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PERIPHERY BEHAVIOUR OF SERIES IN MITTAG-LEFFLER TYPE FUNCTIONS, I

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Abstract: This is a survey on part of author's recent results on the subject. Different families of the the Mittag-Leffler functions and their 3-parametric generalizations are considered. First, asymptotic formulae necessary for proving the main results, are provided. Series defined by means of these families are further studied. Starting with their domains of convergence, the behaviour of such series on the peripheries of their convergence domains is investigated and analogues of the classical results for the power series are proposed.

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1. Introduction

The special functions $E^{\gamma}_{\alpha,\beta}$, defined in the whole complex plane $\mathbb{C}$ by the power

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series

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \tag{1}$$

where $(\gamma)_k$ is the Pochhammer symbol ([1], Section 2.1.1)

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1),$$

arise as natural generalizations of the Mittag-Leffler functions E_α and $E_{\alpha,\beta}$. They were introduced by Prabhakar in 1971 in his paper [15]. As a matter of fact, Prabhakar introduced these functions for positive γ, and in this case they are entire functions of z of order $\rho = 1/Re(\alpha)$, as mentioned e.g. in [3]. For $\gamma = 1$ and $\gamma = \beta = 1$ these functions coincide with the classical Mittag-Leffler functions $E_{\alpha,\beta}$ and E_α, i. e.

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}. \tag{2}$$

The first were introduced by Mittag-Leffler (1902-1905) who investigated some of their properties, while the others first appeared in a paper of Wiman (1905).

In the previous papers [10, 12], the author considered series in systems of Mittag-Leffler type functions and, resp. in [13, 14], series in the multi-index ($2m$-indices) analogues of the Mittag-Leffler functions and some of their special cases, as representatives of the Special Functions of Fractional Calculus, [5]. Their convergence in the complex plane $\mathbb{C}$ is studied and Cauchy-Hadamard, Abel and Tauberian type theorems are proved. In this paper series in Mittag-Leffler functions and their three-parametric Prabhakar generalizations are studied and analogical results for them are discussed. Finding of such a kind of results is provoked by the fact that the solutions of some fractional order differential and integral equations can be written in terms of series (or series of integrals) of Mittag-Leffler type functions (as for example, in Kiryakova [4]). The functions (1) and series in them have recently been used to express solutions of the generalized Langevin equation by Sandev, Tomovski and Dubbeldam [17]. For various anomalous diffusion and relaxation processes, generalized diffusion and Fokker-Planck-Smoluchowski equations with the corresponding memory kernels see survey paper by Sandev, Chechkin, Kantz and Metzler [16].

Consider now the first of the functions (2) for positive indices $\alpha = n \in \mathbb{N}$ and also generalized Mittag-Leffler functions (1) for indices of the kind $\beta = n$;

$n = 0, 1, 2, \dots$, namely:

$$\begin{aligned} E_n(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(nk+1)}, \; n \in \mathbb{N}; \\ E^{\gamma}_{\alpha,\,n}(z) &= \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^k}{k!}, \; \alpha, \; \gamma \in \mathbb{C}, \; Re(\alpha) > 0, \; n \in \mathbb{N}_0. \end{aligned} \tag{3}$$

Since for $\gamma = 1$ the generalized Mittag-Leffler functions reduce to two-parametric Mittag-Leffler functions, all the results discussed in this survey paper hold true for series in the corresponding two-parametric Mittag-Leffler functions.

Note that the coefficients in E_n are all different from zero. However, the situation is not the same in $E^{\gamma}_{\alpha,\,n}$. More detailed observation shows that some coefficients can be zero there, depending on γ and n [11]. Namely, the following remark can be written.

Remark 1. *Given a number γ, suppose that some coefficients in* (3) *equal to zero, that is, there exists a number $p \in \mathbb{N}_0$, such that the representation* (3) *can be written as follows:*

$$E^{\gamma}_{\alpha,n}(z) = z^p \sum_{k=p}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + n)} \frac{z^{k-p}}{k!}. \tag{4}$$

More precisely, as it is given in [11, 12], if γ is different from zero, then $p = 0$ for each positive integer n and $p = 1$ for $n = 0$.

