to as the *scaling function*, such that $\{\phi_{j,n}(x) = 2^{j/2}\phi(2^j x - n),\ n \in \mathbb{Z}\}$ is an orthonormal set for any $j \in \mathbb{Z}$, and the family of subspaces $(V_j|\ j \in \mathbb{Z}) \subset L^2(\mathbb{R})$ defined as $V_j = \text{span}\,\{\phi_{j,n}\}_{n \in \mathbb{Z}}$ defines a multi-resolution analysis, i.e.,

1. $V_{j-1} \subset V_j$ for any $j \in \mathbb{Z}$;

2. $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

3. $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$;

4. $f(x) \in V_0 \iff D_{2^j} f(x) := f(2^j x) \in V_j$ for any $j \in \mathbb{Z}$.

In particular [4, Prop. 5.3.1], there exists a sequence $h_k \in l^2(\mathbb{Z})$ such that the following *scaling equation* holds:

$$\phi(x) = \sum_{n \in \mathbb{Z}} \sqrt{2} h_n \phi(2x - n).$$

Moreover, there exists a (non-unique) sequence $g_k \in l^2(\mathbb{Z})$ such that the function

$$\psi(x) = \sum_{n \in \mathbb{Z}} g_n \phi(2x - n)$$

generates, for each $j \in \mathbb{Z}$, a set of orthonormal functions $\psi_{j,n}(x) = 2^{j/2}\psi(2^j x - n),\ n \in \mathbb{Z}$, whose span W_j satisfies $V_{j+1} = V_j \oplus W_j$. The function ψ is known as the *mother wavelet* associated with ϕ. Given $J > 0$, we can write $V_J = V_0 \oplus W_0 \oplus W_1 \oplus \ldots \oplus W_{J-1}$, which yields the following *multiscale* representation for the orthogonal projection of a function $f \in L^2(\mathbb{R})$ into V_J:

$$f(x) = \sum_{n=-\infty}^{\infty} d_{0,n}\phi_{0,n}(x) + \sum_{j=0}^{J-1}\sum_{n=-\infty}^{\infty} f_{j,n}\psi_{j,n}(x),$$

where the coefficients $d_{j,n}$ and $f_{j,n}$ are given as

$$d_{j,n} = \int_{-\infty}^{\infty} f(x)\phi_{j,n}(x)\,dx, \quad f_{j,n} = \int_{-\infty}^{\infty} f(x)\psi_{j,n}(x)\,dx.$$

We focus in the case where supp(ϕ), the support of ϕ, is compact (and so is supp(ψ)), which implies that only a finite number of coefficients h_k and g_k is non-zero. We can use Mallat's algorithm [11] to compute all the coefficients $d_{0,n}$ and $f_{j,n}$ ($0 \leq j \leq J - 1$) by evaluating only the integrals

$$d_{J,n} = \int_{-\infty}^{\infty} f(x)\phi_{J,n}(x)\,dx = 2^{-J/2}\int_{\text{supp}(\phi)} f(2^{-J}(t+n))\phi(t)\,dt. \quad (1)$$

Barinka et al. [1] considered a weighted Gaussian quadrature rule satisfying

$$\int_{\text{supp}(\phi)} g(x)\phi^c(x)\,dx = \sum_{l=1}^{r} w_l^c g(x_l^c) \quad \forall g \in \mathcal{P}_{2r+1}. \quad (2)$$

The weighting function is defined as $\phi^c(x) = \phi(x) + c\chi_{\text{supp}(\phi)}(x)$, where c is a positive constant that is chosen in order that ϕ^c is a non-negative function (Barinka et al. refer to this procedure as the *lifting trick*). Herein, $\mathcal{P}_m$ denotes the space of real polynomials of degree m and $\chi_I(\cdot)$ denotes the characteristic function over an interval I. Integrals of the form (1) can be approximated with the aid of (2) since

$$\int_{\text{supp}(\phi)} g(x)\phi(x)\,dx = \sum_{l=1}^{r} w_l^c g(x_l^c) - c\sum_{l=1}^{r} w_l^\chi g(x_l^\chi), \quad \forall g \in \mathcal{P}_{2r+1},$$

where the integration points and weights x_l^χ and w_l^χ ($1 \leq l \leq r$) are given by the standard Gauss-Legendre quadrature mapped to supp(ϕ). On the other hand, the integration points $x_1^c, \ldots, x_r^c$ are the roots of the r-th term of the sequence of orthogonal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ satisfying $\mathcal{L}[P_nP_m] = 0$ for $n \neq m$ and $\mathcal{L}[P_n^2] \neq 0$, where

$$\mathcal{L}[f] = \int_{\text{supp}(\phi)} f(x)\phi^c(x)\,dx.$$

These points are the eigenvalues of the Jacobi matrix

$$J_r = \begin{pmatrix} c_1 & \sqrt{\lambda_2} & & & & 0 \\ \sqrt{\lambda_2} & c_2 & \sqrt{\lambda_3} & & & \\ & \sqrt{\lambda_3} & c_3 & \sqrt{\lambda_4} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{\lambda_{r-1}} & c_{r-1} & \sqrt{\lambda_r} \\ 0 & & & & \sqrt{\lambda_r} & c_r \end{pmatrix}, \quad (3)$$

where c_n and λ_n ($n \geq 1$) are the coefficients of the three-term recurrence relation

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n \geq 1, \quad (4)$$
$$P_{-1}(x) = 0, \ P_0(x) = 1.$$

