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# CERTAIN BOUNDS FOR UNIFIED CONVEX FUNCTIONS OF $q$-DIFFERENCE OPERATOR

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**Abstract:** The aim of this paper is to establish coefficient bounds for certain classes of analytic functions of complex order associated with the $q$-derivative operator. Some applications of these results for the functions defined through convolution are also obtained.

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**Key Words:** analytic function, univalent function, Schwarz function, $q$-derivative operator, subordination

## 1. Introduction

Recently, the area of the $q$-analysis has attracted serious attention of researchers. The great interest is due to its applications in various branches of mathematics and physics, as for example, in the areas of ordinary fractional calculus, optimal control problems, $q$-difference and $q$-integral equations and in $q$-transform analysis. The generalized $q$-Taylor formula in the fractional $q$-calculus was introduced by Purohit and Raina [14]. The application of $q$-calculus was initiated by Jackson [5, 6]. He was the first to develop the $q$-integral and $q$-derivative in a systematic way. Later, geometrical interpretation of the $q$-analysis has been recognized through studies on quantum groups. Simply, the quantum calculus

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is ordinary classical calculus without the notion of limits. It defines $q$-calculus and $h$-calculus. Here $h$ ostensibly stands for Planck's constant, while $q$ stands for quantum. Mohammed and Darus [10] studied approximation and geometric properties of these $q$-operators in some subclasses of analytic functions in compact disk, recently Purohit and Raina in [14, 15] have used the fractional $q$-calculus operators in investigating certain classes of functions which are analytic in the open disk, and Purohit [13] also studied these $q$-operators, defined by using the convolution of normalized analytic functions and the $q$-hypergeometric functions. A comprehensive study on applications of $q$-calculus in the operator theory may be found in [2]. Ramachandran et al. [16] have used the fractional $q$-calculus operators in investigating certain bound for $q$-starlike and $q$-convex functions with respect to symmetric points.

Let $\mathcal{A}$ be denote the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

defined on the *open* unit disk $\mathbb{U}$.

Also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the univalent functions in $\mathbb{U}$.

Recalling the principal of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\mathbb{U}$. Then we say that the function $f$ is *subordinate* to $g$, if there exists a Schwartz function $\omega$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))$. We denote this subordination by $f \prec g$, or $f(z) \prec g(z)$. In particular, if the function $g$ is univalent in $\mathbb{U}$, the above subordination is equivalent to (see [3, 9]) $f(0) = g(0) \text{and} f(\mathbb{U}) \subset g(\mathbb{U})$. Also, Jackson [5, 6] $q$-derivative and $q$-integral of a function $f \in \mathcal{A}$ and $0 < q < 1$ defined on a subset of $\mathbb{C}$ are, respectively, given by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ z \neq 0, \tag{2}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (2), we have $D_q f(z) = 1 + \sum\limits_{k=2}^{\infty} [k]_q a_k z^{k-1}$, where $[k]_q = \frac{1-q^k}{1-q}$. If $q \to 1^{-1}$, $[k]_q \to k$. For a function $h(z) = z^k$, we observe that $D_q(h(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}$ and $\lim_{q\to 1} D_q(h(z)) = \lim_{q \to 1}\left([k]_q z^{k-1}\right) = kz^{k-1} = h'(z)$, where $h'$ is the ordinary derivative. As a right inverse, Jackson [5] introduced the $q$-integral

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k z q^k,$$

provided that the series converges. For a function $h(z) = z^k$, we observe that

$$\int_0^z h(t)d_qt = \int_0^z t^k d_qt = \frac{z^{k+1}}{[k+1]_q}, k \neq -1$$

and $\lim_{q\to -1} \int_0^z h(t)d_qt = \lim_{q\to -1} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z h(t)dt$, where $\int_0^z h(t)dt$ is the ordinary integral.

**Definition 1.** Let $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ be a univalent starlike function with respect to 1 which maps the open unit disk $\mathbb{U}$ onto a region in the right half plane which is symmetric with respect to the real axis, and let $B_1 > 0$, $0 \leq \alpha < 1$ and $b \in \mathbb{C} \setminus \{0\}$. The function $f \in S$ is in the class $\mathcal{M}_\alpha(q, b, \phi)$, if

$$1 + \frac{1}{b}\left[(1-\alpha)\left(\frac{zD_qf(z)}{f(z)} - 1\right) + \alpha\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right] \prec \phi(z).$$

We note that

- If $\alpha = 0$, $\lim_{q\to 1^-} \mathcal{M}_0(q, b, \phi) = \mathcal{S}_b(\phi)$ and $\alpha = 1$, $\lim_{q\to 1^-} \mathcal{M}_1(q, b, \phi) = \mathcal{C}_b(\phi)$, [17].
- If $\alpha = 0$, $\lim_{q\to 1^-} \mathcal{M}_0(q, 1, \phi) = \mathcal{S}(\phi)$ and $\alpha = 1$, $\lim_{q\to 1^-} \mathcal{M}_1(q, 1, \phi) = \mathcal{C}(\phi)$, [8].
- If $\alpha = 0$ $\lim_{q\to 1^-} \mathcal{M}_0\left(q, b, \frac{1+(1-2\eta)z}{1-z}\right) = \mathcal{S}^*_\eta(b)$ and $\alpha = 1$,

  $$\lim_{q\to 1^-} \mathcal{M}_1\left(q, b, \frac{1+(1-2\eta)z}{1-z}\right) = \mathcal{C}_\eta(b), (b \in \mathbb{C} \setminus 0, 0 \leq \eta \leq 1),$$

