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T_0 GRAPHS

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Abstract: A simple graph G is said to be T_0 if for any two distinct vertices u and v of G, one of the following conditions hold:

1. At least one of u and v is isolated;
2. There exists an edge e such that either e is incident with u but not with v or e is incident with v but not with u.

In this paper we discuss T_0 graphs and some examples of it. This paper also deals with the sufficient conditions for join of two graphs, middle graph of a graph and corona of two graphs to be T_0. It is established via example that the line graph of a T_0 graph need not be T_0. Moreover, the relations between T_0 graph with its incidence matrix and its adjacency matrix is discussed.

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Key Words: T_0 graph, incidence matrix, adjacency matrix, line graph, corona, middle graph

1. Introduction

All the graphs considered here are finite and simple. In this paper we denote the set of vertices of G by $V(G)$, the set of edges of G by $E(G)$, the maximum degree of G by $\Delta(G)$ and the minimum degree of G by $\delta(G)$.

The *degree* [2] of a vertex v in graph G, denoted by $\deg(v)$, is the number of edges incident with v. A *pendant vertex* [7] in a graph G is a vertex of degree one. A vertex v is *isolated* [2] if $\deg(v) = 0$. By an *empty graph* [5]

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we mean a graph with no edges. A simple graph is said to be *complete* [9] if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n. A graph is *bipartite* [5] if its vertex set can be partitioned into two subsets, X and Y so that every edge has one end in X and other end in Y; such a partition (X, Y) is called a *bipartition* of the bipartite graph. A simple bipartite graph is *complete* if each vertex of X is adjacent to all vertices of Y. A complete bipartite graph with $|X| = m$ and $|Y| = n$ is denoted by $K_{m,n}$. Given two graphs, G and H, we say H is an *induced subgraph* [3] of G if $V(H) \subseteq V(G)$, and two vertices of H are adjacent if and only if they are adjacent in G. In this case if $V(H) = S$, we write $H = G[S]$ or $H = \langle S \rangle$. The *union* [1] of two graphs G_1 and G_2 denoted by $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *line graph* [1] $L(G)$ of a graph G, is the graph whose vertex set is $E(G)$ and edge set is $\{ef : e, f \in E(G)\ and\ e,\ f$ have a vertex in common.$\}$ The *join* [4] of two graphs G_1 and G_2 denoted by $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$. The *corona* [4] of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where i^{th} vertex of G_1 is adjacent to every vertex in i^{th} copy of G_2. The *ring sum* [8] of two graphs G_1 and G_2, denoted by $G_1 \oplus G_2$, is the graph consisting of the vertex set $V(G_1) \cup V(G_2)$ and of edges that are either in G_1 or G_2, but not in both. The *middle graph* [6] of $G = (V(G), E(G))$ is the graph $M(G) = (V(G) \cup E(G), E'(G))$, where $uv \in E'$ if and only if either u is a vertex of G and v is an edge containing u, or u and v are edges having a vertex in common.

2. T_0 Graphs

In this paper we introduce the concept of of T_0 graphs.

Definition 1. A graph G is said to be a T_0-graph if for any two distinct vertices u and v of G, one of the following conditions hold:

1. At least one of u and v is isolated
2. There exists an edge e such that either e is incident with u but not with v or e is incident with v but not with u.

Note that empty graphs are trivially T_0.

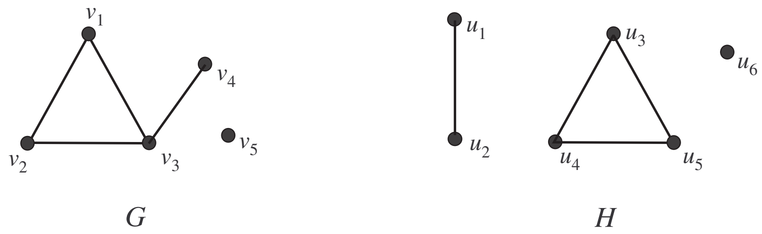

Figure 1: Graphs G and H.

This new concept is termed as 'T_0 graph ', because the topology generated by the collection of all two point sets consisting of the end vertices of edges of a T_0 graph G and singleton sets consisting of its isolated vertices is a T_0 topology on $V(G)$.

The graph G in Figure 1 is T_0, where as the graph H in Figure 1 is not T_0. The failure of the graph H to be T_0 is that K_2 is one of its component. This is the general feature of any T_0 graph. More explicitly we have the following theorem.

Theorem 2. *Let G be a graph with no K_2 as component, then G is a T_0 graph.*

Proof. Let u and v be two distinct non-isolated vertices of G. Since K_2 is not a component of G, there exists a vertex w distinct from u and v which is adjacent to u or v or both. Without loss of generality we can assume that w is adjacent to u. Then the edge $e = uw$ is incident with u but not with v. Therefore, G is T_0. □

Remark 3. The converse of Theorem 2 also holds good. Thus a necessary and sufficient condition for a graph G to be T_0 is that it does not contain K_2 as component.

