International Journal of Applied Mathematics

Volume 29 No. 3 2016, 371-380
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v29i3.8

STABILIZED FEM SOLUTION OF VARIABLE COEFFICIENT CONVECTION-DIFFUSION EQUATION

Selçuk Han Aydın

Department of Mathematics
Karadeniz Technical University
61080 Trabzon, TURKEY

Abstract: The present numerical study deals with the stabilized finite element solution of the variable coefficient convection-diffusion equation. Basically, SUPG type stabilization terms is appended to standard Galerkin finite element formulation for the convection dominated and singularly perturbed boundary layer cases. The proposed stabilized finite element method enables to obtain stable solution and avoids oscillations. The algorithm is presented for different benchmark problems considered in 1-D and 2-D cases. Furthermore, the solutions show the accuracy of the proposed method.

AMS Subject Classification: 58D30, 65N30
Key Words: stabilized FEM, variable coefficient convection-diffusion equation

1. Introduction

Convection-diffusion equations which are also considered as linearized versions of the Navier-Stokes equations, are encountered in many physical and engineering applications. Therefore, considerable amount of researchers have been interested in the solution of the convection-diffusion equations with some other numerical methods especially convection dominated cases and singularly perturbed boundary value problems in which numerical instabilities occur in classical finite element method type solution techniques. Therefore, some stabilized finite element should be considered in order to obtain stable solution.

Received: April 7, 2016

Stabilized methods reduce the oscillations and achieves stability either using a suitable mesh [1, 2] or adding mesh-dependent artificial diffusion terms to physical formulation of the problem which are tuned by a stabilization parameter [3]. These terms enhance the coercivity of the formulation by acting like artificial diffusion in the direction of streamlines and enables the usage of coarse mesh. The amount of such addition is very effective in solutions, so the value should be calculated efficiently. Therefore several numerical tests and researches are performed for calculation of the value of this stabilization parameter.

Most of the studies existing in literature are about constant diffusion coefficient cases. Therefore, in this study we examine the variable coefficient cases of the convection-diffusion equations. In the first part, we give standard Galerkin finite element formulation of variable diffusion coefficient of convection-diffusion equation. Then stabilized formulation based on the Streamline-Upwind/Petrov Galerkin (SUPG) [3] of the equation is obtained. Additionally, an alternative stabilization method called stabilizing subgrid method (SSM) [4, 5] which is based on proper choice of the subgrid point inside each element and calculating the stabilization parameter using this subgrid point. The stabilized finite element methods of SUPG and SSM types have similar formulation except for the value of the stabilization parameter. Accuracy of SSM has been already displayed for the solutions of Navier-Stokes equations [6], magnetohydrodynamic equations [7] and natural convection flow equations [8].

The paper is organized as follows: In Section 2, we describe the variable coefficient of the convection-diffusion equation. In Subsection 2.1, we present the standard Galerkin finite element formulation of the equation. The stabilized forms of the equation are given in Subsection 2.2. We demonstrate and compare the methods in Section 3 by presenting some numerical experiments obtained for some selected benchmark problems defined in 1-dimension and 2-dimensional cases. Finally, a brief conclusion is given in Section 4.

2. Mathematical Model

We consider the steady variable coefficient convection-diffusion equation in a non-dimensional form in a bounded domain $\Omega \subset I\!R^2$ with Dirichlet type boundary conditions on a smooth boundary Γ as,

$$\begin{aligned} -\epsilon(\mathbf{x})\Delta u + \boldsymbol{\beta}(\mathbf{x}) \cdot \nabla u &= f(\mathbf{x})\,, && \text{in } \Omega \\ u|_{\partial\Omega} &= u_0\,, && \text{on } \Gamma \end{aligned} \tag{1}$$

where $\mathbf{x} \in I\!R^2$, ∇ and Δ are gradient and Laplace operators, respectively.

