SYNCHRONIZATION OF GENESIO AND LÜ
CHAOTIC DYNAMICAL SYSTEMS VIA ACTIVE CONTROL

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Abstract: In this paper we have investigated Complete Synchronization, Anti Synchronization and Hybrid Synchronization between the Genesio chaotic dynamical system and Lü chaotic dynamical system using nonlinear active control technique based on Lyapunov Stability theory. Further numerical simulations are carried out to show the effectiveness of the method.

1. Introduction

Chaos is an interesting nonlinear phenomenon and has been intensively studied in the last three decades. Chaos theory has been applied in many scientific disciplines such as Mathematics, Economics, Microbiology, Biology, Ecology, Computer Science and Robotics.

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A chaotic system is a special non linear dynamical system, which has many properties such as the sensitivity to initial conditions as well as an irregular, unpredictable behaviour. During the last few years synchronization in chaotic dynamical systems has received a great deal of attention for the scientists from various fields [2, 5, 6, 18, 41].

In 1990, Pecora and Carroll developed control techniques to synchronize two identical chaotic systems and showed that it was possible for some chaotic systems to be completely synchronized. From then on, chaos synchronization has emerged as an important area of research in many fields such as physical systems [7, 20], chemical systems [11, 25], ecological systems [3] and secure communication [4, 19, 38] etc.

A wide variety of approaches has been used for the synchronization of chaotic systems such as active control method [1, 12, 13, 17, 32], adaptive control [23, 36], backstepping design [37, 39], OGY method [26], sampled- data feedback synchronization method [24, 44], time delay feedback method [28], sliding mode control method [31, 35] and PC method [29], etc. However most of these methods synchronize two identical chaotic systems. But in many practical problems such as laser array and biological systems, it is hardly the case that every component can be assumed to be identical. More and more applications of chaos synchronization in secure communication make it even more important to synchronize two different chaotic systems. In recent years, some works on synchronization of two different chaotic systems have been performed [12, 27, 42]. So far many Synchronization phenomenon have been presented such as complete synchronization [16], phase synchronization [8], anti synchronization [16, 33, 43],generalized synchronization [40], hybrid synchronization [14, 15, 16, 34], projective synchronization [30] and generalized projective synchronization [21], etc.

In this paper we study the chaos synchronization between two different chaotic systems using active nonlinear control method. The stability results derived in this paper are established using Lyapunov stability theory. The Genesio system is taken as the master system and the Lü system as the slave system. This paper is organized as follows. In Section 2, we provide a description of the chaotic systems. In Section 3, we discuss the Complete synchronization, Anti synchronization and Hybrid synchronization between the Genesio chaotic system and the Lü chaotic dynamical system using active control. In Section 4, we discuss the numerical simulations.

2. System Description

The master system is described by the Genesio system, proposed by Genesio et al. [9, 32] capturing many features of chaotic systems. It includes three simple ordinary differential equations that depend on three positive real parameters. The dynamical equations are given by

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -cx_1 - bx_2 - ax_3 + x_1^2, \end{aligned}\right\} \tag{1}$$

where x_1, x_2 and x_3 are state variables and a, b and c are positive constants satisfying $ab < c$. The system exhibits a chaotic behaviour for the parameters values $a = 1.2$, $b = 2.92$ and $c = 6$, shown in Fig. 1.

The slave system is described by the Lü system, [22] is a typical transition system, which connects the Lorenz and Chen attractors and represents the transition from one to the other,that captures many features of chaotic systems. The dynamical equations are given by

$$\left.\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1), \\ \dot{y}_2 &= \gamma y_2 - y_1 y_3, \\ \dot{y}_3 &= -\beta y_3 + y_1 y_2, \end{aligned}\right\} \tag{2}$$

