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CONSTRUCTION OF GREEN FUNCTION FOR BESSEL-HELMHOLTZ EQUATION

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Abstract: In this paper we construct the Green function for a boundary value problem

$$(\Delta_{B_n} + k^2)u(k,x,y) = f(x,y),$$
$$u(k,x,y)|_{\Gamma} = 0,$$

and prove that the limit absorption principle holds for it.

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1. Introduction

Degenerating and singular elliptic equations are one of the most significant

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topics of the modern theory of partial differential equations. The necessity of study of such equations is stipulated by their numerical applications in gas dynamics, theory of shells, theory of elasticity, continuum mechanics, etc. M.V. Keldysh's work [5], where the cases when a characteristic part of the boundary of the domain may become free from boundary conditions, is one of the first papers in this field.

In the last ten years there is a great interest to degenerating and singular equations, including the equations containing the Bessel differential operator

$$B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \quad \gamma > 0.$$

Note that in these directions there exist investigations by I.A. Kiprianov [7], [8], I.A. Kiprianov, V.I. Kononenko [10], I.A. Kiprianov and M.I. Klyuchanchev [9], F.G. Mukhlisov [12], F.G. Mukhlisov, A.Sh. Khismatullin [13], A.Sh. Khusmatullin [6].

The mentioned equations are often encountered in applications, for example in axial symmetry problems of continuum mechanics. One of the intensely developing directions is the investigation of elliptic equations with Bessel operator that in 1967 were called by I.A. Kiprianov B_n-elliptic. The first papers on B_n-elliptic equations are related to the equations of the form

$$\Delta_{B_n} u(x) = \sum_{i=1}^{n-1} \frac{\partial^2 u(x)}{\partial x_i^2} + B_n u(x) = 0.$$

I.A. Kiprianov and his colleagues extensively studied the operator Δ_{B_n} and some of its generalizations.

I.A. Kiprianov showed that the volume potential

$$u(x) = \int_{R_+^n} |y|^{2-n-\gamma} T^y f(x) (y')^\gamma dy$$

is the solution of the B_n-elliptic equation

$$\Delta_{B_n} u(x) = f(x),$$

where

$$T^y f(x) = C_{\gamma,n} \int_0^\pi f\left(x' - y', (x_n^2 - 2x_n y_n \cos\beta + y_n^2)^{\frac{1}{2}}\right) \sin^{\gamma-1}\beta \, d\beta.$$

The study of wave propagation in infinite domains has a great value in physics and mechanics. The long-range wave propagation in atmosphere, sound propagation in sea, waves in pipes belong to such ones. These phenomena reduce to boundary value problems in cylindrical domains for the Helmholts equation. There is a wide range of references devoted to different boundary value problems for partial equations in laminated media.

For distinguishing physically interesting solutions of boundary value problems for elliptic equations in infinite domains with a spectral parameter when this parameter is a point of the continuous spectrum of the problem, three methods are used, more exactly – the limit absorption principle, the limit amplitude principle and the Zommerfeld radiation condition. For brevity, these principles and the Zommerfeld condition are called radiation principles.

2. Main results

Denote by $R^m(y)$, m–dimensional Euclidean space with the point $y = (y_1, \dots, y_m)$, by $R^n(x)$ the same space with the point $x = (x_1, \dots, x_n)$, $R^n_+(x) = \{x \in R^n(x); x = (x', x_n), x_n > 0\}$, $\gamma > 0$.

Let $\Lambda = R^n_+(x) \times \Omega$ be a cylindrical domain in $R^n_+(x) \times R^m(y)$, where Ω is a bounded domain in $R^m(y)$ with a smooth boundary $\partial\Omega$, where $\partial\Omega \in C^{\left[\frac{3m}{2}\right]}$.

Consider in Λ the following boundary value problem

$$(\Delta_{B_n} + k^2)u(k, x, y) = f(x, y), \tag{1}$$

$$u(k, x, y)|_\Gamma = 0, \tag{2}$$

where $f(x, y) \in C_0^\infty(\Lambda)$, k^2 is a constant number not necessarily real, Γ is the boundary of the cylinder Λ.