Before starting to present the numerical methods, let us set the notation. We use standard notation for function spaces: $L^2(\Omega)$ is the space of square integrable functions over the domain Ω, $L_0^2(\Omega)$ is the space of $L^2(\Omega)$ functions with zero mean over Ω, $H^1(\Omega)$ is the Sobolev space of $L^2(\Omega)$ functions whose derivatives are square integrable functions in Ω, and $H_0^1(\Omega)$ is the Sobolev subspace of $H^1(\Omega)$ functions in Ω with zero value on the boundary $\partial\Omega$.

2.1. The Standard Finite Element Method

Consider the steady variable coefficient convection-diffusion equation in a bounded domain $\Omega \in I\!R^2$ with the Eq. (1) with Dirichlet type boundary conditions. Then the weak formulation of the problem can be stated as: Find $u \in V = H_0^1(\Omega)$ such that

$$a(u,v) = (f,v)\,, \qquad \forall v \in V \tag{2}$$

where

$$a(u,v) = \int_\Omega \left[\epsilon \nabla u \cdot \nabla v + (\nabla \epsilon \cdot \nabla u) v\right] d\Omega + \int_\Omega (\boldsymbol{\beta} \cdot \nabla u) v d\Omega$$

and (f,v) denotes the scaler product of f and v in $L_0^2(\Omega)$.

First of all, let's try to explain the derivation of first integral. It is already known that, using the divergence theorem, one can obtain

$$-\int_\Omega v \Delta u d\Omega = \int_\Omega \nabla v \cdot \nabla u d\Omega - \int_{\partial\Omega} v \frac{\partial u}{\partial n} ds.$$

Similarly, equivalence of the integral

$$\int_\Omega v \epsilon(\mathbf{x}) \Delta u d\Omega$$

is obtained again by using divergence theorem as follows:

$$\begin{aligned}
& \textstyle\int_{\partial\Omega} v(\epsilon(\mathbf{x})\nabla u) ds = \int_\Omega \nabla \cdot (v\epsilon(\mathbf{x})\nabla u) d\Omega \\
= \; & \textstyle\int_\Omega \nabla v \cdot (\epsilon(\mathbf{x})\nabla u) d\Omega + \int_\Omega v \nabla \cdot (\epsilon(\mathbf{x})\nabla u) d\Omega \\
= \; & \textstyle\int_\Omega \nabla v \cdot (\epsilon(\mathbf{x})\nabla u) d\Omega + \int_\Omega v(\nabla\epsilon(\mathbf{x}) \cdot \nabla u + \epsilon(\mathbf{x})\nabla^2 u) d\Omega \\
\Rightarrow \; & \textstyle\int_\Omega v(\epsilon(\mathbf{x})\nabla^2 u) d\Omega \\
= \; & \textstyle\int_{\partial\Omega} v(\epsilon(\mathbf{x})\nabla u) ds - \int_\Omega [\epsilon(\mathbf{x})\nabla v \cdot \nabla u + v(\nabla\epsilon(\mathbf{x}) \cdot \nabla u)] d\Omega.
\end{aligned}$$

Now, in order to introduce a finite element method, we begin by partitioning the domain in to finite elements in a standard way and let Ω_h be such a partition of Ω. Then, the Galerkin finite element formulation of the problems reads; Find $u_h \in V_h$ such

$$a(u_h, v_h) = (f, v_h), \qquad \forall v_h \in V_h \tag{3}$$

where $V_h \subset V$ is finite dimensional subspaces defined on the discretization Ω_h.

It is well known that the exact solution of the problem (1) has a boundary layer for convection dominated cases ($\epsilon \ll 1$). In this case, in order to obtain stable solutions with standard Galerkin finite element, a fine and adaptive mesh should be used which increase the computational cost. Alternatively, some stabilized finite element method should be considered in order to eliminate oscillations near boundary layers.

2.2. Stabilized Formulation

It is already stated that, to get rid of unphysical oscillations in the numerical solution, stabilized finite element methods should be considered in the solution procedure which allows to use uniform coarse mesh and linear basis functions. One of the most favorite class of stabilized formulations goes under the name SUPG.