  [4].
- If $\alpha = 0$, $\lim_{q\to 1^-} \mathcal{M}_0\left(q, b, \frac{1+z}{1-z}\right) = \mathcal{S}^*(b)$, [11].
- If $\alpha = 1$, $\lim_{q\to 1^-} \mathcal{M}_1\left(q, b, \frac{1+z}{1-z}\right) = \mathcal{C}(b)$, [11].
- If $\alpha = 0$, $\lim_{q\to 1^-} \mathcal{M}_0\left(q, 1-\eta, \frac{1+z}{1-z}\right) = \mathcal{S}^*(\eta)$ and $\alpha = 1$,

  $$\lim_{q\to 1^-} \mathcal{M}_1\left(q, 1-\eta, \frac{1+z}{1-z}\right) = \mathcal{C}(\eta),$$

  $(0 \leq \eta \leq 1)$, [18].

- If $\alpha = 0$, $\lim_{q \to 1^-} \mathcal{M}_0 \left(q,\, be^{-i\theta} cos\theta,\, \frac{1+z}{1-z}\right) = \mathcal{S}^\theta(b)$ and $\alpha = 1$,
$$\lim_{q \to 1^-} \mathcal{M}_1 \left(q,\, be^{-i\theta} cos\theta,\, \frac{1+z}{1-z}\right) = \mathcal{C}^\theta(b, (\mid \theta \mid \leq \pi/2, b \in \mathbb{C} \setminus 0),$$
[1].

In the present paper, we obtain the Fekete-Szegö inequality for functions in a more general class $\mathcal{M}_\alpha(q,\, b,\, \phi)$ of functions which we define above. Also we give applications of our results to certain functions defined through convolution (or the Hadamard product) and in particular we consider a class $\mathcal{M}_\alpha^\rho(q,\, b,\, \phi)$ of functions defined by fractional derivatives. The motivation of this paper is to generalize the results obtained by Seoudy and Aouf [19]. In order to derive the main result the following lemmas are required.

**Lemma 2.** ([8]) *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with a positive real part in $\mathbb{U}$, then*
$$|c_2 - \mu c_1^2| \leq \begin{cases} -4\mu + 2 & \text{if } \mu \leq 0, \\ 2 & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 2 & \text{if } \mu \geq 1. \end{cases}$$

*When $\mu < 0$ or $\mu > 1$, the equality holds if and only if $p(z)$ is $\dfrac{1+z}{1-z}$ or one of its rotations. If $0 < \mu < 1$, then the equality holds if and only if $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. Equality holds if and only if*
$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\beta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\beta\right)\frac{1-z}{1+z} \quad (0 \leq \beta \leq 1),$$
*or one of its rotations. If $\mu$= 1, the equality holds if and only if $p(z)$ is the reciprocal of one of the function such that the equality holds in the case of $\mu$ = 0. Also the above upper bound is sharp, it can be improved as follows when $0 < \mu < 1$: $|c_2 - \mu c_1^2| + (\mu)|c_1|^2 \leq 2 (0 < \mu \leq 1/2)$ and $|c_2 - \mu c_1^2| + (1-\mu)|c_1|^2 \leq 2 (1/2 < \mu \leq 1)$.*

**Lemma 3.** ([8]0 *If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with a positive real part in $\mathbb{U}$ then for any complex number $\nu$, then $\left|c_2 - \nu c_1^2\right| \leq \max\{1; |2\nu - 1|\}$. The result is sharp for the functions $p(z) = \dfrac{1+z^2}{1-z^2}$ and $p(z) = \dfrac{1+z}{1-z}$.*

## 2. Main Results

Applying Lemma 2, we first derive the following result.

**Theorem 4.** *Let $0 < q < 1, b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If f(z) given by (1) belongs to $\mathcal{M}_\alpha(q, b, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1|b|}{\alpha_1}\left(\frac{B_2}{B_1} - \frac{\alpha_2+\mu\alpha_1}{\alpha_3}B_1 b\right) & \text{if} \quad \mu \leq \sigma_1, \\ \frac{B_1|b|}{\alpha_1} & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{-B_1|b|}{\alpha_1}\left(\frac{B_2}{B_1} - \frac{\alpha_2+\mu\alpha_1}{\alpha_3}B_1 b\right) & \text{if} \quad \mu \geq \sigma_2. \end{cases}$$

*Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then*

$$|a_3 - \mu a_2^2| + \frac{\alpha_3}{B_1^2 b \alpha_1}\left[B_1 - B_2 + \frac{B_1^2 b}{\alpha_3}(\alpha_2 + \mu\alpha_1)\right] | a_2^2 | \leq \frac{B_1 b}{\alpha_1}.$$