The quadrature weights $w_l^c = \lambda_1 u_{l,1}^2$, where $\boldsymbol{u}_l = [u_{l,1}, \ldots, u_{l,r}]^T$ is the normalized eigenvector of J_r corresponding to the eigenvalue x_l^c. The parameters c_n and λ_n are usually calculated from the moments of the weighting function,

$$\mu_k^c = \int_{\text{supp}(\phi)} x^k \phi^c(x)\, dx = \mu_k + c \int_{\text{supp}(\phi)} x^k\, dx, \tag{5}$$

where μ_k, $0 \leq k \leq 2r - 1$, are the moments of the scaling function:

$$\mu_k = \int_{\text{supp}(\phi)} x^k \phi(x)\, dx.$$

Maleknejad et al. [10] considered the case when $\phi = \phi^{D,N}$ and $\psi = \psi^{D,N}$ are respectively the classical Daubechies scaling function and mother wavelet [4] with N vanishing moments, i.e.,

$$\int_0^{2N-1} x^n \psi(x)\, dx = 0, \quad n = 0, 1, \ldots, N - 1. \tag{6}$$

Therein, the support of ϕ is the interval $[0, 2N - 1]$, in order that they seek $\{w_l^c, x_l^c, w_l^\chi, x_l^\chi\}$ such that

$$\int_0^{2N-1} g(x)\phi(x)\, dx = \sum_{l=1}^r w_l^c g(x_l^c) - c \sum_{l=1}^r w_l^\chi g(x_l^\chi), \quad \forall g \in \mathcal{P}_{2r+1}. \tag{7}$$

It follows that

$$x_l^\chi = \frac{2N - 1}{2}(\xi_l + 1), \quad w_l^\chi = \frac{2N - 1}{2} w_l, \tag{8}$$

where ξ_l and w_l $(1 \leq l \leq n)$ are the Gauss-Legendre points and weights, respectively. In order to construct the Jacobi matrix (3) that yields the quadrature points and weights w_l^c and x_l^c, the recursion coefficients c_n and λ_n are computed in [10] through moment determinants [7, Thm. 2.2]. The moments of the weighting function (5) reduce to

$$\mu_k^c = \int_0^{2N-1} x^k \phi^c(x)\, dx = \mu_k + c\frac{(2N - 1)^{k+1}}{k + 1}, \tag{9}$$

whereas the moments

$$\mu_k = \frac{2^{-k}}{1 - 2^{-k}} \sum_{j=0}^{k-1} \binom{k}{j} m_{k-j} \mu_j \tag{10}$$

are defined in terms of the following discrete moments:

$$m_j = \frac{1}{\sqrt{2}} \sum_{n=0}^{2N-1} n^j h_n. \tag{11}$$

We propose two changes in the approach from [10]: we build the Jacobi matrix (3) without resorting to moment determinants (this is achieved by using the classical Chebyshev algorithm [6, Sec. 2.3]) and we shift the support of ϕ from $[0, 2N-1]$ to the interval $[-N+1, N]$, i.e., we consider $\varphi(x) = \phi(x+(N-1))$ and seek $\{w_l^c, x_l^c, w_l^\chi, x_l^\chi\}$ such that

$$\int_{-N+1}^{N} g(x)\varphi(x)\, dx = \sum_{l=1}^{r} w_l^c g(x_l^c) - c \sum_{l=1}^{r} w_l^\chi g(x_l^\chi) \quad \forall g \in \mathcal{P}_{2r+1}. \tag{12}$$

The integration points and weights x_l^χ and w_l^χ ($1 \leq l \leq r$) are now given by the Gauss-Legendre quadrature mapped to the interval $[-N+1, N]$, i.e.,

$$x_l^\chi = \frac{2N-1}{2}\xi_l + \frac{1}{2}, \quad w_l^\chi = \frac{2N-1}{2}w_l.$$

The moments of the weighting function (9) now become

$$\mu_k^c = \mu_k + c\frac{N^{k+1} - (1-N)^{k+1}}{k+1}, \quad \mu_k = \int_{-N+1}^{N} x^k \varphi(x)\, dx.$$

In order to compute the moments μ_k, we multiply the scaling equation

$$\varphi(x) = \sum_{n=-N+1}^{N} \sqrt{2}h_n \varphi(2x - n)$$

by x^k and integrate it from $-N+1$ to N:

$$\mu_k = \sum_{n=-N+1}^{N} \sqrt{2}h_n \int_{-N+1}^{N} x^k \varphi(2x - n)\, dx.$$

After a change of variables in the integral, and using the fact that $\mathrm{supp}(\varphi) = [-N+1, N]$, we find

$$\mu_k = \frac{1}{\sqrt{2}} \sum_{n=-N+1}^{N} h_n \int_{-N+1}^{N} \left(\frac{t+n}{2}\right)^k \varphi(t)\, dt.$$

With the aid of the binomial expansion of $((t+n)/2)^k$, we arrive at

$$\mu_k = \sum_{j=0}^{k} \binom{k}{j} m_{k-j} \int_{-N+1}^{N} \left(\frac{t}{2}\right)^j \varphi(t)\, dt = \sum_{j=0}^{k} 2^{-j} \binom{k}{j} m_{k-j} \mu_j,$$

where the new discrete moments m_j are defined as

$$m_j = \frac{1}{\sqrt{2}} \sum_{n=-N+1}^{N} \left(\frac{n}{2}\right)^j h_n. \tag{13}$$

Similarly to [1, 10], we use the fact that $m_0 = 1$ to obtain

$$\mu_k = \frac{1}{1 - 2^{-k}} \sum_{j=0}^{k-1} 2^{-j} \binom{k}{j} m_{k-j} \mu_j. \tag{14}$$

Note that the factors n^j $(0 \leq N \leq 2N - 1)$ from the discrete moments m_j in (11) grow more rapidly than the factors $(n/2)^j$ from (13), where $-N + 1 \leq n \leq N$ (see also [15, p. 466]). Thus, we expect that (13)-(14) is less susceptible to the amplification of roundoff errors than (10)-(11).

Let us contrast the computations described herein with the procedure outlined in (6)-(11). In analogy with [10, Tab. 9], Fig. 1(a) shows the absolute errors $e_k = |\mu_k - \tilde{\mu}_k|$ between the exact moments μ_k in (10) and the approximate moments $\tilde{\mu}_k$ obtained by expression (7) with $g(x) = x^k$. We employ the scaling coefficients and constants for the "lifting trick" presented in Tables 1 and 4 of [10]. In addition to $r = 7$ (solid lines), we also consider $r = 16$ (dashed lines), for which the accuracy of (10) is lost. Afterwards, we compute $\tilde{\mu}_k$ from (12) with $g(x) = (x + (N - 1))^k$. As shown in Fig. 1(b), this approach was more robust than the approach presented in (6)-(11).