SUPG formulation of the Eq. (1) using linear elements can be written as: Find $u_h \in V_h$ such that

$$a(u_h, v_h) + \sum_{K \in \Omega_h} \tau_K^{SUPG} (\boldsymbol{\beta} \cdot \nabla u_h - f, \boldsymbol{\beta} \cdot \nabla v_h) = (f, v_h), \tag{4}$$

$\forall v_h \in V_h$ with the stabilization parameter [3]

$$\tau_K^{SUPG} = \begin{cases} \frac{h_K}{2|\boldsymbol{\beta}_K|} \text{ if } Pe_K \geq 1 \\ \\ \frac{h_K^2}{12\epsilon_K} \text{ if } Pe_K < 1 \end{cases} \tag{5}$$

where h_K is the diameter of the element K, $Pe_K = \frac{|\boldsymbol{\beta}_K| h_K}{6\epsilon_K}$ is the Peclet number. Since the diffusion coefficient $\epsilon(\mathbf{x})$ and convection coefficient $\boldsymbol{\beta}(\mathbf{x})$ are varying through the element, the parameters ϵ_K and $\boldsymbol{\beta}_K$ are calculated as the average values of $\epsilon(\mathbf{x})$ and $\boldsymbol{\beta}(\mathbf{x})$ on the element nodes, respectively.

The SUPG method is computationally cheap and practically easy to implement and it is one of the effective way of obtaining numerically stable solutions.

An alternative method called the stabilizing subgrid method (SSM) is proposed for calculating the value of stabilization parameter τ_K. The key idea is to enrich the finite element spaces with some appropriate functions so that the resulting numerical method gives rise a stabilized formulation without increasing the size of stiffness matrix (for the details of the method, see [4, 5]).

In SSM, the stabilization parameter τ_K is explicitly given by

$$\tau_K = \frac{1}{|K|} \int_K b_K \, dK \tag{6}$$

where $|K|$ is the length (in 1-D) or area (in 2-D) of element K, and b_K is the unique bubble function defined by the following boundary value problem in K:

$$\begin{cases} -\epsilon \Delta b_K + \beta \cdot \nabla b_K &= 1 \quad \text{in } K \\ b_K &= 0 \quad \text{on } \partial K. \end{cases} \tag{7}$$

It is seen that, the solution of Eq. (7) is as difficult as the original Eq. (1). Therefore, the bubble function b_K should be calculated accurately and also efficiently.

The idea of SSM is obtaining the value of b_K by considering only a single node inside each element. However, location of the node is critical so it should be calculated by considering the boundary layer which determined by the values of ϵ_K and $\boldsymbol{\beta}_K$.

In 1-dimension, the location of the subgrid point call it x_p, for element $K = [x_K, x_{K+1}]$ is given in [4] as

$$x_p = \begin{cases} x_K - \frac{2\epsilon_K}{\boldsymbol{\beta}_K} & \text{if} \quad \epsilon_K \leq \frac{h\boldsymbol{\beta}_K}{6} \\ x_K - \frac{h}{3} & \text{if} \quad \epsilon_K > \frac{h\boldsymbol{\beta}_K}{6} \end{cases} \tag{8}$$

where h is length of the element.

Once the location of subgrid point is calculated, the bubble function b_K is defined explicitly and so after simple calculations, the stabilization parameter τ_K is calculated exactly as

$$\tau_K^{SSM} = \frac{(x_p - x_K)(x_{K+1} - x_p)}{4\epsilon_K}. \tag{9}$$

However, in 2-dimensional case, calculating the bubble function b_K from Eq. (7) is not an easy task. Therefore, the value of b_K and so the value of stabilization parameter can be calculated approximately and used in stabilized formulation.