Additionally, we recall some results related to the asymptotic formulae for 'large' values of indices n for $z, \alpha, \gamma \in \mathbb{C}, \gamma \neq 0$, and $Re(\alpha) > 0$, that are applied in proving the main results. Namely (see e.g. in [10, 11, 12]) there exist entire functions θ_n and $\theta^{\gamma}_{\alpha,\,n}$ such that the functions (3), have the following asymptotic formulae:

$$\begin{aligned} E_n(z) &= 1 + \theta_n(z) \quad (n \in \mathbb{N}), \\ E^{\gamma}_{\alpha,n}(z) &= \frac{(\gamma)_p}{\Gamma(\alpha p + n)} z^p \left(1 + \theta^{\gamma}_{\alpha,\,n}(z)\right) \quad (n \in \mathbb{N}_0), \\ \text{and} \quad \theta_n(z) \to 0, &\quad \theta^{\gamma}_{\alpha,\,n}(z) \to 0 \quad \text{as } n \to \infty; \end{aligned} \tag{5}$$

with the corresponding p, depending on γ and n. Moreover, on the compact subsets of the complex plane $\mathbb{C}$, the convergence is uniform and

$$\theta_n(z) = O\left(\frac{1}{n!}\right), \quad \theta^{\gamma}_{\alpha,\,n}(z) = O\left(\frac{1}{n^{Re(\alpha)}}\right) \quad (n \; \in \mathbb{N}). \tag{6}$$

Remark 2. *If $\gamma = 0$, the functions* (3) *take the simplest form*

$$E^0_{\alpha,\,n}(z) = \frac{1}{\Gamma(n)}$$

for $n \in \mathbb{N}$, and $E^0_{\alpha,\,n}(z) = 0$ for $n = 0$.

Remark 3. *According to the asymptotic formulae* (5) *and* (6), *it follows there exists a positive integer M such that the functions E_n, $E^\gamma_{\alpha,\,n}$ have no zeros for $n > M$, possibly except at the origin.*

2. Series in Mittag-Leffler Type Functions and their Domains of Convergence

Now, let us specify the families of Mittag-Leffler type functions

$$\left\{\widetilde{E}_n(z)\right\}_{n=0}^{\infty}, \quad \left\{\widetilde{E}^\gamma_{\alpha,\,n}(z)\right\}_{n=0}^{\infty}; \quad \alpha, \gamma \in \mathbb{C},\ Re(\alpha) > 0,$$

as below ($\widetilde{E}_0(z) = \widetilde{E}^0_{\alpha,\,0}(z) = 1$, just for completeness), namely:

$$\begin{aligned}
\widetilde{E}_0(z) = 1,\ \widetilde{E}_n(z) = z^n\, E_n(z), \quad n \in \mathbb{N},\\
\widetilde{E}^0_{\alpha,\,0}(z) = 1,\ \widetilde{E}^0_{\alpha,\,n}(z) = \Gamma(n)\ z^n\, E^0_{\alpha,\,n}(z);\ \ n \in \mathbb{N},\\
\widetilde{E}^\gamma_{\alpha,\,n}(z) = \frac{\Gamma(\alpha p + n)}{(\gamma)_p}\ z^{n-p}\, E^\gamma_{\alpha,\,n}(z),\ \ \gamma \neq 0,\ \ n \in \mathbb{N}_0,
\end{aligned}$$

and let us consider series in these functions of the form:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}_n(z), \qquad \sum_{n=0}^{\infty} a_n \widetilde{E}^\gamma_{\alpha,\,n}(z), \tag{7}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