Note that in this direction there exist many investigations of A.N. Tikhonov, A.A. Samarskii [16], L.A. Murave [14], M.V. Federyuk [2], B.A. Iskenderov, Z.G. Abbasov, E.Kh. Eyvazov [3], B.A. Iskenderov, A.I. Mekhtieva [4].

Denote by $C_0^\infty(\Lambda)$ the space of infinitely differentiable finite functions whose support is contained in Λ, by $C^l(D)$ the space of functions continuous together with the derivatives to order l, inclusively in the domain $D \subset R^n_+ \times R^m$.

Definition 1. Let $Jmk^2 \neq 0$, then the function $u(k, x, y) \in C^2(\Lambda) \cap C(\overline{\Lambda})$, convergent to zero as $x \to \infty$, is called a solution of problem (1)–(2), if it satisfies equation (1) in the ordinary sense and vanishes on the boundary Γ of the cylinder Λ.

In the case when k^2 is a point of the continuous spectrum of the problem (1)–(2), there exist three methods to distinguish physically interesting solutions of the problem. These are: the limit absorption principle, limit amplitude or at infinity radiation conditions are imposed on the solution of problem (1)–(2). We will justify these principles and study the conditions for problem (1)–(2).

We determine the Fourier-Bessel transform as follows:

$$F_{B_n}[f(x)](s) = \int\limits_{R_+^n} f(x) \cdot j_{\frac{\gamma-1}{2}}(x_n \cdot s_n) e^{-i(x',s')} x_n^\gamma dx,$$

where $(x', s') = \sum\limits_{m=1}^{n-1} x_m \cdot s_m$.

Using the property of generalized shift, it is easy to show that

$$F_{B_n}(f * g) = F_{B_n}(f) \cdot F_{B_n}(g).$$

Along with the problem (1)–(2), we consider the problem

$$(\Delta_{B_n} + k^2) G(k, x, y) = \delta(x, y), \tag{3}$$

$$G(k, x, y)|_\Gamma = 0, \tag{4}$$

where $x, y \in \Lambda$, $\delta(x)$ is Dirac's delta-function.

Definition 2. Let $Jmk^2 \neq 0$. The decreasing solution of the problem (3)–(4) as $x \to \infty$ (for every $y \in \Omega$) will be called the Green function of problem (1)–(2).

Theorem 3. ([1]) *Let*

$$F(t) = \int\limits_0^a \varphi(\tau) e^{-t\tau} d\tau, \quad a > 0,$$

where $\varphi(\tau)$ *is an analytic function regular at the points of the interval* $0 < \tau \leq a$, *while in the vicinity of the point* $\tau = 0$ *is representable by the series*

$$\varphi(\tau) = \tau^\alpha (a_0 + a_1 \tau + \cdots), \quad \alpha > -1.$$

Then as $t \to +\infty$,

$$F(t) = \sum_{p=1}^{\infty} \frac{\Gamma(\alpha+1)}{t^{\alpha+p}} a_{p-1},$$

where $\Gamma(\alpha)$ *is Euler's Gamma-function.*

The following asymptotic estimation of the Hankel functions holds for large values of the argument and in what follows we will use it ([15]):

$$H_{\mu}^{(1,2)}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i\left(z - \frac{\pi\mu}{2} - \frac{\pi}{4}\right)}(1 + O(z^{-1})). \tag{5}$$

As it was noted above, for $k^2 > 0$ for distinguishing physically interesting solutions of problem (1)–(2) the limit absorption principle is used.

Definition 4. We say that for the problem (1)–(2) the limit absorption principle holds, if for the solution of this problem converging to zero at infinity, with the complex parameter $k_\varepsilon^2 = k^2 + i\varepsilon$ $(\varepsilon \neq 0)$ there exists the limit

$$u(k,x,y) = \lim_{\varepsilon \to 0} u(k_\varepsilon, x, y)$$

uniformly with respect to x, y at every compact from Λ, and $u(k,x,y)$ satisfies the limit problem.