It is also possible to use standard Galerkin finite element formulation over the new mesh which is obtained by partitioning each element to triangles using the subgrid node inside the element as considered in SSM. In 2-dimension, generally, the domain is discretized in to triangular or rectangular elements. Details of how the location of subgrid point is determined are given for triangular elements in [4] and for rectangular elements in [5].

3. Results and Discussion

In this section, we present some numerical results with SUPG and SSM methods in order to show the efficiency of stabilization compared to standard Galerkin FEM.

3.1. Test Problem in 1-D

Consider the variable coefficient convection-diffusion equation with Dirichlet type boundary conditions as,

$$\begin{aligned} -x^2u'' + wxu' &= 0\,, \qquad 0 < x < 1, \\ u(0) = 1 \quad &, \quad u(1) = 0 \end{aligned} \tag{10}$$

where w is positive constant and the exact solution of the equation is

$$u(x) = 1 - x^{w+1}. \tag{11}$$

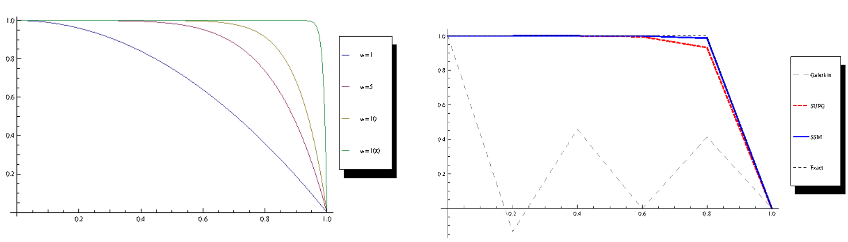


Figure 1: Exact solution of the problem 10 for different values of w (left) and solutions of the problem 10 for $w = 100$

Notice that, for small values of w, standard Galerkin finite element method will yield good results. However, as w getting large there is a boundary layer around x = 1 as displayed in Figure 1 which shows exact solutions for different values of w. It is seen from the figure that, standard Galerkin FEM produces oscillations which is not exist in physical configuration of the problem. The problem is also solved by using both SUPG and SSM methods and obtained solutions are compared with the exact one. We will concentrate on the solutions near the boundary layer in which the difference between the SUPG and SSM can be seen more clearly. The comparison is performed on coarse mesh (5 linear elements, 6 nodes). Additionally, the SSM produces accurate results compared to the SUPG method although the stabilization is pronounced in both of the methods (no oscillations and disturbances in the solutions). The accuracy of SSM is based on the location of the subgrid point and the value of the stabilization parameter which are displayed in Table 1.

Table 1: Location of the Mid point, subgrid point and values of stabilization parameter for SUPG and SSM

Interval	Mid Point	Subgrid Point	τ_K^{SUPG}	τ_K^{SSM}
$[0.0, 0.2]$	0.1	0.196	0.01	0.0098
$[0.2, 0.4]$	0.3	0.3933	0.0033	0.0032
$[0.4, 0.6]$	0.5	0.5896	0.002	0.001896
$[0.6, 0.8]$	0.7	0.7857	0.0014	0.001326
$[0.8, 1.0]$	0.9	0.9817	0.0011	0.001009

3.2. Test Problem in 2-D

As a second test problem, we consider the 2-dimensional L-Shape flow problem which is the variable diffusion coefficient version of the problem considered in [9] with Dirichlet type boundary conditions

$$-\varepsilon(x^2 + y^2)\Delta u + (-y, x) \cdot \nabla u = 0 \tag{12}$$

where ε is very small positive number in the range $10^{-2} \leq \varepsilon \leq 10^{-4}$. The difficulty of the problem source from the interaction between a boundary layer and corner singularity. Domain of the problem with the boundary conditions and uniformly discretized both triangular and rectangular meshes are displayed in Figure 2.