where y_1, y_2 and y_3 are state variables and α, β and γ are positive constants. The system exhibits a chaotic behaviour for the parameters values $\alpha = 36$, $\beta = 3$ and $\gamma = 20$ as shown Fig. 2.
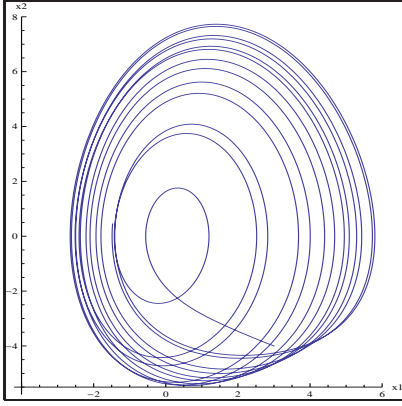
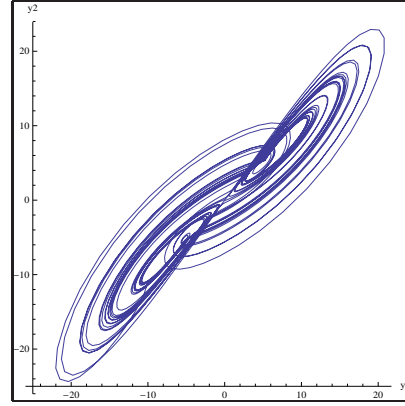Fig.1 Genesio chaotic system with
$a = 1.2$, $b = 2.92$ and $c = 6$

Fig.2 Lü chaotic system with
$\alpha = 36$, $\beta = 3$ and $\gamma = 20$

3. Complete Synchronization, Anti Synchronization and Hybrid Synchronization between Genesio Chaotic System and Lü Chaotic System

3.1. Complete Synchronization

Complete synchronization is characterized by the equality of state variables evolving in time. As the master system we consider the Genesio dynamical system defined by

$$\left.\begin{aligned}
\dot{x_1} &= x_2, \\
\dot{x_2} &= x_3, \\
\dot{x_3} &= -cx_1 - bx_2 - ax_3 + x_1^2,
\end{aligned}\right\} \tag{3}$$

where x_1, x_2 and x_3 are state variables and a, b and c are unknown, real, constant parameters satisfying $ab < c$.

As the slave system, we consider the Lü dynamical system described by

$$\left.\begin{aligned}
\dot{y_1} &= \alpha(y_2 - y_1) + u_1, \\
\dot{y_2} &= \gamma y_2 - y_1 y_3 + u_2, \\
\dot{y_3} &= -\beta y_3 + y_1 y_2 + u_3,
\end{aligned}\right\} \tag{4}$$

where y_1, y_2 and y_3 are state variables, α, β and γ are unknown, real, constant parameters and u_1, u_2 and u_3 are control functions to be designed. The synchronization errors e_i, $i = 1, 2, 3$ are defined by $e_i = y_i - x_i$. From (3) and (4), we obtain the error dynamics as follows:

$$\left.\begin{aligned}
\dot{e}_1 &= \alpha(y_2 - y_1) - x_2 + u_1, \\
\dot{e}_2 &= \gamma y_2 - y_1 y_3 - x_3 + u_2, \\
\dot{e}_3 &= -\beta y_3 + y_1 y_2 + \mathrm{c} x_1 + \mathrm{b} x_2 + \mathrm{a} x_3 - x_1^2 + u_3.
\end{aligned}\right\} \tag{5}$$

We choose the nonlinear controllers as

$$\left.\begin{aligned}
u_1 &= -\alpha(y_2 - y_1) + x_2 - k_1 e_1, \\
u_2 &= -\gamma y_2 + y_1 y_3 + x_3 - k_2 e_2, \\
u_3 &= \beta y_3 - y_1 y_2 - \mathrm{c} x_1 - \mathrm{b} x_2 - \mathrm{a} x_3 + x_1^2 - k_3 e_3,
\end{aligned}\right\} \tag{6}$$

where k_1, k_2 and k_3 are positive constants. Substituting (6) into (5), we get

$$\left.\begin{aligned}
\dot{e}_1 &= -k_1 e_1, \\
\dot{e}_2 &= -k_2 e_2, \\
\dot{e}_3 &= -k_3 e_3.
\end{aligned}\right\} \tag{7}$$

Consider the Lyapunov function defined by

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \tag{8}$$

Differentiating (8) along the trajectories of (7) we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 \tag{9}$$

which is a negative definite function on $\mathbb{R}^2$. Thus by Lyapunov stability theory [10], the error dynamics (7) is found to be asymptotically stable and converges to zero exponentially with time and thus complete synchronization is achieved between master system (3) and slave system (4). Hence the non-identical Genesio system (3) and Lü system (4) are exponentially and globally synchronized using nonlinear active control law.