For constructing the solution of problem (1)–(2) with the complex parameter k_ε^2, we perform the Fourier-Bessel transform with respect to x and get the following boundary value problem

$$[\Delta_y + (k_\varepsilon^2 - |s|^2)]\widehat{u}(k_\varepsilon, x, y) = \widehat{f}(s, y),$$

$$\widehat{u}(k,x,y)|_{\partial\Omega} = 0,$$

where $\widehat{u}(k_\varepsilon, x, y) = F_{B_n}[u(k_\varepsilon, x, y)](s)$ is the Fourier-Bessel transform with respect to x, and Δ_y is the Laplace operator with respect to the variable y.

Consider the following operator $L = -\Delta_y$ with the domain of definition

$$D(L) = \left\{v : v \in C^2(\Omega) \cap C(\overline{\Omega}),\ \Delta v \in L_2(\Omega),\ v|_{\partial\Omega} = 0\right\}.$$

For the operator L it holds the following theorem.

Theorem 5. ([18]) *The set of eigenvalues $\{\lambda_\ell\}$ of the operator L is denumerable and has no finite limit points, each eigenvalue λ_ℓ has a finite multiplicity and the least one eigenvalue is prime. The eigenfunctions $\{\varphi_\ell(y)\}$ may be chosen as real and orthonormed. Any function from $L_2(\Omega)$ expands in regularly convergent Fourier series with respect to $\{\varphi_\ell(y)\}$.*

As it follows from this theorem for eigenvalues of the operator L, we have the chain of the inequalities:

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_\ell \le \ldots, \quad \text{as} \quad \ell \to \infty.$$

As $k_\varepsilon^2 - |s|^2 \neq \lambda_\ell$ for any $s \in R^n$ and $\widehat{u}(k_\varepsilon, x, y) \in D(L)$, then using Theorem 3 for solving problem (1)–(2), we get

$$\widehat{u}(k_\varepsilon, x, y) = R_{k_\varepsilon^2 - |s|^2} \widehat{f}(s, y) = \sum_{l=1}^{\infty} \frac{C_\ell(s)\varphi_\ell(y)}{\lambda_\ell - k_\varepsilon^2 + |s|^2}, \tag{6}$$

where R_μ is a resolvent of the operator L,

$$C_\ell(s) = \int_\Omega \widehat{f}(s, y)\varphi_\ell(y)dy. \tag{7}$$

Note that λ_ℓ and $\varphi_\ell(y)$ are independent of s.

Now, the solution of the problem (1)–(2) is determined as the Fourier-Bessel inverse transform of $\widehat{u}(k, x, y)$,

$$u(k_\varepsilon, x, y) = \int_{R_+^n} \widehat{u}(k_\varepsilon, s, y) j_{\frac{\gamma-1}{2}}(x_n \cdot s_n) e^{-i(x', s')} s_n^\gamma ds$$

$$= \sum_{l=1}^{\infty} \varphi_\ell(y) \int_{R_+^n} \frac{C_\ell(s) j_{\frac{\gamma-1}{2}}(x_n \cdot s_n) e^{-i(x', s')}}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_n^\gamma ds. \tag{8}$$

Here, the term by term integration is valid by the uniform convergence of series (6) and its derivatives ([11]). Taking (7) into account in (8), we get

$$u(k_\varepsilon, x, y) = \sum_{l=1}^{\infty} \varphi_\ell(y) \int_{R_+^n} \left[f_\ell(\xi) \int_{R_+^n} \frac{e^{i(\xi', s')} j_{\frac{\gamma-1}{2}}(x_n s_n) j_{\frac{\gamma-1}{2}}(\xi_n s_n)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_n^\gamma ds \right] \xi_n^\gamma d\xi, \tag{9}$$

where

$$f_\ell(\xi) = \int_\Omega f(\xi, y)\varphi_\ell(y)dy. \tag{10}$$

Denote

$$J_\ell(k_\varepsilon, x, \xi)$$

$$= \lim_{N\to\infty} \int\limits_{|s|\le N} \frac{e^{i(\xi'-x',s')} j_{\frac{\gamma-1}{2}}(x_n s_n) j_{\frac{\gamma-1}{2}}(\xi_n s_n)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_n^\gamma ds. \qquad (11)$$