The change of subgrid point in SSM is displayed in Figure 3. In diffusion dominated case ($\varepsilon = 10^{-2}$), stabilization is not needed. Therefore, subgrid point

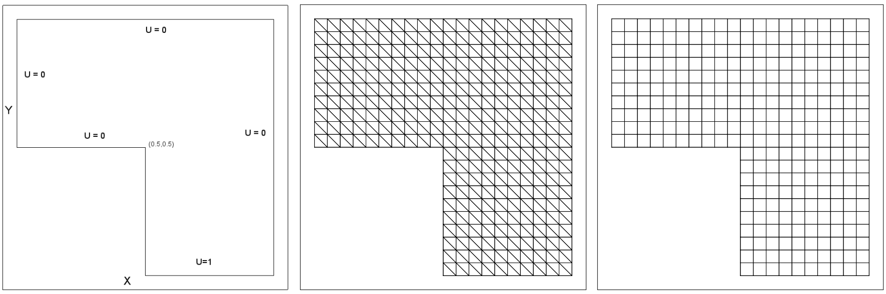
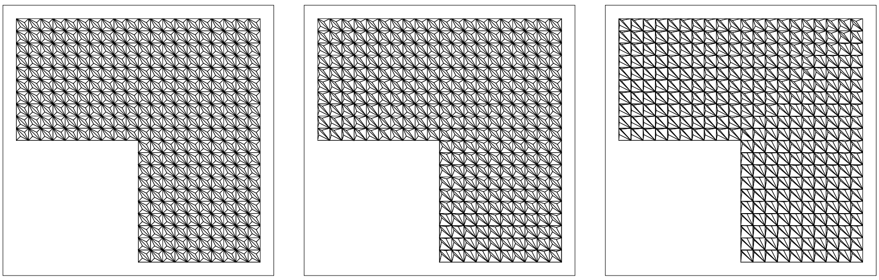


Figure 2: Definition of L-shape flow problem and sample uniform global mesh for both triangular and rectangular elements



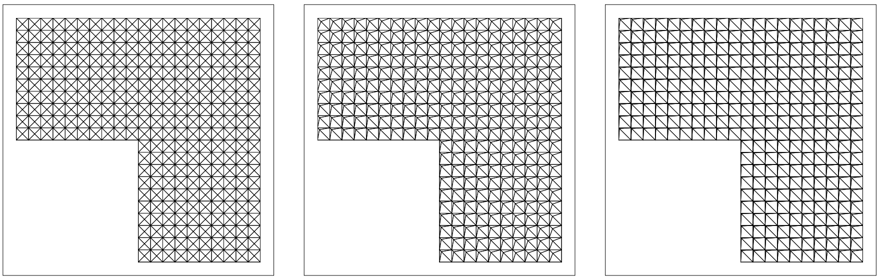
(a) $\epsilon = 0.01$ (b) $\epsilon = 0.0025$ (c) $\epsilon = 0.001$

Figure 3: Location of subgrid point for the triangular mesh

is stays in gravitational center of the element (Figure 3(a)). But, as ε getting smaller, problem turns to advection dominated case. In this case, arrangement of the subgrid point is firstly seen upper left boundary and re-entrant L-corner (Figure 3(b)), then for very small values $\varepsilon(= 10^{-3})$, stabilization becomes effective in all of the domain (Figure 3(c)).

Similar observation is also seen for rectangular elements (Figure 4). In Figure 4(a), subgrid point for all elements stays at the center of rectangle. Then, for $\epsilon = 0.0025$ location of subgrid point is arranged especially near the boundary layers (Figure 4(b)), and for $\epsilon = 0.001$ it is almost at the left upper corner of rectangular elements.