3. Gaussian Quadratures for the CDV Basis

Let us briefly review the CDV scaling functions. The starting point is the Daubechies' symmlet [5] scaling function ϕ. Note that the coefficients $h_k = h_k^{(N)}$ are now different from [10, Tab. 1-2].

Let us consider a scaling parameter $j \in \mathbb{Z}$ large enough so that $2^j - 2N \geq 0$, and introduce the $2^j - 2N$ interior scaling functions

$$\phi_{j,n}^{int}(x) = 2^{j/2} \phi(2^j x - n), \quad N \leq n \leq 2^j - N,$$

with support within $[0, 1]$. The next step is to add N functions at each end of the boundary. At the left end, we introduce the left edge scaling functions $\phi_{j,n}^{left}(x) = 2^{j/2} \phi_n^{left}(2^j x)$ for $0 \leq n \leq N - 1$, where
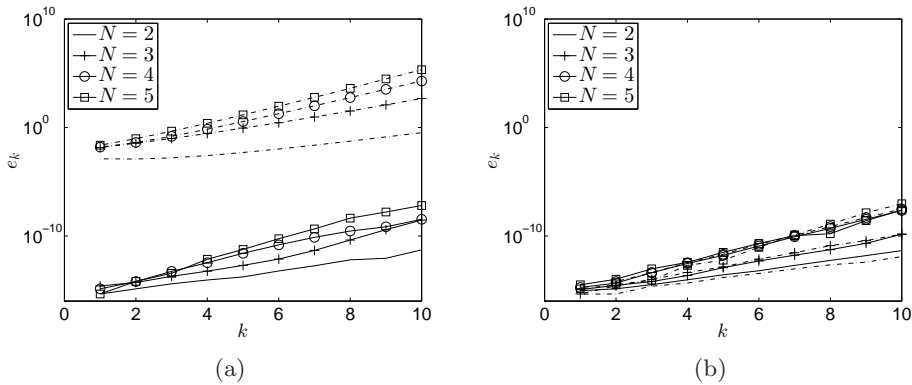
Figure 1: Error $e_k = |\mu_k - \tilde{\mu}_k|$ of the continuous moments with $N = 2, \ldots, 5$, considering $r = 7$ (solid lines) and $r = 16$ (dashed lines). Figures (a) and (b) refer to quadratures (7) and (12), respectively.

$$2^{-1/2}\phi_n^{left}(x) = \sum_{l=0}^{N-1} H_{n,l}^{left}\phi_l^{left}(2x) + \sum_{l=N}^{N+2n} H_{n,l}^{left}\phi(2x - l), \qquad (15)$$

whereas for the right end we consider, for $-N \leq n \leq -1$, the right edge scaling functions $\phi_{j,-n}^{right}(x) = 2^{j/2}\phi_{-n}^{right}(2^j x)$,

$$2^{-1/2}\phi_n^{right}(x) = \sum_{l=-N}^{-1} H_{n,l}^{right}\phi_l^{right}(2x) + \sum_{l=-N+2n+1}^{-N-1} H_{n,l}^{right}\phi(2x - l). \qquad (16)$$

We have that $\mathrm{supp}(\phi_n^{left}) = [0, N + n] \subset [0, 2N - 1]$ for $0 \leq n \leq N - 1$ and $\mathrm{supp}(\phi_n^{right}) = [n - N + 1, 0] \subset [-2N + 1, 0]$, $n = -1, \ldots, -N$. The coefficients $H_{n,l}^{left}$ and $H_{n,l}^{right}$, as well as the symmlet scaling coefficients h_k, have been tabulated for several values of the parameter N and are available, for instance, at http://www.nr.com/contrib/.

The family of subspaces $(V_j)_{j \geq 1 + \log_2(N)}$, $V_j = \{\theta_{j,n}(x)\}_{n=0}^{2^j - 1}$, where

$$\begin{aligned} \theta_{j,n}(x) &= \begin{cases} \phi_{j,n}^{left}(x), & n = 0, \ldots, N - 1, \\ \phi_{j,n}^{int}(x), & n = N, \ldots, 2^j - N - 1, \end{cases} \\ \theta_{j,2^j-n}(x) &= \phi_{j,-n}^{right}(x - 1), & n = 1, \ldots, N, \end{aligned} \qquad (17)$$

is dense in $L^2([0,1])$ [2].

In this work, we focus in a single member V_J of the family (V_j), i.e., we seek Gaussian quadratures for integrals in the form

$$d_{J,n} = \int_{\text{supp}(\theta_{J,j})} f(x)\theta_{J,n}(x)\,dx. \tag{18}$$

When $n = N, \ldots, 2^J - N - 1$, we have that (18) reduces to

$$d_{J,n} = 2^{-J/2} \int_{-N+1}^{N} f(2^{-J}(t+n))\phi(t)\,dt.$$

The procedure to compute the quadrature points and weights for any $g \in \mathcal{P}_{2r+1}$ such that

$$\int_{-N+1}^{N} g(x)\phi(x)\,dx = \sum_{l=1}^{r} w_l^{c,int} g(x_l^{c,int}) - c^{int} \sum_{l=1}^{r} w_l^{\chi,int} g(x_l^{\chi,int}),$$

is the same as in (12)-(14). The only difference is the positive constant c^{int} in the auxiliary weighting function

$$\phi^{c,int}(x) = \phi(x) + c^{int}\chi_{[-N+1,N]}(x), \tag{19}$$

which must be updated according to the symmlet scaling function. The values of $c^{int} = c_N^{int}$ with $N = 2, 3, 4$ were found using a standard cascade algorithm [4] and are given as follows:

$$c_2^{int} = 0.36602, \quad c_3^{int} = 0.38635, \quad c_4^{int} = 0.32296.$$

For $n \leq N - 1$, we have

$$d_{J,n} = \int_{-\infty}^{\infty} f(x)2^{J/2}\phi_n^{left}(2^J x)\,dx = 2^{-J/2} \int_{0}^{2N-1} f(2^{-J}t)\phi_n^{left}(t)\,dt.$$