*Further, if $\sigma_3 \leq \mu \leq \sigma_2$, then*

$$|a_3 - \mu a_2^2| + \frac{\alpha_3}{B_1^2 b \alpha_1}\left[B_2 + B_1 - \frac{B_1^2 b}{\alpha_3}(\alpha_2 + \mu\alpha_1)\right] | a_2^2 | \leq \frac{B_1 b}{\alpha_1},$$

*where*

$$\sigma_1 = \frac{(B_2 - B_1)\alpha_3 - \alpha_2 B_1^2 b}{B_1^2 b \alpha_1},$$
$$\sigma_2 = \frac{(B_2 + B_1)\alpha_3 - \alpha_2 B_1^2 b}{B_1^2 b \alpha_1}$$

*and*

$$\sigma_3 = \frac{B_2 \alpha_3 - \alpha_2 B_1^2 b}{B_1^2 b \alpha_1},$$

*and assume that*

$$\alpha_1 = ([2]_q[3]_q - [3]_q + 1)\,\alpha + [3]_q - 1, \tag{3}$$
$$\alpha_2 = \left([2]_q - [2]_q^2 - 1\right)\alpha - [2]_q + 1 \tag{4}$$

*and*

$$\alpha_3 = ([2]_q + \alpha - 1)^2. \tag{5}$$

*The results are sharp.*

*Proof.* If $f \in \mathcal{M}_\alpha(q,\, b,\, \phi)$, then by the Definition 1, we have

$$1+\frac{1}{b}\left[(1-\alpha)\left(\frac{zD_qf(z)}{f(z)}-1\right)+\alpha\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right]\prec\phi(z).$$

By the subordination principle, there exists a Schwartz function $\omega(z)$ analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ in $\mathbb{U}$ such that

$$1+\frac{1}{b}\left[(1-\alpha)\left(\frac{zD_qf(z)}{f(z)}-1\right)+\alpha\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right]=\phi(\omega(z)).$$

Let a function $p(z)$ be defined by

$$\begin{aligned} p(z) &= 1+\frac{1}{b}\left[(1-\alpha)\left(\frac{zD_qf(z)}{f(z)}-1\right)+\alpha\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right)\right] \\ &= 1+b_1z+b_2z^2+\cdots. \end{aligned}$$

Hence $b_1=\frac{([2]_q+\alpha-1)}{b}a_2$, and

$$\begin{aligned} b_2=\frac{1}{b}\{&[([2]_q[3]_q-[3]_q+1)\,\alpha+[3]_q-1]\,a_3 \\ &+\left[\left([2]_q-[2]_q^2-1\right)\alpha-[2]_q+1\right]a_2^2\}. \end{aligned}$$

Since $\phi(z)$ is univalent and $p\prec\phi$, then the function $p_1(z)=\frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))}=1+c_1z+c_2z^2+\cdots$ is analytic and has positive real part in $U$. Thus we have

$$p(z)=\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{6}$$

and from the equation (6)

$$b_2=\frac{1}{2}B_1\left(c_2-\frac{1}{2}c_1^2\right)+\frac{1}{4}B_2c_1^2,$$

and

$$b_1=\frac{1}{2}B_1c_1.$$

Hence, we have $a_2=\frac{bB_1c_1}{2([2]_q+\alpha-1)}$ and

$$\begin{aligned} a_3=&\frac{bB_1}{2\left[([2]_q[3]_q-[3]_q+1)\,\alpha+[3]_q-1\right]} \\ &\times\left[c_2-\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{\left([2]_q-[2]_q^2-1\right)\alpha-[2]_q+1}{([2]_q+\alpha-1)^2}B_1b\right)c_1^2\right]. \end{aligned}$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{bB_1}{2\left[\left([2]_q[3]_q - [3]_q + 1\right)\alpha + [3]_q - 1\right]}\left(c_2 - \nu c_1^2\right), \tag{7}$$

where

$$\nu = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{\alpha_2 + \alpha_1\mu}{\alpha_3}B_1 b\right). \tag{8}$$

Theorem 4 follows now by an application of Lemma 2. To show that these bounds are sharp, we define the functions $\mathcal{K}_{\phi n}(n = 2, 3, 4, ...)$ by

$$1 + \frac{1}{b}\left[(1-\alpha)\left(\frac{zD_q\mathcal{K}_{\phi n}(z)}{f(z\mathcal{K}_{\phi n}(z))} - 1\right) + \alpha\left(\frac{D_q(zD_q\mathcal{K}_{\phi n}(z))}{D_q\mathcal{K}_{\phi n}(z)}\right)\right] = \phi(z^{n-1})$$

with $\mathcal{K}_{\phi n}(0) = 0 = \mathcal{K}'_{\phi n}(0) - 1$.

