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APPLICATION OF MVIM TO ANALYSIS AN ILL-POSED PROBLEM

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Abstract: In this paper, we consider the inverse heat conduction problem (IHCP) with a single sensor location inside the heat conduction body and solve it by applying modified variational iteration method (MVIM).

Although the IHCP is ill-posed problem, we can find the numerical-analytic solution of the problem using the MVIM. This method has an excellent estimation of the solution compared with finite difference method even x vanishing. The approximations of the temperature and the heat flux at $x = 0$ are considered. Moreover, MVIM does not require any discretization, linearization or small perturbation, thus it can be considered as an efficient method to solve this problem. To show the strength of the method, some examples are given.

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Key Words: IHCP, MVIM, ill-posed, stable numerical solution

1. Introduction

The inverse heat conduction problem (IHCP) is defined as the estimation of the surface heat flux history given one or more measured temperature histories inside a heat-conducting body. IHCPs have many applications in various branches of science and engineering.

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In remote sensing, oil exploration, nondestructive evaluation of material and determination of the earth's interior structure. One of the applications may be the determination of the surface heat flux histories of re-centering heat shield [1].

In nature, inverse problems are unstable because the unknown parameters and solutions have to be determined indirectly. So, for approximating the solution in any numerical algorithm serve ill-posedness of the problem. A number of techniques have been proposed for solving the inverse problem, including the finite difference method [2], the method of fundamental solutions [3] and [4] compared with some methods that the initial error has more influence on the error and unstable scheme, the IHCP can be solved by VIM exactly. The VIM was first suggested by Ji-Huah He [5]. This method is based on the use of Lagrange multipliers for the identification of optimal values of parameters in a functional. This method construct a rapidly convergent sequence to the exact solution.

This paper is organized as follows: In Section 2, description of the problem is presented. Numerical procedure is introduced in Section 3. In Section 4, some examples are given. A conclusion of paper is considered in Section 5.

2. Formulation of an IHCP

Consider the inverse heat conduction model:

$$T_t(x,t) = T_{xx}(x,t); (x,t) \in (0,1) \times (0,t_{max}), \tag{1}$$

$$T(x,0) = f(x); 0 < x_1 \leq x \leq 1, \tag{2}$$

$$BT(1,t) = h(t); t \in (0,t_{max}), \tag{3}$$

$$|T(x,t)| \leq C; (x,t) \in (0,1) \times (0,t_{max}), \tag{4}$$

and the overspecified condition:

$$T(x_1,t) = g(t); (x_1,t) \in (0,1) \times (0,t_{max}), \tag{5}$$

where C is a known positive constant, $f(x)$ is piecewise-continuous known function while $h(t)$ and $g(t)$ are infinitely differentiable known functions, the location $x_1 \in (0,1)$ is the sensor assumed to be measured and to have negligible error and t_{max} represents the final time of interest for the time evolution of the problem, B is a boundary operator ($B = \frac{\partial^i}{\partial x^i}; i = 0,1$).

In this model, the surface temperature $T(0,t)$ and heat flux $T_x(0,t)$ are unknown which remain to be determined from semi inter or temperature measurements.

The problem (1)-(5) may be divided into two separated problems.

First, the direct problem is:

$$T_t = T_{xx}; x_1 < x < 1, 0 < t < t_{max}, \tag{6}$$

$$T(x,0) = f(x); x_1 \leq x \leq 1, \tag{7}$$

$$T(x_1,t) = g(t); \quad BT(1,t) = h(t);\ 0 < t < t_{max}, \tag{8}$$

$$|T(x,t)| \leq C; x_1 \leq x \leq 1, 0 < t < t_{max}. \tag{9}$$

This problem may be analyzed as a direct problem of the body from $x = x_1$ to $x = 1$ with known boundary conditions. Existence and uniqueness of the solution of problem (6)-(9) is mentioned in [6]. By solving (6)-(9), can be obtained T on $[x_1, 1]$. So $T_x(x_1,t) = k(t)$ is easily determined.