From the definition of T_0 graphs we have,

- if G is a T_0 graph with no isolated vertices, then any supergraph of G is T_0.
- n-regular graphs are T_0 for every $n \neq 1$

- the cycle C_n is T_0 for every n
- the path P_n is T_0 if $n \neq 2$
- the complete graph K_n is T_0 if $n \neq 2$.
- the complete bipartite graph K_{mn} is T_0 if and only if both m and n are different from 1.

Theorem 4. *Let G_1 and G_2 be two isomorphic graphs. If G_1 is T_0, so is G_2.*

Proof. Given that G_1 and G_2 are isomorphic. Therefore, there exist bijections $f : V_1 \to V_2$ and $g : E_1 \to E_2$, such that $g(uv) = f(u)f(v)$ for every $uv \in E_1$. Let u and v be two distinct vertices of G_2. If one of u and v is isolated then there is nothing to prove.

So suppose both u and v are non-isolated. Since f is a bijection there exist two distinct vertices x and y of G_1 such that $f(x) = u$ and $f(y) = v$. Since G_1 is T_0, there exists an edge e which is incident with x but not with y or which is incident with y but not with x. Without loss of generality assume that e is incident with x but not with y. Let us suppose that $e = xp$. Then $p \neq y$, therefore $f(p) \neq f(y)$ and $g(e) = f(x)f(p) = uf(p)$ is an edge of G_2 incident with u but not with v. Hence the theorem holds. □

3. Incidence Matrix and Adjacency Matrix

Theorem 5. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$, and edge set $E(G) = \{e_1, e_2, \dots e_m\}$. Let $M = (m_{ij})$ be its incidence matrix. Then a necessary and sufficient condition that the graph G to be T_0 is that for any index j if there exists a pair of distinct indices (r, s) such that $m_{rj} = m_{sj} = 1$, then there should necessarily exists an index $i \neq j$ such that $m_{ri} = 1$ or $m_{si} = 1$.*

Proof. Suppose the hypothesis of the theorem holds. Then all the components of G are different from K_2. Therefore, by Theorem 2, G is T_0. Conversely assume that G is a T_0 graph. Suppose there exists an index j and a pair (r, s) of distinct indices such that $m_{rj} = m_{sj} = 1$ and $m_{ri} = m_{si} = 0$ for all $i \neq j$. The existence of such indices imply that the edge e_j is incident with the vertices v_r and v_s and the edge e_i is not incident with v_r and v_s for any $i \neq j$. Which

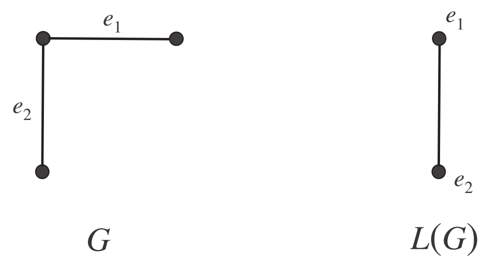

Figure 2: Graph G and its line graph $L(G)$.

implies $e = v_r v_s$ is an edge of G and no other edges of G is incident with v_r or with v_s. Therefore, the graph induced by $\{v_r, v_s\}$ is K_2. Hence G contains K_2 as component. Therefore, G can not be T_0, a contradiction. Therefore, for any index j if there exist a pair of distinct indices (r, s) such that $m_{rj} = m_{sj} = 1$, then there should necessarily exists an index i such that $m_{ri} = 1$ or $m_{si} = 1$. □

Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$. Let $A = (a_{ij})$ be its adjacency matrix. Then G is a graph with K_2 as component if and only if there exist some pair of distinct indices (r, s) such that $a_{rs} = 1, \sum_{j=1}^{n} a_{rj} = 1, \sum_{j=1}^{n} a_{sj} = 1$. Therefore, we have:

Theorem 6. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots v_n\}$. Let $A = (a_{ij})$ be its adjacency matrix. Then G is a T_0 graph if and only if there does not exist a pair (r, s) of distinct indices such that $a_{rs} = 1, \sum_{j=1}^{n} a_{rj} = 1, \sum_{j=1}^{n} a_{sj} = 1$.*

4. Line Graph of a Graph

Figure 2 shows that line graph of a T_0 graph need not be T_0.