The functions $\mathcal{F}_\delta$ and $\mathcal{G}_\delta(0 \leq \delta \leq 1)$ are defined by

$$1 + \frac{1}{b}\left[(1-\alpha)\left(\frac{zD_q\mathcal{F}_\delta(z)}{\mathcal{F}_\delta(z)} - 1\right) + \alpha\left(\frac{D_q(zD_q\mathcal{F}_\delta(z))}{D_q\mathcal{F}_\delta(z)}\right)\right] = \phi\left(\frac{z(z+\delta)}{1+\delta z}\right) \quad \text{with} \mathcal{F}_\delta(0) = 0 = \mathcal{F}'_\delta(0) - 1 \quad \text{and}$$

$$1 + \frac{1}{b}\left[(1-\alpha)\left(\frac{zD_q\mathcal{G}_\delta(z)}{\mathcal{G}_\delta(z)} - 1\right) + \alpha\left(\frac{D_q(zD_q\mathcal{G}_\delta(z))}{D_q\mathcal{G}_\delta(z)}\right)\right] = \phi\left(-\frac{1+\delta z}{z(z+\delta)}\right) \quad \text{with} \quad \mathcal{G}_\delta(0) = 0 = \mathcal{G}'_\delta(0) - 1.$$

Clearly, the functions $\mathcal{K}_{\phi n}, \mathcal{F}_\delta$ and $\mathcal{G}_\delta$ are in $\mathcal{M}_\alpha(q, b, \phi)$. Also we write $\mathcal{K}_\phi =: \mathcal{K}_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is $\mathcal{K}_\phi$, or one of its rotations. When $\sigma_1 \leq \mu \leq \sigma_2$, then the equality holds if and only if $f$ is $\mathcal{K}_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f$ is $\mathcal{F}_\delta$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f$ is $\mathcal{G}_\delta$ or one of its rotations. □

Next we derive the following result.

**Theorem 5.** *Let $0 < q < 1, b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 \neq 0$. If f(z) given by (1) belongs to $\mathcal{M}_\alpha(q, b, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \frac{B_1 \mid b \mid}{\alpha_1} \max\left\{1; \frac{B_2}{B_1} + \left(\frac{\alpha_1 - \alpha_2}{\alpha_3}\mu\right) B_1 b\right\},$$

*where $\alpha_1, \alpha_2$ and $\alpha_3$ are defined by (3), (4) and (5). The result is sharp.*

*Proof.* Our result now follows by an application of Lemma 3. The result is sharp for the functions $\frac{zD_qf(z)}{f(z)} = \phi(z^2)$ and $\frac{zD_qf(z)}{f(z)} = \phi(z)$. This completes the proof of Theorem 5. ☐

For the special case $b = 1$ and $\lim_{q\to 1^-}$, the function $\mathcal{M}_\alpha(q, b, \phi)$ given by Definition 1 has the form respectively:

$$\mathcal{M}_\alpha(q, b, \phi) := \mathcal{M}_\alpha(q, 1, \phi) \tag{9}$$

and

$$\mathcal{M}_\alpha(q, b, \phi) := \mathcal{M}_\alpha(b, \phi). \tag{10}$$

Consequently, from Theorems 4 and 5, we can deduce respectively the following corollaries which represent the sharp upper bound for the function defined above.

**Corollary 6.** *Let $0 < q < 1$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \geq 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha(q, 1, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{\alpha_1}\left(\frac{B_2}{B_1} - \frac{\alpha_2+\mu\alpha_1}{\alpha_3}B_1\right) & \text{if} \quad \mu \leq \beta_1, \\ \frac{B_1}{\alpha_1} & \text{if} \quad \beta_1 \leq \mu \leq \beta_2, \\ \frac{-B_1}{\alpha_1}\left(\frac{B_2}{B_1} - \frac{\alpha_2+\mu\alpha_1}{\alpha_3}B_1\right) & \text{if} \quad \mu \geq \beta_2, \end{cases}$$

*where $\alpha_1, \alpha_2$ and $\alpha_3$ are given by (3), (4) and (5) and*

$$\beta_1 = \frac{(B_2 - B_1)\,\alpha_3 - \alpha_2 B_1^2}{B_1^2 \alpha_1},$$

$$\beta_2 = \frac{(B_2 + B_1)\,\alpha_3 - \alpha_2 B_1^2}{B_1^2 \alpha_1}$$

*and*

$$\beta_3 = \frac{B_2\alpha_3 - \alpha_2 B_1^2}{B_1^2 \alpha_1}.$$

*Further, if $\beta_1 \leq \mu \leq \beta_3$, then*

$$|a_3 - \mu a_2^2| + \frac{\alpha_3}{B_1^2\alpha_1}\left[B_2 - B_1 + \frac{B_1^2}{\alpha_3}(\alpha_2 + \mu\alpha_1)\right] \leq \frac{B_1}{\alpha_1}.$$

*Further, if $\beta_3 \le \mu \le \beta_2$, then*

$$|a_3 - \mu a_2^2| + \frac{\alpha_3}{B_1^2\alpha_1}\left[B_2 + B_1 - \frac{B_1^2}{\alpha_3}(\alpha_2 + \mu\alpha_1)\right] \le \frac{B_1}{\alpha_1}.$$