Then the second problem is the following IHCP:

$$T_t = T_{xx}; 0 < x < x_1, 0 < t < t_{max}, \tag{10}$$

$$T(x_1,t) = g(t);\ T_x(x_1,t) = k(t);\ 0 < t < t_{max}, \tag{11}$$

$$|T(x,t)| \leq C; x_1 \leq x \leq 1, 0 < t < t_{max}. \tag{12}$$

The problem (10)-(12) is a Cauchy problem. It is ill-posed. The solution of the problem (10)-(12) exists and is unique but it is not always stable. To prove that the solution of the problem (10)-(12) is not always stable, we can construct an example of this behavior by considering solutions:

$$T_n(x,t) = n^{-2}(e^{-n(x-x_1)}\cos[n(x_1 - x) + 2n^2t] + (x - x_1)^2 t);\ n = 1, 2, 3, ...$$

These sequences of $T_n(x,t)$ are the solutions of the following problems:

$$\frac{\partial T_n(x,t)}{\partial t} = \frac{\partial^2 T_n(x,t)}{\partial x^2} - n^{-2}[2t - (x - x_1)^2];\ 0 < x < x_1,$$

$$T_n(x_1,t) = n^{-2}\cos 2n^2 t; \quad t > 0,$$

$$\frac{\partial T_n(x_1,t)}{\partial x} = n^{-1}(\sin 2n^2 t - \cos 2n^2 t);\ t > 0.$$

Note that, as n tends to infinity, the data $T_n(x_1,t)$ and $\frac{\partial T_n(x_1,t)}{\partial x}$ for every $t \geq 0$ and $h_n(x,t) = -n^{-2}[2t - (x - x_1)^2]$ for every $t > 0$ and $0 < x < x_1$ tends uniformly to zero.

The functions T_n assume values $n^{-2}e^{-n(x-x_1)}$ for every $x < x_1$ which tends to infinity as n tends to infinity, where as T_n for every $x > x_1$, tends uniformly to zero. Therefore the problem (10)-(12) is unstable. Hence, the problem (1)-(5) is unstable.

3. Description of the Method

The VIM is used for solving a wide range of general non-linear differential equations of the form [7, 8]:

$$Ly(x) + Ny(x) = g(x); \tag{13}$$

where L and N are linear and non-linear operators respectively, $g(x)$ is a known analytical inhomogeneous term and y is a unknown function. According to VIM, we can construct a correction functional as follows [8, 9]:

$$y_{n+1}(x) = y_n(x) + \int_{x_1}^{x} \lambda(\xi, x)[Ly_n(\xi) + N\tilde{y}_n(\xi) - g(\xi)]d\xi, \ n \geq 0,$$

where λ is a general Lagrange multiplier [7, 8, 11], optimally determined by using the variational theory [7, 8, 12], the index n denotes the $n-$th order approximation and $\tilde{y}_n$ is a restricted variation i.e. $\delta\tilde{y}_n = 0$ [7, 9, 12]. Now, we need to determine the Lagrangian multiplier λ. Therefore we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. Then the solution of the differential equation is considered as the fixed point of the following functional under the suitable choice of the initial term $y_0(t)$ [7, 10-12]:

$$y_{n+1}(x) = y_n(x) + \int_{x_1}^{x} \lambda(\xi, x)[Ly_n(\xi) + Ny_n(\xi) - g(\xi)]d\xi, \ n \geq 0. \tag{14}$$

Consequently, the exact solution may be obtained by using the Banach's fixed point theorem [10]:

$$y(x) = \lim_{n\to\infty} y_n(x),$$

because according to the above theorem for the nonlinear mapping:

$$A[y] = y(x) + \int_{x_1}^{x} \lambda(\xi, x)[Ly(\xi) + Ny(\xi) - g(\xi)]d\xi,$$

$$|A[y] - A[w]| \leq \gamma|y - w|, \quad 0 < \gamma < 1, \ \forall y, w \in X,$$

a sufficient condition for the convergence of the VIM is strictly contraction of $A : X \to X$ where X is a Banach's space. Furthermore the sequence (14) i.e. $y_{n+1}(x) = A[y_n(x)]$ converges to the fixed point of A with an arbitrary choice of $y_0 \in X$ which is also the solution of the nonlinear equation (13) (See Theorem 2.3 in [10], pp. 123 and Theorem 1 in [14], pp. 2529).