Proposition 7. *Let G be a graph. If the path P_2 is not a component of G, then its line graph $L(G)$ is T_0.*

Proof. Let e_1 and e_2 be two distinct vertices of $L(G)$. If one of them is isolated then there is nothing to prove. So suppose that both of them are non-isolated vertices of $L(G)$. If e_1 and e_2 are not adjacent in $L(G)$, then again

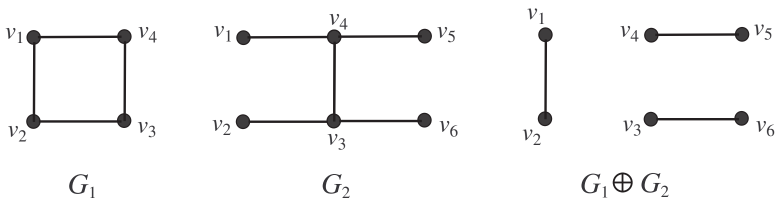

Figure 3: The ring sum of two T_0 graphs.

there is nothing to prove. If e_1 and e_2 are adjacent in $L(G)$, then e_1 and e_2 incident in G. Since P_2 is not a component of G we can find an edge e of G which is adjacent to e_1 or e_2 or both. Suppose e is adjacent to e_1 in G. Then e_1e is an edge of $L(G)$ incident with the vertex e_1 but not with e_2. Hence the proposition. □

5. Operations on Graph

In this section we deal with union, ring sum and join of two graphs.

Proposition 8. *The union of T_0 graphs is T_0.*

As the ring sum of two graphs with disjoint vertex sets is just their union, it is immediate from Proposition 8 the following.

Proposition 9. *The ring sum of two graphs with disjoint vertex set is T_0 if and only if both of them are T_0.*

Example 10.

Figure 3 shows that, the ring sum of two T_0 graphs need not be T_0 and Figure 4 shows that the ring sum of two non-T_0 graph may be T_0.

Example 11.

Next, we consider the case of join of graphs.

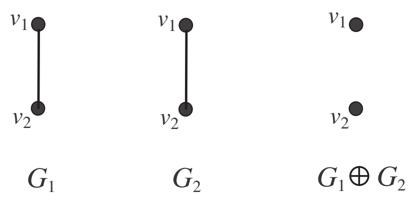

Figure 4: The ring sum of two non-T_0 graphs.

Theorem 12. *Let G_1 and G_2 be two graphs. Then $G_1 \vee G_2$ is T_0 if and only if either $|V(G_1)| \geq 2$ or $|V(G_2)| \geq 2$.*

Proof. Suppose $G_1 \vee G_2$ is T_0. If possible $|V(G_1)| = |V(G_2)| = 1$, then $G_1 \vee G_2 \cong K_2$, which is not T_0, a contradiction. Conversely, suppose that either $|V(G_1)| \geq 2$ or $|V(G_2)| \geq 2$. Without loss of generality we can assume that $|V(G_2)| \geq 2$. Since each vertex of G_1 is adjacent to every vertex of G_2 in $G_1 \vee G_2$ and $|V(G_2)| \geq 2$, K_2 cannot be a component of $G_1 \vee G_2$. Therefore, by Theorem 2, $G_1 \vee G_2$ is T_0. □

6. Corona and Middle Graph

From the definition of corona of two graphs we have:

Proposition 13. *If G_1 is any graph and G_2 a graph with $|V(G_2)| \geq 2$ or if G_1 is a graph with no isolated vertices and G_2 any graph, then $G_1 \circ G_2$ is T_0.*

Proof. In the first case, since each vertex v of G_1 is adjacent to every vertex of the copy of G_2 corresponding to v and $|V(G_2)| \geq 2$, K_2 cannot be a component of $G_1 \circ G_2$ Therefore, $G_1 \circ G_2$ is T_0.

In the second case, each vertex v of G_1 is adjacent to at least one vertex of G_1. Therefore, each vertex of $G_1 \circ G_2$ which belongs to the copy of G_1 is adjacent to at least two vertices of $G_1 \circ G_2$ and each vertex of $G_1 \circ G_2$ which belongs to the copy of G_2 is adjacent to a vertex of the copy of G_1. Therefore, $G_1 \circ G_2$ cannot contain K_2 as a component. Hence $G_1 \circ G_2$ is T_0. □

The middle graph of a graph also behaves very nicely with the T_0-axiom.

Theorem 14. *The middle graph $M(G)$ of any graph G is T_0.*

Proof. Let u and v be two distinct vertices of $M(G)$.

Suppose u and $v \in V(G)$. If one of them say u is isolated in G, then u is also an isolated vertex of $M(G)$. So suppose both u and v are non-isolated vertices of G. Let e be an edge of G incident with u. Then ue is an edge of $M(G)$ incident with u but not with v.

Now suppose $u \in V(G)$ and $v \in E(G)$. Let w be a vertex of G distinct from u such that v is incident with w in G. Then vw is an edge of $L(G)$ incident with v but not with u.

Finally suppose $u, v \in E(G)$. Since $u \neq v$, there exist a vertex w of G such that u is incident with w in G. Then uw is an edge of $M(G)$ incident with u but not with v

Hence the theorem holds. □

7. Conclusions

In this paper T_0 graphs have been discussed with examples. Sufficient conditions for join of two graphs, middle graph of a graph, corona of two graphs to be T_0 have also been discussed. It is established via example that the line graph of a T_0 graph need not be T_0. Furthermore, the relations of T_0 graph with its incidence matrix and adjacency matrix is discussed.

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