*The results are sharp.*

**Corollary 7.** *Let $0 < q < 1$ and $b \in \mathbb{C}\backslash\{0\}$ and $\phi(z) = 1+B_1z+B_2z^2+\cdots$ with $B_1 \neq 0$. If f(z) given by (1) belongs to $\mathcal{M}_\alpha(q, 1, \phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{B_1}{\alpha_1}\max\left\{1; \frac{B_2}{B_1} + \left(\frac{\alpha_1 - \alpha_2}{\alpha_3}\mu\right)B_1\right\},$$

*where $\alpha_1, \alpha_2$ and $\alpha_3$ are denoted by (3), (4) and (5). The result is sharp.*

**Corollary 8.** *Let $b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1z + B_2z^2 + \cdots$ with $B_1 > 0$. If f(z) given by (1) belongs to $\mathcal{M}_\alpha(b, \phi)$ with $b > 0$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{B_1|b|}{2(1+\alpha)}\left(\frac{B_2}{B_1} - \gamma_4 B_1 b\right) & \text{if } \mu \le \gamma_1, \\ \frac{B_1|b|}{2(1+\alpha)} & \text{if } \gamma_1 \le \mu \le \gamma_2, \\ -\frac{B_1|b|}{2(1+\alpha)}\left(\frac{B_2}{B_1} - \gamma_4 B_1 b\right) & \text{if } \mu \ge \gamma_2, \end{cases}$$

*where*

$$\gamma_1 = \frac{(B_2 - B_1)(1+\alpha)^2 + (3\alpha+1)B_1^2 b}{2B_1^2 b(1+\alpha)},$$
$$\gamma_2 = \frac{(B_2 + B_1)(1+\alpha)^2 + (3\alpha+1)B_1^2 b}{2B_1^2 b(1+\alpha)},$$
$$\gamma_3 = \frac{B_2(1+\alpha)^2 + (3\alpha+1)B_1^2 b}{2B_1^2 b(1+\alpha)}$$

*and*

$$\gamma_4 = \frac{2\mu(1+\alpha) - (3\alpha+1)}{(1+\alpha)^2}. \tag{11}$$

*Further, if $\gamma_1 \le \mu \le \gamma_3$, then*

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)}{2B_1^2 b}\left[B_1 - B_2 + \gamma_4 B_1^2 b\right] \mid a_2^2 \mid \le \frac{B_1 b}{2(1+\alpha)}.$$

*Further, if $\gamma_3 \le \mu \le \gamma_2$, then*

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)}{2B_1^2 b}\left[B_1 + B_2 + \gamma_4 B_1^2 b\right] \mid a_2^2 \mid \le \frac{B_1 b}{2(1+\alpha)}.$$

*Each of these results are sharp.*

Further, we have the following corollary.

**Corollary 9.** *Let $b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 \neq 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha(b, \phi)$, then*

$$|a_3 - \mu a_2^2| \le \frac{B_1 \mid b \mid}{2(1+\alpha)} \max\left\{1; \frac{B_2}{B_1} + \left(\frac{3\alpha+1}{(1+\alpha)^2} + \frac{2}{(1+\alpha)}\mu\right) B_1 b\right\}.$$

*The result is sharp.*

For the special case $b = 1$ in Corollaries 8 and 9, the function $\mathcal{M}_\alpha(b, \phi)$ given by (10) has the form $\mathcal{M}_\alpha(1, \phi)$ hence, we can deduce the following results.

**Example 10.** Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha(1, \phi)$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{B_1}{2(1+\alpha)}\left(\frac{B_2}{B_1} - \gamma_4 B_1\right) & \text{if} \quad \mu \le \delta_1, \\ \frac{B_1}{2(1+\alpha)} & \text{if} \quad \delta_1 \le \mu \le \delta_2, \\ -\frac{B_1}{2(1+\alpha)}\left(\frac{B_2}{B_1} - \gamma_4 B_1\right) & \text{if} \quad \mu \ge \delta_2, \end{cases}$$

where $\gamma_4$ is given by (11) and

$$\delta_1 = \frac{(B_2 - B_1)(1+\alpha)^2 + (3\alpha+1)B_1^2}{2B_1^2(1+\alpha)},$$

$$\delta_2 = \frac{(B_2 + B_1)(1+\alpha)^2 + (3\alpha+1)B_1^2}{2B_1^2(1+\alpha)}$$

and

$$\delta_3 = \frac{B_2(1+\alpha)^2 + (3\alpha+1)B_1^2}{2B_1^2(1+\alpha)}.$$

Further, if $\delta_1 \leq \mu \leq \delta_3$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)}{2B_1^2}\left[B_1 - B_2 + \gamma_4 B_1^2\right] \mid a_2^2 \mid \leq \frac{B_1}{2(1+\alpha)}.$$

Further, if $\delta_3 \leq \mu \leq \delta_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)}{2B_1^2}\left[B_1 + B_2 + \gamma_4 B_1^2\right] \mid a_2^2 \mid \leq \frac{B_1}{2(1+\alpha)}.$$

The results are sharp.

**Example 11.** Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 \neq 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha(1, \phi)$, then

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(1+\alpha)} \max\left\{1; \frac{B_2}{B_1} + \left(\frac{3\alpha+1}{(1+\alpha)^2} + \frac{2}{(1+\alpha)}\mu\right) B_1\right\}.$$

**Remark 12.** For the special case $\alpha = 0$, Theorems 4 and 5 represent results, similar to these obtained by Seoudy and Aouf [19, Theorem 3 and Theorem 1].