In analogy with (19), the weighting function ϕ_n^c is defined as

$$\phi_n^c(x) = \phi_n^{left}(x) + c_n^{left}\chi_{[0,2N-1]}(x) \quad (0 \leq n \leq N - 1),$$

where the constants c_n^{left} (Table 1) are estimated by evaluating the left edge functions as described in the appendix. We consider a quadrature in the form

$$\int_{0}^{2N-1} g(x)\phi_n^{left}(x)\,dx = \sum_{l=1}^{r} w_{l,n}^{c,left} g(x_{l,n}^{c,left}) - c_n^{left} \sum_{l=1}^{r} w_l^{\chi,left} g(x_l^{\chi,left}) \tag{20}$$

	c_0^{left}	c_1^{left}	c_2^{left}	c_2^{left}
$N = 2$	1.09955	0.35627		
$N = 3$	1.23758	0.91284	0.400477	
$N = 4$	0.04959	0.92431	0.261993	0.15470

Table 1: Left edge constants c_n^{left}, $0 \leq n \leq N - 1$.

for any $g \in \mathcal{P}_{2r+1}$, where the integration points and weights $x_l^{\chi,left}$ and $w_l^{\chi,left}$ ($1 \leq l \leq r$) associated with the characteristic function $\chi_{[0,2N-1]}$ are the same as in (8).

Taking into account Fig. 1(a), the fact that the support of the left edge functions is within $[0, 2N - 1]$ may lead to numerical instabilities. For this reason, we consider the modified moments

$$\mu_{n,k}^c = \mu_{n,k}^{left} + c_n^{left} \frac{N^{k+1} - (1 - N)^{k+1}}{k + 1}, \tag{21}$$

$$\mu_{n,k}^{left} = \int_0^{2N-1} (x - (N - 1))^k \phi_n^{left}(x) \, dx \tag{22}$$

for $0 \leq n \leq N - 1$. These moments are defined with respect the polynomial sequence $p_k(x) = (x - (N - 1))^k$, which satisfies the recursive relation

$$p_{k+1}(x) = (x - a_k)p_k(x) - b_k p_{k-1}(x), \quad a_k = N - 1, \, b_k = 0 \tag{23}$$

for $k \geq 0$, and $p_{-1}(x) = 0$ and $p_0(x) = 1$. The entries c_n and λ_n of the Jacobi matrix (3) are now computed with the *modified* Chebyshev algorithm [7, Sec. 2.4]. Let us proceed with the computation of the modified moments (22). We have from (15) that

$$\mu_{n,k}^{left} = 2^{1/2} \sum_{l=0}^{N-1} H_{n,l}^{left} \int_0^{2N-1} (x - (N - 1))^k \phi_l^{left}(2x) \, dx$$

$$+ 2^{1/2} \sum_{l=N}^{N+2n} H_{n,l}^{left} \int_0^{2N-1} (x - (N - 1))^k \phi(2x - l) \, dx.$$

By changing variables in the integrals above, we arrive at

$$\mu_{n,k}^{left} = 2^{-1/2} \sum_{l=0}^{N-1} H_{n,l}^{left} \int_0^{4N-2} \left(\frac{t}{2} - (N - 1) \right)^k \phi_l^{left}(t) \, dt$$

$$+ 2^{-1/2} \sum_{l=N}^{N+2n} H_{n,l}^{left} \int_{-l}^{4N-l-2} \left(\frac{t + l}{2} - (N - 1) \right)^k \phi(t) \, dt.$$

Since $\mathrm{supp}(\phi) \subset [-l, 4N - l - 2]$ for any $N \leq l \leq N + 2n$ and $\mathrm{supp}(\phi_l^{left}) \subset [0, 4N - 2]$ for any $0 \leq l \leq N - 1$, we have that

$$
\begin{aligned}
2^{1/2} \mu_{n,k}^{left} &= \sum_{l=0}^{N-1} H_{n,l}^{left} \int_0^{2N-1} \left(\frac{t}{2} - (N-1) \right)^k \phi_l^{left}(t)\, dt \\
&+ \sum_{l=N}^{N+2n} H_{n,l}^{left} \int_{-N+1}^{N} \left(\frac{t}{2} + \frac{l}{2} - (N-1) \right)^k \phi(t)\, dt.
\end{aligned}
$$

We again use the binomial formula and consider the moments (14) to conclude that

$$
\begin{aligned}
2^{1/2} \mu_{n,k}^{left} &= \sum_{l=0}^{N-1} H_{n,l}^{left} \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1-N}{2} \right)^{k-j} 2^{-j} \mu_{l,j}^{left} \\
&+ \sum_{l=N}^{N+2n} H_{n,l}^{left} \sum_{j=0}^{k} 2^{-j} \binom{k}{j} \left(\frac{l}{2} - (N-1) \right)^{k-j} \mu_j \\
&= \sum_{j=0}^{k} \binom{k}{j} \left(\frac{1-N}{2} \right)^{k-j} M_{n,j}^{left} + \sum_{j=0}^{k} 2^{-j} \binom{k}{j} m_{n,k-j}^{left} \mu_j,
\end{aligned}
$$

where, in analogy with (13),

$$
M_{n,j}^{left} = 2^{-j} \sum_{l=0}^{N-1} H_{n,l}^{left} \mu_{l,j}^{left}, \quad m_{n,j}^{left} = \sum_{l=N}^{N+2n} \left(\frac{l}{2} - (N-1) \right)^j H_{n,l}^{left}.
$$

Thus we have, for each $k \geq 0$, the linear system

$$
\begin{aligned}
2^{1/2} \mu_{n,k}^{left} - \sum_{l=0}^{N-1} \frac{H_{n,l}^{left}}{2^k} \mu_{l,k}^{left} &= \sum_{j=0}^{k-1} \binom{k}{j} \left(\frac{1-N}{2} \right)^{k-j} M_{n,j}^{left} \\
&+ \sum_{j=0}^{k} 2^{-j} \binom{k}{j} m_{n,k-j}^{left} \mu_t \quad (0 \leq n \leq N-1).
\end{aligned}
$$

Once we find the moments $\mu_{n,k}^{left}$, we evaluate the moments of the weighting function from (21) and apply the modified Chebyshev algorithm to recover the quadrature points $x_{l,n}^{c,left}$ and weights $w_{l,n}^{c,left}$ for $\phi_n^c(x)$.