Our objective is to study the convergence of the series (7) in the complex plane and to propose theorems, corresponding to the classical results for the power series. Beginning with the domain of convergence of the series (7), we recall [10, 12] that it is the open disk $D(0; R) = \{z : \ |z| < R, z \in \mathbb{C}\}$ with a radius of convergence

$$R = \left(\limsup_{n\to\infty} (\ |a_n|\ )^{1/n}\right)^{-1}. \tag{8}$$

More precisely, both series are absolutely convergent in the disk $D(0;R)$ and they are divergent in the domain $|z|>R$. The cases $R=0$ and $R=\infty$ fall in the general case. Farther, analogously to the classical Abel lemma, if any of the series (7) converges at the point $z_0\neq 0$, then it is absolutely convergent in the disk $D(0;|z_0|)$. Moreover, inside the disk $D(0;R)$, i.e. on each closed disk $|z|\leq r<R$ (R defined by (8)), the series is uniformly convergent.

3. Behaviour 'Near' the Periphery of the Domains of Convergence

Let $z_0\in\mathbb{C}$, $0<|z_0|=R<\infty$, and g_φ be an arbitrary angular domain with size $2\varphi<\pi$ and with a vertex at the point $z=z_0$, which is symmetric with respect to the straight line passing through the origin and z_0. Let d_φ be the part of the angular domain g_φ, closed between the angle arms and the arc of the circle, centered at the point 0 and touching the arms of the angle. Another interesting result is the Abel type theorem, analogical to the classical Abel theorem for the power series. It refers to the uniform convergence of the series (7) in the set d_φ and the existence of the limits of their sums at the point z_0 from the boundary $C(0;R)$, provided $z\in D(0;R)\cap g_\varphi$, i.e. the limit of the sum of any of these series, is equal to the corresponding series sum at the point z_0. Namely, the following theorem is valid (for the lines of proof and details one can see in [12]).

Theorem 4 (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers and let $0<R<\infty$ be defined by (8). If $F(z)$ is the sum of any of the series (7) in the domain $D(0;R)$, i.e.:*

$$F(z)=\sum_{n=0}^{\infty}a_n\widetilde{E}_n(z), \quad \textit{respectively} \quad F(z)=\sum_{n=0}^{\infty}a_n\widetilde{E}_{\alpha,n}^{\gamma}(z), \tag{9}$$

and this series converges at the point z_0 of the boundary of $D(0;R)$, then:

(i) *The corresponding relation holds*

$$\lim_{z\to z_0}F(z)=\sum_{n=0}^{\infty}a_n\widetilde{E}_n(z_0), \tag{10}$$

respectively

$$\lim_{z\to z_0}F(z)=\sum_{n=0}^{\infty}a_n\widetilde{E}_{\alpha,n}^{\gamma}(z_0), \tag{11}$$

provided $z\in D(0;R)\cap g_\varphi$;

(ii) *The corresponding series is uniformly convergent in the domain* d_φ.

4. (E, z_0)-Summation

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \qquad n = 0, 1, 2, ... \tag{12}$$

To define its Abel summability ([2, p.20, 1.3 (2)]), we also consider the power series $\sum\limits_{n=0}^{\infty} a_n z^n$. The series (12) is called A-summable, if the series $\sum\limits_{n=0}^{\infty} a_n z^n$ converges in the disk $D = \{z : z \in \mathbb{C}, |z| < 1\}$ and moreover there exists $\lim\limits_{z\to 1-0} \sum\limits_{n=0}^{\infty} a_n z^n = S$. The complex number S is called A-sum of the series (12). The A-summation is regular, i.e. if the series (12) converges, then it is A-summable and its A-sum is equal to its usual sum. In general, the A-summability of the series (12) does not imply its convergence. But, with additional conditions on the growth of the general term of the series (12), the convergence can be ensured.

Note that each of the functions $\widetilde{E}_n$, $\widetilde{E}^{\gamma}_{\alpha,n}$ $(n \in \mathbb{N})$, being an entire function not identically zero, has no more than finite number of zeros in the closed and bounded set $|z| \leq R$ ([6, Vol. 1, Ch. 3, §6, 6.1, p.305]). Moreover, because of Remark 3, no more than finite number of these functions have some zeros, possibly except for zero.