**Remark 13.** For the special case $\alpha = 1$, Theorems 4 and 5 represent results, similar to these obtained by Seoudy and Aouf [19, Theorem 4 and Theorem 2].

**Remark 14.** For $\alpha = 0$and $q \to 1^-$, Theorem 5 provides similar results to those recently obtained by Ravichandran et al. [17, Theorem 4.1].

**Remark 15.** For $\alpha = 0$and $q \to 1^-$, Theorem 4 gives similar results to those recently obtained by Seoudy and Aouf [19, Corollary 3].

**Remark 16.** For $\alpha = 1$ and $q \to 1^-$, Theorems 4 and 5 correspond to the results recently obtained by Seoudy and Aouf [19, Corollary 2 and Corollary 4].

**Remark 17.** For $\alpha = 0, q \to 1^-$ and $\alpha = 1, q \to 1^-$, Theorems 4 and 5 correspond to the results obtained by Ma and Minda [8] for the known classes of starlike and convex functions which were mentioned as $\mathcal{S}(\phi)$ and $\mathcal{C}(\phi)$ respectively.

## 3. Applications to Analytic Functions Defined by using Fractional Calculus Operators and Convolution

**Definition 18.** Let $f(z)$ be analytic in a simply connected region of the $z$-plane containing the origin. The fractional derivative for $f(z)$ of order $\rho$ is defined by

$$\mathcal{D}_z^{\rho} f(z) = \frac{1}{\Gamma(1-\rho)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\rho}} d\zeta \quad (0 \leq \rho < 1),$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\rho}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$.

With the help of Definition 18 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivatsava [12] introduced the operator $\Omega^{\rho} : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned}(\Omega^{\rho} f)(z) &= \Gamma(2-\rho) z^{\rho} \mathcal{D}_z^{\rho} f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\rho)}{\Gamma(k+1-\rho)} a_k z^k, \quad (\rho \neq 2, 3, 4 \ldots).\end{aligned} \tag{12}$$

The class $\mathcal{M}_{\alpha}^{\rho}(q, b, \phi)$ consists of functions $f \in \mathcal{A}$ for which $\Omega^{\rho} f \in \mathcal{M}_{\alpha}(q, b, \phi)$, and note that the class $\mathcal{M}_{\alpha}^{\rho}(q, b, \phi)$ is a special case of the class $\mathcal{M}_{\alpha}^{g}(q, b, \phi)$ when

$$g(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\rho)}{\Gamma(k+1-\rho)} z^k. \tag{13}$$

For $f(z)$ given by (1) and $g(z)$ given by $g(z) = z + \sum_{k=2}^{\infty} g_k z^k$, the convolution of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k g_k z^k.$$

Since $f \in \mathcal{M}_{\alpha}^{g}(q,\, b,\, \phi)$ if and only if $f * g \in \mathcal{M}\alpha(q,\, b,\, \phi)$, we obtain the coefficient estimates for functions in the class $\mathcal{M}_{\alpha}^{g}(q,\, b,\, \phi)$, from the corresponding estimates for functions in the class $\mathcal{M}_{\alpha}(q,\, b,\, \phi)$.

Applying Theorem 4 for the function $(f * g)(z)$, we get the following theorem after choosing the suitable parameter $\mu$:

**Theorem 19.** *Let $0 < q < 1, b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If f(z) given by (1) belongs to $\mathcal{M}_{\alpha}^{g}(q,\, b,\, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1|b|}{g_3\alpha_1}\left(\frac{B_2}{B_1} - \eta_3 B_1 b\right) & \text{if} \quad \mu \leq \eta_1, \\ \frac{B_1|b|}{g_3\alpha_1} & \text{if} \quad \eta_1 \leq \mu \leq \eta_2, \\ \frac{-B_1|b|}{g_3\alpha_1}\left(\frac{B_2}{B_1} - \eta_3 B_1 b\right) & \text{if} \quad \mu \geq \eta_2, \end{cases}$$

*where $\alpha_1, \alpha_2$ and $\alpha_3$ are denoted by (3), (4) and (5), and*

$$\begin{aligned} \eta_1 &= \frac{(B_2 - B_1)\,\alpha_3 - \alpha_2 B_1^2 b}{B_1^2 g_3 b \alpha_1} g_2^2, \\ \eta_2 &= \frac{(B_2 + B_1)\,\alpha_3 - \alpha_2 B_1^2 b}{B_1^2 g_3 b \alpha_1} g_2^2 \end{aligned}$$

*and*

$$\eta_3 = \frac{\alpha_2 + \mu\alpha_1}{\alpha_3 g_2^2} g_3.$$

*The result is sharp.*

When $g$ corresponds to the Owa-Srivastava operator given in (12), we have

$$g_2 := \frac{\Gamma(3)\Gamma(2-\rho)}{\Gamma(3-\rho)} = \frac{2}{2-\rho} \tag{14}$$

and

$$g_3 := \frac{\Gamma(4)\Gamma(2-\rho)}{\Gamma(4-\rho)} = \frac{6}{(2-\rho)(3-\rho)}. \tag{15}$$

Combining (14) and (15), Theorem 19 is reduced as follows.