It remains to compute the right-end coefficients

$$
d_{J,2^J - n} = 2^{-J/2} \int_{-2N+1}^{0} f(2^{-J} t + 1) \phi_{-n}^{right}(t)\, dt \quad (1 \leq n \leq N).
$$

Again, for the "lifting trick" we define

$$\phi_n^c(x) = \phi_n^{right}(x) + c_n^{right}\chi_{[-2N+1,0]}(x) \quad (-N \le n < -1).$$

The positive constants c_n^{right} (Table 2) are chosen such that $\phi_n^c > 0$ and evaluated as described in the appendix. We seek, for $-N \le n < -1$,

$$\int_{-2N+1}^0 g(x)\phi_n^{right}(x)\,dx = \sum_{l=1}^r w_{l,n}^c g(x_{l,n}^c) - c_n^{right}\sum_{l=1}^r w_l^{\chi,right}g(x_l^{\chi,right})$$

for any $g \in \mathcal{P}_{2r+1}$. The integration points and weights $x_l^{\chi,right}$ and $w_l^{\chi,right}$ ($1 \le l \le r$) are given by the Gauss-Legendre quadrature mapped to the interval $[-2N+1, 0]$, i.e.,

$$x_l^{\chi,right} = \frac{2N-1}{2}(\xi_l - 1), \quad w_l^{\chi,right} = \frac{2N-1}{2}w_l.$$

	c_{-1}^{right}	c_{-2}^{right}	c_{-3}^{right}	c_{-4}^{right}
$N = 2$	0.0000	0.42407		
$N = 3$	0.0000	0.73006	0.16226	
$N = 4$	0.0302	1.12700	0.22025	0.18134

Table 2: Right edge constants c_n^{right}, $-N \le n < -1$.

We consider the modified moments of the weighting function with respect to $p_k(x) = (x+N)^k$,

$$\mu_{n,k}^c = \mu_{n,k}^{right} + c_n^{right}\frac{N^{k+1} - (1-N)^{k+1}}{k+1} \quad (-N \le n < -1),$$

where $\mu_{n,k}^{right}$ are the modified moments of the right edge functions:

$$\mu_{n,k}^{right} = \int_{-2N+1}^0 (x+N)^k\phi_n^{right}(x)\,dx \quad (-N \le n < -1). \tag{24}$$

Proceeding as in the left-end case, we find

$$2^{1/2}\mu_{n,k}^{right} = \sum_{l=-N}^{-1}\frac{H_{n,l}^{right}}{2^k}\mu_{l,k}^{right} + \sum_{j=0}^k 2^{-j}\binom{k}{j}m_{n,k-j}^{right}\mu_j,$$

where

$$M_{n,j}^{right} = 2^{-j} \sum_{l=-N}^{-1} H_{n,l}^{left} \mu_{l,j}^{right}, \quad m_{n,k}^{right} = \sum_{l=2n+1-N}^{-N-1} \left(\frac{l}{2} + N\right)^k H_{n,l}^{right}.$$

We have, for each $k \geq 0$, the linear system

$$2^{1/2} \mu_{n,k}^{right} - \sum_{l=-N}^{-1} \frac{H_{n,l}^{right}}{2^k} \mu_{l,k}^{right} = \sum_{j=0}^{k-1} \binom{k}{j} \left(\frac{N}{2}\right)^{k-j} M_{n,j}^{right}$$

$$+ \sum_{j=0}^{k} 2^{-j} \binom{k}{j} m_{n,k-j}^{right} \mu_j \quad (-N \leq n \leq -1).$$

Now we have $\mu_{n,k}^{right}$, we find $\mu_{n,k}^c$ from (21) and apply again the modified Chebyshev algorithm to recover the quadrature points and weights for $\phi_n^c(x)$. Since the modified moments are defined with respect to $p_k(x) = (x + N)^k$, the parameters a_k and b_k from the recursion (23) are given as $a_k = -N$ and $b_k = 0$, $k \geq 0$.

4. Application to a Fredholm Integral Equation

Let $K \in L^2([0,1] \times [0,1])$ be a symmetric and nonnegative-definite covariance kernel. We have ([9]) that K admits the spectral decomposition

$$K(x,y) = \sum_{k=1}^{\infty} \lambda_k u_k(x) u_k(y),$$

where the nonnegative eigenvalues λ_k and the orthonormal eigenfunctions u_k ($k \geq 1$) are the solutions of the homogeneous Fredholm integral equation

$$\int_0^1 K(x,y) u_k(y)\, dy = \lambda_k u_k(x), \quad x \in [0,1]. \tag{25}$$

Let $\mathcal{B} = \{v_1, \ldots, v_m\} \subset L^2([0,1])$ be a set of linearly independent functions and $\mathcal{V} = \text{span}(\mathcal{B})$. The Galerkin approximation to (25) in $\mathcal{V}$ consists of finding $\lambda_k^h \in \mathbb{R}$ and $u_k^h(x) \in V_h$ ($1 \leq k \leq m$) such that

$$a(u_k^h, v_h) = \lambda_k^h \langle u_k^h, v_h \rangle \quad \forall v_h \in \mathcal{V}. \tag{26}$$

$$a(u,v) = \int_0^1 \int_0^1 K(x,y) u(y) v(x)\, dy\, dx, \quad \langle u,v \rangle = \int_0^1 u(x) v(x)\, dx. \tag{27}$$

In matrix form, we have the generalized eigenvalue problem $\boldsymbol{K}\boldsymbol{u}_k = \lambda_k^h \boldsymbol{W}\boldsymbol{u}_k$, where the unknown vector $\boldsymbol{u}_k$ are the coordinates of the k-th approximate eigenvector in the basis $\mathcal{B}$, and the matrices $\boldsymbol{K}$ and $\boldsymbol{W}$ are defined by the coefficients $K_{i,j} = a(v_j, v_i)$ and $W_{i,j} = \langle v_j, v_i \rangle$, $1 \le i, j \le m$.