Let $z_0 \in \mathbb{C}$, $|z_0| = R$, $0 < R < \infty$, $\widetilde{E}_n(z_0) \neq 0$ and $\widetilde{E}^{\gamma}_{\alpha,n}(z_0) \neq 0$. For the sake of brevity, denote

$$E^*_n(z; z_0) = \frac{\widetilde{E}_n(z)}{\widetilde{E}_n(z_0)}, \quad E^*_{\alpha,n,\gamma}(z; z_0) = \frac{\widetilde{E}^{\gamma}_{\alpha,n}(z)}{\widetilde{E}^{\gamma}_{\alpha,n}(z_0)}.$$

Analogously to the definition of A-summability, taking the family $\{E^*_n(z; z_0)\}$ (respectively $\{E^*_{\alpha,n,\gamma}(z; z_0)\}$) instead of the Taylor family $\{z^n\}$, (E, z_0)-summability is defined by means of the corresponding family [10, 12].

Definition 5. The series (12) is said to be (E, z_0)-summable by means of the family $\{E^*_n(z; z_0)\}$ (respectively $\{E^*_{\alpha,n,\gamma}(z; z_0)\}$), if the series

$$\sum_{n=0}^{\infty} a_n E^*_n(z; z_0), \quad \text{respectively} \quad \sum_{n=0}^{\infty} a_n E^*_{\alpha,n,\gamma}(z; z_0), \tag{13}$$

converges in the disk $D(0;R)$ and, moreover, there exists the limit

$$\lim_{z\to z_0}\sum_{n=0}^{\infty}a_nE_n^*(z;z_0),\quad \text{respectively}\quad \lim_{z\to z_0}\sum_{n=0}^{\infty}a_nE_{\alpha,n,\gamma}^*(z;z_0), \tag{14}$$

provided z remains in the segment $[0,z_0)$, i.e. z radially tends to z_0.

Remark 6. *Every (E,z_0)-summation is regular, and this property is just a particular case of Theorem 4.*

5. Tauberian Type Theorems

The Tauberian theorem is a statement that relates the Abel summability and the standard convergence of a numerical series by means of some assumptions imposed on the general term of the series under consideration. In fact, such type of statement is inverse to the Abel theorem. A classical result in this direction is given by Theorem 85 in [2].

Tauber type theorems are given e.g. for summations by means of Laguerre polynomials by Rusev and Bessel type functions by the author [7, 8]. The validity of such a type of assertion is also extended to series of the kind (13) in Mittag-Leffler functions and their three-parametric generalizations. Namely, the following result holds (for details, see [10, 12]).

Theorem 7 (of Tauber type)**.** *If the numerical series* (12) *is* (E,z_0)*-summable by means of the family* $\{E_n^*(z;z_0)\}$ *(respectively* $\{E_{\alpha,n,\gamma}^*(z;z_0)\}$*) and*

$$\lim_{n\to\infty} na_n = 0, \tag{15}$$

then it is convergent.

At first sight, it seems that the condition $a_n = o(1/n)$ is essential. Nevertheless, Littlewood succeeded to weaken it and obtained the strengthened version of the Tauber theorem ([2, Theorem 90]). Similar theorem is proved for series in Bessel type functions, see e.g. [9]. A Littlewood generalization of the $o(n)$ version of the Tauber type theorem (Theorem 7) is given in this part of the survey as well. The proof can be found in [12].

Theorem 8 (of Littlewood type)**.** *If the numerical series* (12) *is* (E, z_0)*-summable by means of the family* $\{E_n^*(z; z_0)\}$ (*respectively* $\{E_{\alpha,n,\gamma}^*(z; z_0)\}$) *and*

$$a_n = O(1/n)$$

then the series (12) *converges.*

6. Conclusion

As a conclusion of this survey, all the discussed results can be briefly summarized in the following way. The basic properties of the series, objects of this survey, are sufficiently 'close' to the corresponding classical results for the power series with the same coefficients, i.e. their behaviour is quite similar to the one of the widely used power series.

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