Passing in (11) to spherical coordinates, and taking into account the spherical symmetry $(\lambda_\ell - k_\varepsilon^2 + |s|^2)^{-1}$, we have:

$$J_\ell(k_\varepsilon, x, \xi) = \lim_{N\to\infty} \int\limits_0^N |s'|^{n-2} \left(\int\limits_\Omega e^{i|\xi'-x'||s'|\cos\theta d\omega} \right)$$

$$\times \left(\int\limits_0^{\sqrt{N^2+|s'|^2}} \frac{j_{\frac{\gamma-1}{2}}(x_n s_n) j_{\frac{\gamma-1}{2}}(\xi_n s_n)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_n^\gamma ds_n \right) d|s'|, \qquad (12)$$

where Ω_r is a sphere of radius r centered at the origin, θ is an angle between the directions of the vectors $\xi - x$ and s, while $\widetilde{\omega}$ is a point of a unit sphere. Since

$$\int\limits_{\Omega_r} e^{i|\xi-x||s|cos\theta)} d\widetilde{\omega} = (2\pi)^{\frac{n-1}{2}-1} (|\xi - x||s|)^{1-\frac{n-1}{2}} j_{\frac{n-1}{2}-1}(|\xi - x||s|), \qquad (13)$$

where $j_{\frac{n-1}{2}-1}(z)$ is the Bessel function of order $\frac{n}{2} - 1$, substituting (13) in (12), we get

$$J_\ell(k_\varepsilon, x, \xi) = (2\pi)^{\frac{n-3}{2}} |\xi' - x'|^{\frac{1-n}{2}} \lim_{N\to\infty} \int\limits_0^N |s'|^{\frac{n-3}{2}} j_{\frac{n-3}{2}}(|\xi - x'||s'|)$$

$$\times \left(\int\limits_0^{\sqrt{N^2+|s'|^2}} \frac{j_{\frac{\gamma-1}{2}}(x_n s_n) j_{\frac{\gamma-1}{2}}(\xi_n s_n)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_n^\gamma ds_n \right) d|s'|$$

$$= (2\pi)^{\frac{n-3}{2}} |\xi' - x'|^{\frac{1-n}{2}} \left[2^{\frac{\gamma-1}{2}} \Gamma\left(\frac{\gamma+1}{2}\right) \right]^2 (\xi_n s_n)^{\frac{1-\gamma}{2}}$$

$$\times \lim_{N\to\infty} \int\limits_0^N \left(\int\limits_0^{\sqrt{N^2+|s'|^2}} |s'|^{\frac{n-1}{2}} J_{\frac{n-3}{2}}(|\xi - x'||s'|) \frac{J_{\frac{\gamma-1}{2}}(x_n s_n) J_{\frac{\gamma-1}{2}}(\xi_n s_n)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_n ds_n \right) d|s'|$$

$$= (2\pi)^{\frac{n-3}{2}}|\xi'-x'|^{\frac{1-n}{2}}\left[2^{\frac{\gamma-1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)\right]^2(\xi_n s_n)^{\frac{1-\gamma}{2}}\lim_{N\to\infty}\int\limits_0^N\frac{r^{\frac{n+1}{2}}}{\lambda_\ell-k_\varepsilon^2+r^2}$$

$$\times\left(\int\limits_0^{\frac{\pi}{2}}J_{\frac{n-3}{2}}(|\xi'-x'|r\sin\varphi)J_{\frac{\gamma-1}{2}}(rx_n\cos\varphi)J_{\frac{\gamma-1}{2}}(r\xi_n\cos\varphi)\sin^{\frac{n-1}{2}}\varphi\cos\varphi d\varphi\right)dr. \tag{14}$$