3.2. Anti Synchronization

Anti synchronization is characterized by the disappearance of the sum of relevant variables evolving in time. To observe anti synchronization between the master system and the slave system, we first define the synchronization errors e_i, $i = 1, 2, 3$ by $e_i = y_i + x_i$. From (3) and (4), we obtain the error dynamics as follows:

$$\left.\begin{aligned}
\dot{e}_1 &= \alpha(y_2 - y_1) + x_2 + u_1, \\
\dot{e}_2 &= \gamma y_2 - y_1 y_3 + x_3 + u_2, \\
\dot{e}_3 &= -\beta y_3 + y_1 y_2 - c x_1 - b x_2 - a x_3 + x_1^2 + u_3.
\end{aligned}\right\} \tag{10}$$

We choose the nonlinear controllers as

$$\left.\begin{aligned}
u_1 &= -\alpha(y_2 - y_1) - x_2 - k_1 e_1, \\
u_2 &= -\gamma y_2 + y_1 y_3 - x_3 - k_2 e_2, \\
u_3 &= \beta y_3 - y_1 y_2 + c x_1 + b x_2 + a x_3 - x_1^2 - k_3 e_3,
\end{aligned}\right\} \tag{11}$$

where k_1, k_2 and k_3 are positive constants. Substituting (11) into (10), we get

$$\left.\begin{aligned}
\dot{e}_1 &= -k_1 e_1, \\
\dot{e}_2 &= -k_2 e_2, \\
\dot{e}_3 &= -k_3 e_3.
\end{aligned}\right\} \tag{12}$$

Consider the Lyapunov function defined by

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \tag{13}$$

Differentiating (13) along the trajectories of (12) we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 \tag{14}$$

which is a negative definite function on $\mathbb{R}^2$. Thus by Lyapunov stability theory [10], the error dynamics (12) is found to be asymptotically stable and converges to zero exponentially with time and hence anti synchronization is achieved between master system (3) and slave system (4).

Thus the non-identical Genesio system (3) and Lü system (4) are exponentially and globally anti synchronized for all initial conditions using nonlinear controller defined by (11).

3.3. Hybrid Synchronization

In Hybrid synchronization of chaotic systems, one part of the system is synchronized and the other part is anti synchronized so that complete synchronization and anti synchronization exist together in the system. This coexistence of complete synchronization and anti synchronization is useful in secure communication and chaotic encryption schemes. The Hybrid synchronization errors are defined by

$$\left.\begin{aligned}
e_1 &= y_1 - x_1, \\
e_2 &= y_2 + x_2, \\
e_3 &= y_3 - x_3.
\end{aligned}\right\} \tag{15}$$

Here, we see that one part of the two chaotic systems is completely synchronized and the other part is anti synchronized so that complete synchronization and anti synchronization exist together in the synchronization of systems (3) and (4).

From (3) and (4) we obtain the error dynamics as follows:

$$\left.\begin{aligned}
\dot{e}_1 &= \alpha(y_2 - y_1) - x_2 + u_1, \\
\dot{e}_2 &= \gamma y_2 - y_1 y_3 + x_3 + u_2, \\
\dot{e}_3 &= -\beta y_3 + y_1 y_2 + cx_1 + bx_2 + ax_3 - x_1^2 + u_3.
\end{aligned}\right\} \tag{16}$$

We choose the nonlinear controllers as

$$\left.\begin{aligned}
u_1 &= -\alpha(y_2 - y_1) + x_2 - k_1 e_1, \\
u_2 &= -\gamma y_2 + y_1 y_3 - x_3 - k_2 e_2, \\
u_3 &= \beta y_3 - y_1 y_2 - cx_1 - bx_2 - ax_3 + x_1^2 - k_3 e_3,
\end{aligned}\right\} \tag{17}$$

where k_1, k_2 and k_3 are positive constants. Substituting (17) into (16), we get

$$\left.\begin{aligned}
\dot{e}_1 &= -k_1 e_1, \\
\dot{e}_2 &= -k_2 e_2, \\
\dot{e}_3 &= -k_3 e_3.
\end{aligned}\right\} \tag{18}$$