If the basis $\mathcal{B}$ is orthonormal (such as the basis (17)), then the problem reduces to the standard eigenvalue problem, i.e.,

$$\boldsymbol{K}\boldsymbol{u}_k = \lambda_k^h \boldsymbol{u}_k, \quad K_{i,j} = a(v_j, v_i), \quad 1 \le i, j \le m. \tag{28}$$

We consider $\mathcal{V} = V_J$, $m = 2^J$, and $v_i = \theta_{J,i-1}$ ($1 \le i \le m$), with $\theta_{J,0}, \ldots \theta_{J,2^J-1}$ defined in (17). Moreover, we approximate $K_{i,j}$ with the product quadrature

$$
\begin{aligned}
K_{i,j} &= \int_{\mathrm{supp}(\theta_{J,j})} \left(\int_{\mathrm{supp}(\theta_{J,i})} K(x,y)\theta_{J,i}(y)\, dy \right) \theta_{J,j}(x)\, dx \\
&\approx \sum_{l=1}^{r} \tilde{K}^{(j)}(x_{l,i}^c)w_{l,i}^c - c_i \sum_{l=1}^{r} \tilde{K}^{(j)}(x_{l,i}^\chi)w_{l,i}^\chi,
\end{aligned}
\tag{29}
$$

where $\tilde{K}^{(j)}(x)$ is the quadrature for the inner integral given as

$$\tilde{K}^{(j)}(x) = \sum_{l=1}^{r} K(x, x_{l,j}^c)w_{l,j}^c - c_j \sum_{l=1}^{r} K(x, x_{l,j}^\chi)w_{l,j}^\chi, \tag{30}$$

in analogy with the Gaussian quadratures for (18) presented in Sec. 3. The quadrature points and weights in (29)-(30) are written as follows: if $0 \le i \le N-1$, then

$$
\begin{aligned}
x_{l,i}^c &= 2^{-J} x_{l,n}^{c,left}, \quad w_{l,i}^c = 2^{-J/2} w_{l,n}^{c,left}, \\
x_{l,i}^\chi &= 2^{-J} x_l^{\chi,left}, \quad 2^{-J} w_{l,i}^\chi = x_l^{\chi,left}, \qquad c_i = c_i^{left};
\end{aligned}
$$

if $N \le i \le 2^J - N - 1$, then

$$
\begin{aligned}
x_{l,i}^c &= 2^{-J}(x_l^{c,int} + i), \quad w_{l,i}^c = 2^{-J/2} w_l^{c,int}, \\
x_{l,i}^\chi &= 2^{-J}(x_l^{\chi,int} + i), \quad 2^{-J} w_{l,i}^\chi = x_l^{\chi,int}, \qquad c_i = c^{int};
\end{aligned}
$$

and if $i = 2^J - n$, $1 \le n \le N$, then

$$
\begin{aligned}
x_{l,i}^c &= 2^{-J}(x_{l,n}^{c,right} + 1), \quad w_{l,i}^c = 2^{-J/2} w_{l,n}^{c,right}, \\
x_{l,i}^\chi &= 2^{-J}(x_l^{\chi,right} + 1), \quad 2^{-J/2} w_{l,i}^\chi = x_l^{\chi,right}, \quad c_i = c_n^{right}.
\end{aligned}
$$

For comparison purposes, we also consider the basis $\{\phi_{J,n}^{Haar}(x)\}_{n=0}^{2^J-1}$ generated by the Haar scaling function $\phi^{Haar}(x) = \chi_{[0,1)}(x)$. As usual (see, e.g., [14]), the entries of $K_{i,j}$ in (28) are computed with the midpoint rule

$$K_{i,j} \approx 2^{-J} K(\overline{x}_{J,j}, \overline{x}_{J,i}), \quad \overline{x}_{J,i} = \frac{2i+1}{2(2^{-J})}. \tag{31}$$

4.1. Numerical Examples

In our experiments, we considered the Gaussian covariance kernel with variance σ^2 and correlation parameter η:

$$K(x,y) = \sigma^2 \exp(-|x-y|^2/\eta^2). \tag{32}$$

The reference solution was computed using the spectral element method [12] of degree 16 on a spatial mesh of $2^{12}+1$ nodes. In our simulations we employ different correlation lengths ($\eta = 0.1$ and 0.01) to quantify the influence of the scale of correlation on the quality of simulation. For a coherent comparison of the two covariance models, the degree of variability is kept in $\sigma^2 = 1$.

Let us first compare the computational costs of building the matrix $\boldsymbol{K}$ in the eigenvalue system (28) with the CDV and Haar bases. Figure 2 shows the ratio between CPU times of CDV and Haar as a function of the number of integration points r. We consider the correlation length $\eta = 0.1$. Note that the relative cost in terms of CPU time is $\mathcal{O}(r^2)$, which is expected from the fact that the Galerkin method with Haar basis functions needs one kernel evaluation to compute each entry of the eigenvalue system (see (31)), whereas the CDV bases require $(2r)^2$ kernel evaluations in (29)-(30).

We denote as $e^{J,N}$ the relative error of approximation of the 10-th eigenvalue λ_{10} of (25) by the Galerkin method with the basis $\mathcal{V} = V_J$ with N vanishing moments, i.e.,

$$e^{J,N} = \frac{\|\lambda_{10} - \lambda_{10}^{J,N}\|}{\|\lambda_{10}\|}.$$

Figure 3 illustrates how the eigenvalue error $e^{J,N}$ depends on the scaling parameter J, the number of vanishing moments N, the correlation length η, and the number of integration points r. Note that the convergence rate increases with N, in particular the performance of CDV basis is better than that of the Haar basis. Furthermore we needed more integration points in the case $\eta = 0.01$. Although the covariance function is smooth, it rapidly decreases to zero as $|x-y|$ increases, demanding a higher resolution in the computation of the double integral (27) near the diagonal $x = y$.
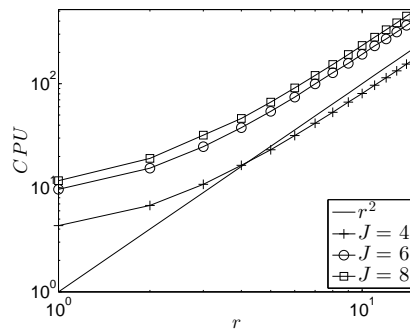
Figure 2: Ratio of CPU times of CDV and Haar considering $\eta = 0.1$ and $N = 3$.