The internal integral in (14) is the Sonine second determined integral. Therefore we have

$$J_\ell(k_\varepsilon,x,\xi)$$

$$= (2\pi)^{\frac{n-3}{2}}2^{\frac{\gamma-1}{2}}\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}(\xi_n s_n)^{\frac{1-\gamma}{2}}(\xi_n s_n)^{\frac{\gamma-1}{2}}\lim_{N\to\infty}\int\limits_0^N\frac{r^{n+\gamma-3}}{\lambda_\ell-k_\varepsilon^2+r^2}$$

$$\times\left(\int\limits_0^{\pi}\frac{J_{\frac{n+\gamma-2}{2}}(\sqrt{|\xi'-x'|^2r^2+\xi_n^2r^2+x_n^2r^2-2r^2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2r^2+\xi_n^2r^2+x_n^2r^2-2r^2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}\right)dr$$

$$= 2^{\frac{n+\gamma-4}{2}}\pi^{\frac{n-3}{2}}\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}\lim_{N\to\infty}\int\limits_0^N\frac{r^{\frac{n+\gamma}{2}-3}}{\lambda_\ell-k_\varepsilon^2+r^2}$$

$$\times\left(\int\limits_0^{\pi}\frac{J_{\frac{n+\gamma-2}{2}}(r\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}\right)dr.$$

Let $n+\gamma$ be an odd number. Then $z^{\frac{n+\gamma}{2}-2}J_{\frac{n+\gamma}{2}-1}(z)$ is an even function. Therefore,

$$J_{l_N}(k_\varepsilon,x,\xi) = 2^{\frac{n+\gamma}{2}-3}\pi^{\frac{n-3}{2}}\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}\lim_{N\to\infty}\int\limits_{-N}^N\frac{r^{\frac{n+\gamma}{2}-3}}{\lambda_\ell-k_\varepsilon^2+r^2}$$

$$\times\left(\int\limits_0^{\pi}\frac{J_{\frac{n+\gamma-2}{2}}(r\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}\right)dr.$$

Expressing the Bessel function by the Hankel function (see [15])

$$J_\nu(z) = \frac{1}{2}\left(H_\nu^{(1)}(z)+H_\nu^{(2)}(z)\right), \tag{15}$$

we get

$$J_{l_N}(k_\varepsilon, x, \xi) = 2^{\frac{n+\gamma}{2}-4}\pi^{\frac{n-3}{2}} \frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)} \lim_{N\to\infty} \int_{-N}^{N} \frac{r^{\frac{n+\gamma}{2}-3}}{\lambda_\ell - k_\varepsilon^2 + r^2}$$

$$\times \left(\int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(r\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}} \right.$$

$$\left. + \int_0^\pi \frac{H^{(2)}_{\frac{n+\gamma-2}{2}}(r\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi}))\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}} \right) dr$$

$$\equiv J_l^{(1)}(k_\varepsilon, x, \xi) + J_l^{(2)}(k_\varepsilon, x, \xi). \tag{16}$$

Taking into account the analyticity of the integrand function in (14) and asymptotic (15) of the Hankel function as $|s| \to +\infty$, applying the residue theorem and tending $N \to +\infty$, we get

$$J_l^{(1)}(k_\varepsilon, x, \xi) = 2^{\frac{n+\gamma}{2}-4}\pi^{\frac{n-3}{2}} \frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)} \lim_{N\to\infty} \int_{-N}^{N} \frac{r^{\frac{n+\gamma}{2}-3}}{\lambda_\ell - k_\varepsilon^2 + r^2}$$

$$\times \left(\int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(r\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}} \right)$$

$$= 2^{\frac{n+\gamma}{2}-4}\pi^{\frac{n-3}{2}} \frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$\times \lim_{N\to\infty} \int_0^\pi \frac{\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}$$

$$\times \left(\int_{-N}^{N} \frac{r^{\frac{n+\gamma}{2}-2} H^{(1)}_{\frac{n+\gamma-2}{2}}(r\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})}{\lambda_\ell - k_\varepsilon^2 + r^2} dr \right) d\varphi$$

$$= i2^{\frac{n+\gamma}{2}-4}\pi^{\frac{n-1}{2}} \frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)} \left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}}$$

$$\times \int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}};$$

$$J_l^{(2)}(k_\varepsilon, x, \xi) = -i2^{\frac{n+\gamma}{2}-4}\pi^{\frac{n-1}{2}}\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(-i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}}$$