Consider the Lyapunov function defined by

$$V(e) = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2). \tag{19}$$

Differentiating (19) along the trajectories of (18), we get

$$\dot{V}(e) = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 \qquad (20)$$

which is a negative definite function on $\mathbb{R}^2$. Thus by Lyapunov stability theory
[10], the hybrid synchronization errors e_i , $i = 1, 2, 3$ converges to zero expo-
nentially with time. In the similar manner, the 5 other hybrid synchronization
phenomena can be described. We find that in all the cases after defining the
control functions u_i, $i = 1, 2, 3$ the linear error system obtained is identical to
the one defined in (18). Hence the hybrid synchronization is attainable in all
the cases. Thus the non-identical Genesio system (3) and Lü system (4) are
exponentially and globally hybrid synchronized using nonlinear active control
law.

4. Numerical Simulations

For simulations, the parameters of the Genesio Chaotic system are chosen as
$a = 1.2$, $b = 2.92$ and $c = 6$. The parameters of Lü chaotic system are chosen
as $\alpha = 36$, $\beta = 3$ and $\gamma = 20$. The initial conditions of the master and slave
system are chosen as $x(0) = (3, -4, 2)$ and $y(0) = (-10, -11, 5)$, respectively.
For complete synchronization, the control functions defined by (6) are used and
the systems (3) and (4) are simulated, for anti synchronization, the control
functions defined by (11) are used and the systems (3) and (4) are simulated
and for hybrid synchronization, the control functions defined by (17) are used
and the systems (3) and (4) are simulated. The results of the simulation of
the two non identical chaotic systems without active control are shown in Fig.
3: (a) displays the x_1 and y_1, (b) displays the x_2 and y_2, and (c) displays
the x_3 and y_3, where x_1, x_2 and x_3 are state variables of the Genesio Chaotic
system and y_1, y_2 and y_3 are state variables of the Lü chaotic system. Fig. 4
$(a) - (c)$ display the same sequence of signals, when completely synchronized
using the active control. The corresponding time series analysis graph of the
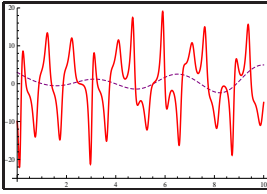synchronization errors are given in Fig. 5 $(a) - (c)$.

Fig.3 (a) State trajectories x_1 $(--)$ and y_1 $(-)$ without active controllers
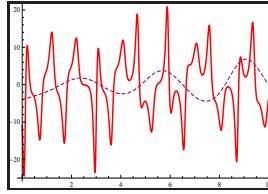

Fig.3 (b) State trajectories x_2 $(--)$ and y_2 $(-)$ without active controllers
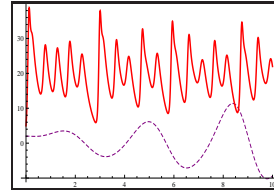

Fig.3 (c) State trajectories x_3 $(--)$ and y_3 $(-)$ without active controllers
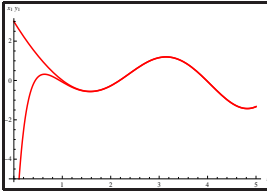

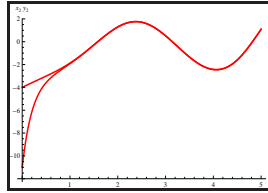Fig. 4 (a) State trajectories x_1 and y_1 with active controllers

Fig.4 (b) State trajectories x_2 and y_2 with active controllers
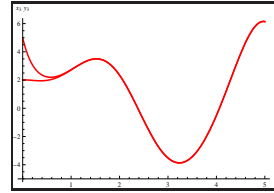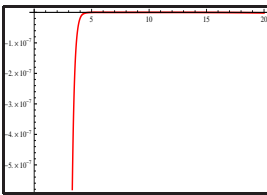

Fig.4 (c) State trajectories x_3 and y_3 with active controllers

Fig.5 (a) Time series analysis of $e_1(t)$
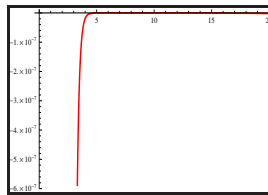
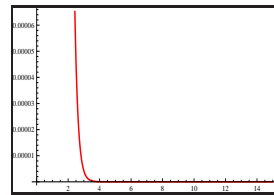Fig.5 (b) Time series analysis of $e_2(t)$