Fig. 4 shows the approximation of the tenth eigenfunction considering the Haar and CDV bases in contrast to the reference solution (GLL). In this experiment we consider correlation lengths $\eta = 0.1$, $r = 7$ integration points, and the scaling parameter $J = 6$. In this case, all approximations are satisfactory (even the Haar approximation, which accurately approximates the reference eigenfunction at the element midpoints). The smoothness of the approximation increases with N. On the other hand, we note that the improvement is not significant from $N = 3$ to $N = 4$, indicating the fast convergence of the Galerkin method with the CDV basis. Similar qualitative results (not shown herein) were observed with $\eta = 0.01$.

5. Conclusions

In this work, we proposed quadrature rules for integrals involving the Cohen-Daubechies-Vial scaling functions and employed them to design a high-order Galerkin method for the homogeneous Fredholm integral equation of the second kind. The use of the modified Chebyshev algorithm rendered these quadratures more robust than the ones proposed in [10] for the standard Daubechies' family of scaling functions as noted in Fig. 1. We can note from Fig. 3(c)-(d) that an accurate computation of the wavelet coefficients may be crucial on Fredholm integral equations if the kernel exhibits a strong decay. We note that, by a tensor product of the basis functions in (17) and the quadratures (29)-(30), Galerkin method (26) can be extended to the two-dimensional case [13]. The methodology employed herein may be applied to the wavelet functions of the CDV family as well as other orthogonal families.
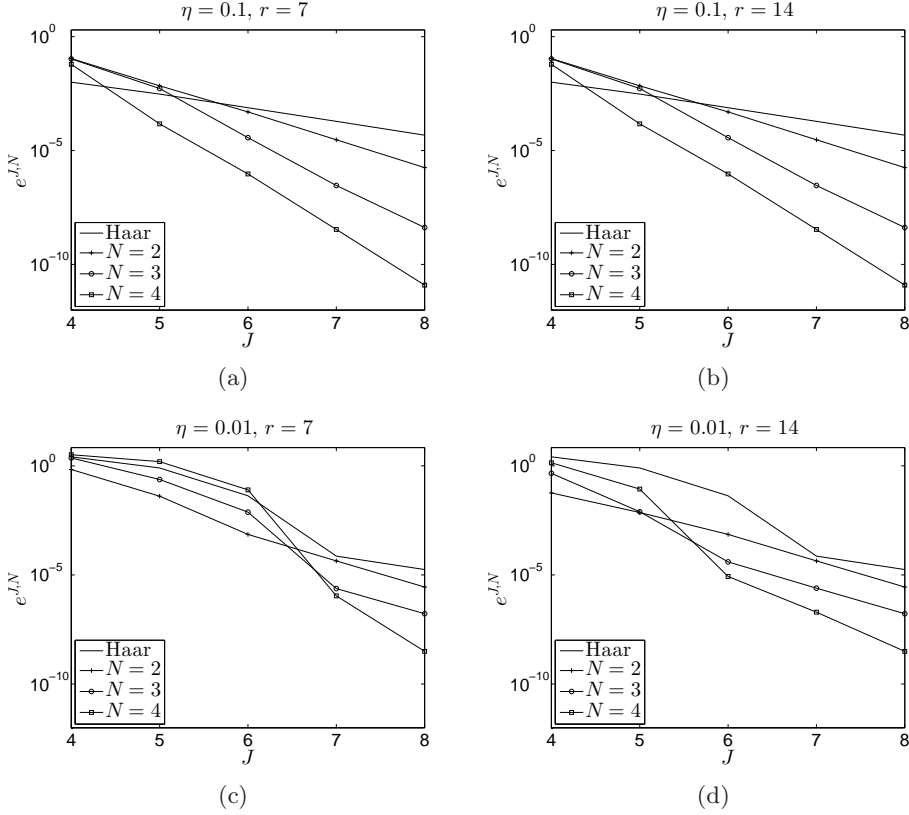
$$\eta = 0.1, \, r = 7 \qquad \eta = 0.1, \, r = 14$$

(a)   (b)

$$\eta = 0.01, \, r = 7 \qquad \eta = 0.01, \, r = 14$$

(c)   (d)

Figure 3: Relative eigenvalue errors for Gaussian kernel (32) when $\eta = 0.1$ and 0.01.

A. Recursive Evaluation of the Scaling Functions

This appendix regards the recursive evaluation of the left and right edge functions present in the Cohen-Daubechies-Vial basis. For conciseness, let us focus on the left edge functions ϕ_n^{left}. The starting point is to evaluate $\phi_n^{left}(k)$ for the integers $0 \leq k \leq 2N - 1$.

Let $0 \leq n \leq N - 1$ be fixed. If $N \leq k \leq 2N - 1$, then $2k > 2N - 1$ and thus $\phi_n^{left}(2k) = 0$. It follows from (15) that, for $N \leq k \leq 2N - 1$,