$$\times \int_0^\pi \frac{H^{(2)}_{\frac{n+\gamma-2}{2}}(-i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}$$

$$= -i2^{\frac{n+\gamma}{2}-4}\pi^{\frac{n-1}{2}}\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}}$$

$$\times \int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}.$$

Since (see [15])

$$H^{(2)}_{\frac{n+\gamma-2}{2}}(-z) = (-1)^{\frac{n+\gamma-2}{2}}H^{(1)}_{\frac{n+\gamma-2}{2}}(z),$$

then from (14)–(16) it follows that

$$J_\ell(k_\varepsilon, x, \xi) = i2^{\frac{n+\gamma}{2}-3}\pi^{\frac{n-1}{2}}\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}}$$

$$\times \int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}. \quad (17)$$

Proceeding in the same way as above, at even n for $J_\ell(k_\varepsilon, x, \xi)$, we get formula (7). Substituting (17) in (9), we get

$$u(k_\varepsilon, x, y) = i2^{\frac{n+\gamma}{2}-3}\pi^{\frac{n-1}{2}}\frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}\sum_{\ell=1}^\infty\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}}\varphi_\ell(y)$$

$$\times \int_{R_+^n}\int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_n^2 + x_n^2 - 2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}$$

$$\times f_\ell(\xi)\xi_n d\xi. \tag{18}$$

Now, substituting the value of $f_\ell(\xi)$ from (10) in (18) and distinguishing the function $f(x)$, we get

$$u(k_\varepsilon, x, y) = i2^{\frac{n+\gamma}{2}-3}\pi^{\frac{n-1}{2}} \frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)} \sum_{\ell=1}^{\infty}\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}} \varphi_\ell(y)$$

$$\times \int_{R_+^n}\int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}$$

$$\times \int_\Omega f(\xi, z)\varphi_\ell(z)dz\xi_n d\xi = i2^{\frac{n+\gamma}{2}-3}\pi^{\frac{n-1}{2}} \frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

$$\times \int\cdots\int_\Lambda \sum_{\ell=1}^{\infty}\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}} \varphi_\ell(y)\varphi_\ell(z)\times$$

$$\times \int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}}$$

$$\times f(\xi, z)\varphi_\ell(z)d\Lambda.$$

The function

$$G(k_\varepsilon, x-\xi, y, z) = i2^{\frac{n+\gamma}{2}-3}\pi^{\frac{n-1}{2}} \frac{\Gamma^2\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma}{2}\right)\Gamma\left(\frac{1}{2}\right)} \sum_{\ell=1}^{\infty}\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+\gamma-6}{2}} \varphi_\ell(y)\varphi_\ell(z)$$

$$\times \int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi})\sin^{\gamma-1}\varphi d\varphi}{(|\xi'-x'|^2+\xi_n^2+x_n^2-2x_n\xi_n\cos\varphi)^{\frac{n+\gamma-2}{4}}} \tag{19}$$

is the Green function of problem (1)–(2) with a complex parameter k_ε^2. Thus, we proved the following theorem.

Theorem 6. *The Green function of problem* (1)–(2) *is an analytic function of k, except for the denumerable number of points $k = \pm\lambda_\ell^{\frac{1}{2}}$, $\ell = 1, 2, \ldots$, being the branching points and for it expansion* (19) *does not hold, where λ_ℓ are eigenvalues of the operator L, $\varphi_\ell(y)$ are appropriate eigenfunctions.*

From this theorem we have the following statement.

Theorem 7. *The solution of problem* (1)–(2) *with a complex parameter* k_ε^2 *is represented in the form*

$$u(k_\varepsilon, x, y) = \int \cdots \int_\Lambda G(k_\varepsilon, x - \xi, y, z) f(\xi, z) d\Lambda, \tag{20}$$

and this solution is unique, where $G(k_\varepsilon, x - \xi, y, z)$ *is determined by formula* (19).