4. Numerical Results and Discussion

According to He's classical variational iteration scheme and (13), we can construct a correction functional concerning equation (1) in x-direction as follows:

$$T_{n+1}(x,t)=T_n(x,t)+\int_{x_1}^{x}\lambda(x,\xi)(\frac{\partial^2}{\partial\xi^2}T_n(\xi,t)-\frac{\partial}{\partial t}T_n(\xi,t))d\xi=0,\ n\geq 0. \quad (15)$$

To find the optimal value of λ from (15), we have:

$$\delta T_{n+1}(x,t)=\delta T_n(x,t)+\delta\int_{x_1}^{x}\lambda(x,\xi)(\frac{\partial^2}{\partial\xi^2}T_n(\xi,t)-\frac{\partial}{\partial t}T_n(\xi,t))d\xi=0,$$

which after some calculation, we obtain the following stationary conditions:

$$\lambda''(x,\xi)=0,\ \lambda'(x,\xi)|_{\xi=x}=-1,\ \lambda(x,\xi)|_{\xi=x}=0.$$

The Lagrange multiplier can be easily identified:

$$\lambda(x,\xi)=\xi-x. \quad (16)$$

Here, we obtain the following iteration formula:

$$T_{n+1}(x,t)=T_n(x,t)+\int_{x_1}^{x}(\xi-x)(\frac{\partial^2}{\partial\xi^2}T_n(\xi,t)-\frac{\partial}{\partial t}T_n(\xi,t))d\xi=0,\ n\geq 0, \quad (17)$$

where T_0 may be selected as any function that just satisfies, at least the initial or boundary conditions [7-12] but according to the Adomian's decomposition method (ADM) in x-direction which is equivalent to the VIM in x-direction [15], we assume $LT_0(x,t)=0$ or $T_0(x,t)=g(t)+(x-x_1)k(t)$, for simplicity, as the initial approximation [9]. Therefore using (17), according to the Banach's fixed point theorem, we can find the solution of the problem (11)-(14) as a convergent sequence [13, 14]. Also, we can consider T_n as an approximation of the exact solution for sufficiently large values of n. Consequently, from the solution of the problem (11)-(14), we can obtain the approximate solutions $T_x(0,1)$ and $T(0,t)$.

The example has been chosen so that its analytical solution exists. The application of the problem is demonstrated in [2]. The result of the present method is remarkably effective.

Example. Problem (1)-(5) with assumptions:

$$x_1=0.5,\quad f(x)=x^2,\quad B=\frac{\partial}{\partial x},\quad h(t)=2,\quad g(t)=1/4+2t;$$

has the exact solution $T(x,t)=x^2+2t$.

By solving the direct problem (6)-(9), we obtain $k(t) = 1$. Using (17), the recursive relation for problem (10)-(12) is:

$$T_0(x,t) = g(t) + (x - x_1)k(t) = 2t + x - 0.5,$$

$$T_{n+1}(x,t) = T_n(x,t) + \int_{x_1}^{x} (\xi - x)\left(\frac{\partial^2}{\partial \xi^2} T_n(\xi,t) - \frac{\partial}{\partial t} T_n(\xi,t)\right) d\xi = 0,\ n \geq 0.$$

By the above iteration formula and after some simplifications, we obtain the following successive approximations:

$$T_0(x,t) = 2t + x - 0.5$$

$$T_1(x,t) = T_0(x,t) + \int_{x_1}^{x} (\xi - x)(0 - 2)d\xi = T_0(x,t) + (0.5 - x)^2 = 2t + x^2,$$

$$T_2(x,t) = T_1(x,t) + \int_{x_1}^{x} (\xi - x)(2 - 2)d\xi = T_1(x,t),$$

$$\vdots$$

Finally $T_n(x,t) = x^2 + 2t$. Then we can write:

$$T(x,t) = \lim_{n->\infty} T_n(x,t) = x^2 + 2t,$$

that we obtain the exact solution after two iterations.

5. Conclusion

In this paper, the modified variational iteration method (MVIM) is used for finding solution pair of the inverse heat source equation. The modified variational iteration method is show to be a powerful numerical method for the solution of ill-posed problems. Also, the method applied does not require discretization of the region, as in the case of classical methods based on the finite difference method, the boundary element method or the other methods. The simplicity of the method and the obtained results show that this method is effective and simple. It is also shown that this method provides an exact solution for the inverse problem (1)-(5).

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