Solutions with standard Galerkin and stabilized finite element method for $\epsilon = 0.01, 0.001$ and 0.0001 are displayed in Figures 5-6. As stated above, for diffusion dominated case, both solutions are almost same. But, convection dominated case, standard Galerkin FEM exhibits oscillations whereas the stabilized



(a) $\epsilon = 0.01$ (b) $\epsilon = 0.0025$ (c) $\epsilon = 0.001$

Figure 4: Location of subgrid point for the rectangular mesh

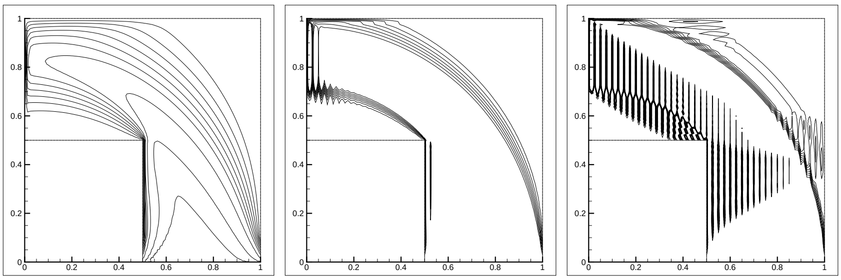


Figure 5: Standard Galerkin FEM solutions

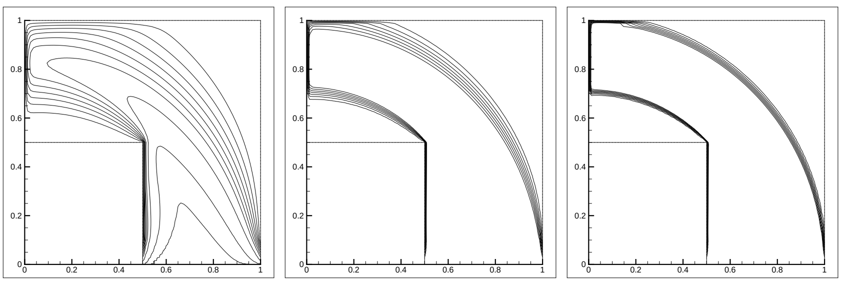


Figure 6: Stabilized FEM solutions

FEM eliminates these disturbances.

4. Conclusion

A stabilized finite element method for the approximate solution of the variable coefficient convection-diffusion equations is considered in the framework of

SUPG type stabilization. Numerical results indicate that, proposed method is convergent and obtained solutions are in a good agreement with the exact or results given in previous studies by other methods.

References

[1] T. Linß, Layer-adapted meshes for convection-diffusion problems, *Comput. Methods Appl. Mech. Engrg.*, **192**, No 9-10 (2003), 1061-1105.

[2] Feng-Nan Hwang, A new submesh strategy in the two-level finite element method for the advective-diffusive equation, *M. Sc. Thesis*, University of Colorado, Denver (1999).

[3] A.N. Brooks, T.J.R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.*, **32** (1982), 199-2592.

[4] A.I. Nesliturk, A stabilizing subgrid for convection-diffusion problem, *Math. Mod. Meth. Appl. S.*, **16**, No 2 (2006), 211-231.

[5] A.I. Nesliturk, On the choice of stabilizing subgrid for convection-diffusion problem on rectangular grids, *Comput. Math. Appl.*, **59** (2010), 3687-3699.

[6] A.I. Nesliturk, S.H. Aydın, M. Tezer-Sezgin, Two-level finite element method with a stabilizing subgrid for the incompressible Navier-Stokes equations, *Int. J. Num. Meth. Fl.*, **58**, No 5 (2007), 551-572.

[7] S.H. Aydın, A.I. Nesliturk, M. Tezer-Sezgin, Two-level finite element method with a stabilizing subgrid for the incompressible MHD equations, *Int. J. Num. Meth. Fl.*, **62**, No 2 (2010), 188-210.

[8] S.H. Aydın, Two-level finite element method with a stabilizing subgrid for the natural convection flow simulation in different geometries, *Numer. Heat Tr. A-Appl.*, **59** (2011), 799-813.

[9] F. Brezzi, D. Marini, A. Russo, Applications of the pseudo residual-free bubbles to the stabilization of convection-diffusion problems, *Comput. Methods Appl. Mech. Engrg.*, **166** (1998), 51-63.