**Theorem 20.** *Let $0 < q < 1, b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If f(z) given by (1) belongs to $\mathcal{M}_\alpha^g(q,\, b,\, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\rho)(3-\rho)B_1|b|}{6\alpha_1}\left(\frac{B_2}{B_1} - \lambda_3 B_1 b\right) & \text{if} \quad \mu \leq \lambda_1, \\ \frac{(2-\rho)(3-\rho)B_1|b|}{6\alpha_1} & \text{if} \quad \lambda_1 \leq \mu \leq \lambda_2, \\ \frac{-(2-\rho)(3-\rho)B_1|b|}{6\alpha_1}\left(\frac{B_2}{B_1} - \lambda_3 B_1 b\right) & \text{if} \quad \mu \geq \lambda_2, \end{cases}$$

*where $\alpha_1, \alpha_2$ and $\alpha_3$ are represented by (3), (4) and (5), and*

$$\lambda_1 = \left(\frac{2(3-\rho)}{3(2-\rho)}\right)\frac{(B_2 - B_1)\,\alpha_3 - \alpha_2 B_1^2 b}{B_1^2 b \alpha_1},$$

$$\lambda_2 = \left(\frac{2(3-\rho)}{3(2-\rho)}\right)\frac{(B_2 + B_1)\,\alpha_3 - \alpha_2 B_1^2 b}{B_1^2 b \alpha_1}$$

*and*

$$\lambda_3 = \left(\frac{2(3-\rho)}{3(2-\rho)}\right)\frac{\alpha_2 + \mu\alpha_1}{\alpha_3}. \tag{16}$$

*The result is sharp.*

For the special case $b = 1$ and $\lim_{q \to 1^-}$, the function $\mathcal{M}_\alpha(q,\, b,\, \phi)$ given by Definition 1 has the form respectively denoted by (9) and (10). Consequently, from Theorems 19 and 20, we can deduce respectively the following corollaries which gave the sharp upper bound for the function mentioned above.

**Corollary 21.** *Let $0 < q < 1$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If f(z) given by (1) belongs to $\mathcal{M}_\alpha^g(q,\, 1,\, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{g_3\alpha_1}\left(\frac{B_2}{B_1} - \frac{\alpha_2 + \mu\alpha_1}{\alpha_3 g_2^2} B_1 g_3\right) & \text{if} \quad \mu \leq \mu_1, \\ \frac{B_1}{g_3\alpha_1} & \text{if} \quad \mu_1 \leq \mu \leq \mu_2, \\ \frac{B_1}{g_3\alpha_1}\left(\frac{B_2}{B_1} - \frac{\alpha_2 + \mu\alpha_1}{\alpha_3 g_2^2} B_1 g_3\right) & \text{if} \quad \mu \geq \mu_2, \end{cases}$$

*where $\alpha_1, \alpha_2$ and $\alpha_3$ are defined by (3), (4) and (5), and*

$$\mu_1 = \frac{(B_2 - B_1)\,\alpha_3 - \alpha_2 B_1^2}{B_1^2 g_3 \alpha_1} g_2^2$$

*and*

$$\mu_2 = \frac{(B_2 + B_1)\,\alpha_3 - \alpha_2 B_1^2}{B_1^2 g_3 \alpha_1} g_2^2.$$

*The result is sharp.*

**Corollary 22.** *Let $0 < q < 1$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha^g(q, 1, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\rho)(3-\rho)B_1}{6\alpha_1}\left(\frac{B_2}{B_1} - \lambda_3 B_1\right) & \text{if} \quad \mu \leq \zeta_1, \\ \frac{(2-\rho)(3-\rho)B_1}{6\alpha_1} & \text{if} \quad \zeta_1 \leq \mu \leq \zeta_2, \\ \frac{-(2-\rho)(3-\rho)B_1}{6\alpha_1}\left(\frac{B_2}{B_1} - \lambda_3 B_1\right) & \text{if} \quad \mu \geq \zeta_2, \end{cases}$$

*where $\alpha_1, \alpha_2$, $\alpha_3$ and $\lambda_3$ are represented by (3), (4), (5), and (16), and*

$$\zeta_1 = \left(\frac{2(3-\rho)}{3(2-\rho)}\right) \frac{(B_2 - B_1)\,\alpha_3 - \alpha_2 B_1^2}{B_1^2 b \alpha_1}$$

*and*

$$\zeta_2 = \left(\frac{2(3-\rho)}{3(2-\rho)}\right) \frac{(B_2 + B_1)\,\alpha_3 - \alpha_2 B_1^2}{B_1^2 b \alpha_1}.$$