Taking into account the asymptotics (5) as $z \to \infty$ of the function $H^{(1)}_{\frac{n+\gamma}{2}-1}(z)$, we get that series (19) and its derivatives contained in equation (1), converge uniformly with respect to ε for $|x - \xi| > 0$, and therefore passing in (20) and in its derivatives to limit as $\varepsilon \to 0$, we get that $u(k, x, y)$ is the solution of problem (1)–(2). Thus, we proved the following theorem.

Theorem 8. *For the problem* (1)–(2) *it holds the limit absorption principle (for* $n = 1, 2$; $k = \pm\lambda_\ell^{\frac{1}{2}}$, $l = 1, 2, \ldots$*).*

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References

[1] M.A. Evgrafov, *Asymptotic Estimations and Entire Functions*, Fizmatgiz, Moscow (1962), 200 (In Russian).

[2] M.F. Federyuk, Helmholtz equation is waveguide (driving off a boundary condition from infinity), *Zhurnal vychislitelnoy matematiki i matem. fiziki*, **12**, No 2 (1972), 374-387 (In Russian).

[3] B.A. Iskenderov, Z.G. Abbasov, E.Kh. Eyvazov, Radiation principles for Helmholtz equations in cylindrical domain, *Doklady AN Azerb. SSR*, **36**, No 4 (1980), 8-11 (In Russian).

[4] B.A. Iskenderov, A.I. Mekhtieva, Radiation principles for Helmholtz equation in multi-dimensional layer with impedance boundary conditions, *Diff. Uravn.*, **29**, No 8 (1983), 1462-1464 (In Russian).

[5] M.B. Keldysh, On some cases of degeneration of elliptic type equations on the boundary of domain, *Doklady Acad. Nauk SSSR*, **77**, No 2 (1951), 181-183 (In Russian).

[6] A.Sh. Khismatullin, Solving boundary value problems for some degenerating B-elliptic equations by potentials method, *Ph.D. Thesis*, Kazan (2008), 107 (In Russian).

[7] I.A. Kiprianov, On a class of singular elliptic operators, *Diff. Uravn.*, **7**, No 11 (1971), 2066-2077 (In Russian).

[8] I.A. Kiprianov, *Singular Elliptic Boundary Value Problems*, Nauka, Fizmalit, Moscow (1997), 208 (In Russian).

[9] I.A. Kiprianov, M.I. Klyuchanchev, On singular integrals generated by a generalized shift operator, *Sib. Math. J.*, **11**, No 5 (1970), 1060-1083 (In Russian).

[10] I.A. Kiprianov, V.I. Konenko, Fundamental solution of B-elliptic equations, *Diff. Uravn.*, **3** (1967), 114-129 (In Russian).

[11] V.P. Mikhailov, *Partial Differential Equations*, Nauka (1976), 391 (In Russian).

[12] F.G. Mukhlisov, On existence and uniqueness of the solution of some partial equations with Bessel's differential operator, *Izvestia Vuzov Matem.*, No 11 (1984), 63-66 (In Russian).

[13] F.G. Mukhlisov, A.Sh. Khismatullin, On potentials for a degenerating B-elliptic equation, *Vestnik Samarskogo Gos. Techn. Un-ta Ser. Fiz. Mat. Nauki*, **27** (2004), 5-9 (In Russian).

[14] L.A. Muravei, Asymptotic behavior for large time values of solutions of second and third boundary value problems for a wave equation with two space variables, *Proc. of Steklov Institute of Mathematics*, **126** (1973), 73-144 (In Russian).

[15] A.V. Nikiforov, V.V. Uvarov, *Special Functions of Mathematical Physics*, Moscow, 1974, 303 (In Russian).

[16] A.N. Tikhonov, A.A. Samarskii, On radiation principle, *Zhurn. eksper. i theor. fiziki*, **8**, No 2 (1948), 243-248 (In Russian).

[17] I.N. Vekua, On methaharmonic functions, *Proc. of the Institute of Mathematics, Tbilisi*, **12** (1943), 105-174 (In Russian).

[18] V.S. Vladimirov, *Equations of Mathematical Physics*, Fizmatgiz, Moscow (1981), 512 (In Russian).

[19] A.G. Zemanyan, *Integral Transformations of Distributions*, Nauka, Moscow (1974), 400 (In Russian).