Fig.5 (c) Time series analysis of $e_3(t)$

Fig. 6 $(a) - (c)$ display the same sequence of signals, when the signals are anti synchronized using the active control. The corresponding time series analysis graph of the synchronization errors are given in Fig. 7 $(a) - (c)$.
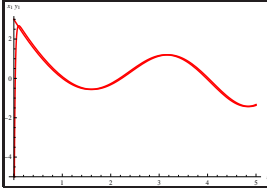
Fig.6 (a) State trajectories x_1 and y_1 with active controllers
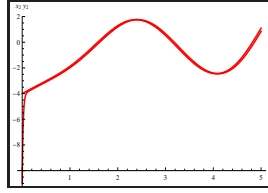
Fig.6 (b) State trajectories x_2 and y_2 with active controllers
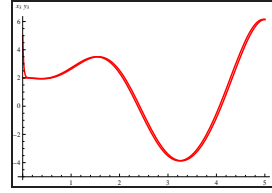
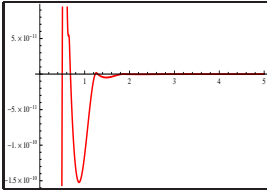Fig.6 (c) State trajectories x_3 and y_3 with active controllers

Fig.7 (a) Time series analysis of $e_1(t)$

Fig.7 (b) Time series analysis of $e_2(t)$

Fig.7 (c) Time series analysis of $e_3(t)$

Fig. 8 $(a) - (c)$ display the same sequence of signals, when the signals are hybrid synchronized using the active control. The corresponding time series analysis graph of the synchronization errors are given in Fig. 9 $(a) - (c)$.

Fig.8 (a) State trajectories x_1 and y_1 with active controllers

Fig.8 (b) State trajectories x_2 and y_2 with active controllers

Fig.8 (c) State trajectories x_3 and y_3 with active controllers

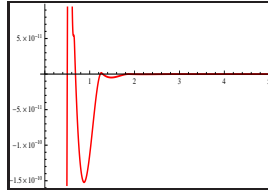Fig.9 (a) Time series analysis of $e_1(t)$
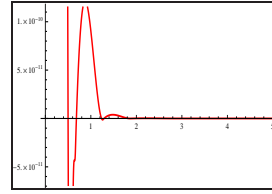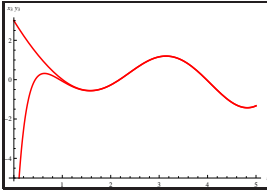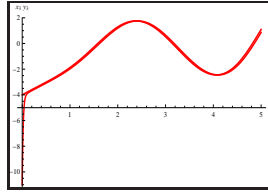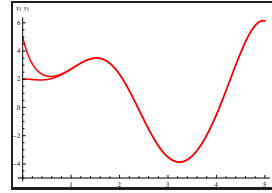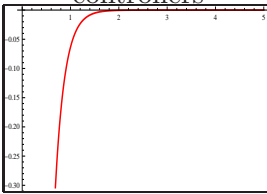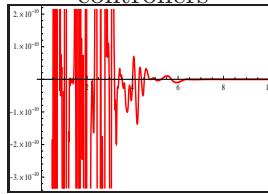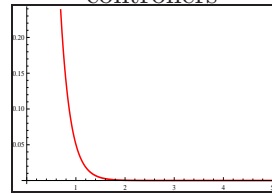
Fig.9 (b) Time series analysis of $e_2(t)$

Fig.9 (c) Time series analysis of $e_3(t)$

5. Conclusion

In this paper, we have investigated synchronization behavior between the Genesio chaotic system and the Lü chaotic system via active control method and Lyapunov stability theory. Since the Lyapunov exponents are not needed for these calculations, the nonlinear control method is very effective and convenient to achieve global chaos complete synchronization, anti synchronization and hybrid synchronization between the systems studied in this paper. The results are validated by numerical simulations using *Mathematica* and *Matlab* both.

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