*The result is sharp.*

**Corollary 23.** *Let $b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha(b, \phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1|b|}{2g_3(1+\alpha)}\left(\frac{B_2}{B_1} - \frac{\gamma_4}{g_2^2} B_1 b g_3\right) & \text{if} \quad \mu \leq \nu_1, \\ \frac{B_1|b|}{2g_3(1+\alpha)} & \text{if} \quad \nu_1 \leq \mu \leq \nu_2, \\ -\frac{B_1|b|}{2g_3(1+\alpha)}\left(\frac{B_2}{B_1} - \frac{\gamma_4}{g_2^2} B_1 b g_3\right) & \text{if} \quad \mu \geq \nu_2, \end{cases}$$

*where $\gamma_4$ is given by (11) and*

$$\nu_1 = \frac{(B_2 - B_1)\,(1+\alpha)^2 + (3\alpha+1) B_1^2 b}{2 g_3 B_1^2 b (1+\alpha)} g_2^2$$

*and*

$$\nu_2 = \frac{(B_2 + B_1)\,(1+\alpha)^2 + (3\alpha+1) B_1^2 b}{2 g_3 B_1^2 b (1+\alpha)} g_2^2.$$

**Corollary 24.** *Let $b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha^g(b, \phi)$, then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \rho_3 \left(\frac{B_2}{B_1} - \frac{3(2-\rho)\gamma_4}{2(3-\rho)} B_1 b\right) & \text{if} \quad \mu \le \rho_1, \\ \rho_3 & \text{if} \quad \rho_1 \le \mu \le \rho_2, \\ -\rho_3 \left(\frac{B_2}{B_1} - \frac{3(2-\rho)\gamma_4}{2(3-\rho)} B_1 b\right) & \text{if} \quad \mu \ge \rho_2, \end{cases}$$

*where $\gamma_4$ is given by (11) and*

$$\rho_1 = \left(\frac{2(3-\rho)}{3(2-\rho)}\right) \frac{(B_2 - B_1)(1+\alpha)^2 + 2(3\alpha+1)B_1^2 b}{B_1^2 b(1+\alpha)},$$

$$\rho_2 = \left(\frac{2(3-\rho)}{3(2-\rho)}\right) \frac{(B_2 + B_1)(1+\alpha)^2 + 2(3\alpha+1)B_1^2 b}{B_1^2 b(1+\alpha)}$$

*and*

$$\rho_3 = \frac{(2-\rho)(3-\rho)B_1 \mid b \mid}{12(1+\alpha)}$$

*The result is sharp.*

For the special case $b = 1$ in Corollaries 23 and 24, we can deduce the following results.

**Example 25.** Let $b \in \mathbb{C} \setminus \{0\}$ and $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$ and $B_2 \ge 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha(1, \phi)$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{(2-\rho)(3-\rho)B_1}{12(1+\alpha)} \left(\frac{B_2}{B_1} - \frac{3(2-\rho)\gamma_4}{2(3-\rho)} B_1\right) & \text{if} \quad \mu \le \tau_1, \\ \frac{(2-\rho)(3-\rho)B_1}{12(1+\alpha)} & \text{if} \quad \tau_1 \le \mu \le \tau_2, \\ \frac{-(2-\rho)(3-\rho)B_1}{12(1+\alpha)} \left(\frac{B_2}{B_1} - \frac{3(2-\rho)\gamma_4}{2(3-\rho)} B_1\right) & \text{if} \quad \mu \ge \tau_2, \end{cases}$$

where $\gamma_4$ is given by (11) and

$$\tau_1 = \frac{(B_2 - B_1)(1+\alpha)^2 + (3\alpha+1)B_1^2}{2g_3 B_1^2(1+\alpha)} g_2^2$$

and

$$\tau_2 = \frac{(B_2 + B_1)(1+\alpha)^2 + (3\alpha+1)B_1^2}{2g_3 B_1^2(1+\alpha)} g_2^2.$$

The results are sharp.

**Example 26.** Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ with $B_1 > 0$. If $f(z)$ given by (1) belongs to $\mathcal{M}_\alpha^g(1, \phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\rho)(3-\rho)B_1}{12(1+\alpha)} \left( \frac{B_2}{B_1} - \frac{3(2-\rho)}{2(3-\rho)} \gamma_4 B_1 \right) & \text{if} \quad \mu \leq \omega_1, \\ \frac{(2-\rho)(3-\rho)B_1}{12(1+\alpha)} & \text{if} \quad \omega_1 \leq \mu \leq \omega_2, \\ \frac{-(2-\rho)(3-\rho)B_1}{12(1+\alpha)} \left( \frac{B_2}{B_1} - \frac{3(2-\rho)}{2(3-\rho)} \gamma_4 B_1 \right) & \text{if} \quad \mu \geq \omega_2, \end{cases}$$

where $\gamma_4$ is given by (11) and

$$\omega_1 = \left( \frac{2(3-\rho)}{3(2-\rho)} \right) \frac{(B_2 - B_1)(1+\alpha)^2 + 2(3\alpha+1)B_1^2}{B_1^2(1+\alpha)}$$

and

$$\omega_2 = \left( \frac{2(3-\rho)}{3(2-\rho)} \right) \frac{(B_2 + B_1)(1+\alpha)^2 + 2(3\alpha+1)B_1^2}{B_1^2 b(1+\alpha)}.$$

The result is sharp.

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