$$\phi_n^{left}(k) \;\; = \;\; \sqrt{2} \sum_{l=N}^{N+2n} H_{n,l}^{left} \phi(2k - l).$$
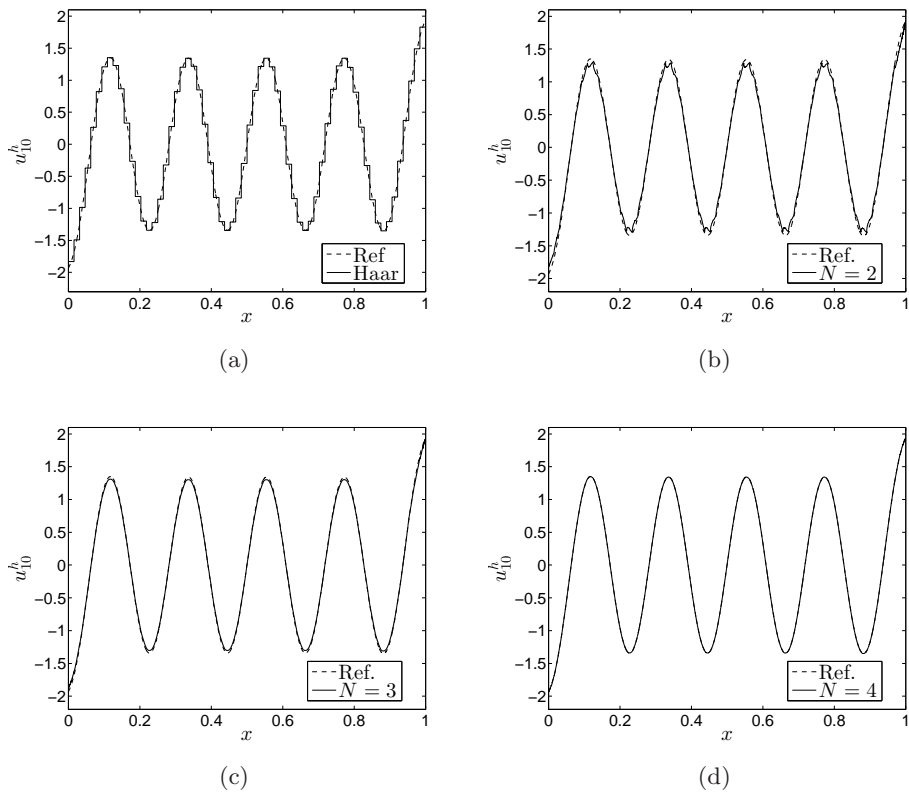
Figure 4: Tenth reference (Ref.) eigenfunction of Gaussian kernel (32) and its Galerkin approximation (Haar and CDV with $N = 2, \ldots, 4$) when $\eta = 0.1$.

For $1 \leq k \leq N - 1$, we compute $\phi_n^{left}(k)$ backwards:

$$2^{-1/2}\phi_n^{left}(N-1) = \sum_{l=0}^{N-1} H_{n,l}^{left} \phi_l^{left}(2N-2) + \sum_{l=N}^{N+2n} H_{n,l}^{left} \phi(2N-2-l),$$

$$2^{-1/2}\phi_n^{left}(N-2) = \sum_{l=0}^{N-1} H_{n,l}^{left} \phi_l^{left}(2N-4) + \sum_{l=N}^{N+2n} H_{n,l}^{left} \phi(2N-4-l),$$

$$\vdots$$

$$2^{-1/2}\phi_n^{left}(1) = \sum_{l=0}^{N-1} H_{n,l}^{left} \phi_l^{left}(2) + \sum_{l=N}^{N+2n} H_{n,l}^{left} \phi(2-l).$$

It remains to compute $\phi_n^{left}(0)$. Taking $x = 0$ into (15), we find

$$2^{-1/2}\phi_n^{left}(0) - \sum_{l=0}^{N-1} H_{n,l}^{left} \phi_l^{left}(0) = \sum_{l=N}^{N+2n} H_{n,l}^{left} \phi(-l), \quad 0 \leq n \leq N-1.$$

Since $\text{supp}(\phi) = [-N+1, N]$, we have $\phi(-l) = 0$ for any $l \geq N$ and the equation above reduces to

$$\sum_{l=0}^{N-1} H_{n,l}^{left} \phi_l^{left}(0) = 2^{-1/2}\phi_n^{left}(0), \quad 0 \leq n \leq N-1. \tag{33}$$

Let us recall from [2] that the space V_J spanned by (17) is constructed in order to generate all polynomials of degree $\leq N - 1$. In particular, we have for the constant function $p(x) = 1$ that

$$1 = \sum_{n=0}^{2^j-1} \langle 1, \theta_{j,n} \rangle \theta_{j,n}(x), \quad \langle 1, \phi \rangle = \int_{\text{supp}(\phi)} \phi(x)\, dx. \tag{34}$$

For $x = 0$, the interior and right edge functions vanish, thus

$$\sum_{n=0}^{N-1} 2^j \left(\int_0^{2N-1} \phi_n^{left}(2^j t)\, dt \right) \phi_n^{left}(0) = \sum_{n=0}^{N-1} \mu_{n,0}^{left} \phi_n^{left}(0) = 1, \tag{35}$$

where the moments $\mu_{n,0}^{left}$ have been found in Sec. 3. It follows from (35) that there exists a non-trivial solution $\boldsymbol{v} = [\phi_0^{left}(0), \ldots, \phi_{N-1}^{left}(0)]^T$ to (33), which is

an eigenvector corresponding to $\lambda = 2^{-1/2}$ of the eigenvalue problem

$$A\boldsymbol{v} = \lambda\boldsymbol{v}, \quad A = \begin{bmatrix} H_{0,0}^{left} & H_{0,1}^{left} & \cdots & H_{0,N-1}^{left} \\ H_{1,0}^{left} & H_{1,1}^{left} & \cdots & H_{1,N-1}^{left} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N-1,0}^{left} & H_{N-1,1}^{left} & \cdots & H_{N-1,N-1}^{left} \end{bmatrix}. \tag{36}$$

Once we find an eigenvector $\boldsymbol{w}$ of (36) corresponding to $\lambda = 2^{-1/2}$, we retrieve $\boldsymbol{v}$ by a scaling of $\boldsymbol{w}$ in order that (35) holds:

$$\boldsymbol{v} = \alpha\boldsymbol{w}, \quad \alpha = \left(\sum_{n=0}^{N-1} \mu_{n,0}^{left} w_n \right)^{-1}.$$

Now we have $\phi_n^{left}(k)$ for the integers $0 \le k \le 2N - 1$, we employ (15) to compute $\phi_n^{left}((2k-1)/2)$ for any $1 \le k \le 2N - 1$, and iteratively repeat this process until we find $\phi_n^{left}(x_k)$, $x_k = 2^{-M}(2N-1)k$, for any $0 \le k \le 2^M$, where M is the desired resolution level.

Acknowledgements

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