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# BLOW-UP FOR DISCRETIZATIONS OF SOME REACTION-DIFFUSION EQUATIONS WITH A NONLINEAR CONVECTION TERM

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**Abstract:** This paper concerns the study of the numerical approximation for the following parabolic equations with a nonlinear convection term

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u^q(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \\ u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \end{cases}$$

where $q \ge 1$ and $p \ge q + 1$.

We find some conditions under which the solution of the discrete form of the above problem blows up in a finite time and estimate its numerical blow-up time. We also prove that the numerical blow-up time converges to the real one, when the mesh size goes to zero. Finally, we give some numerical experiments to illustrate ours analysis.

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## 1. Introduction

Consider the following boundary value problem

$$u_t(x,t) = u_{xx}(x,t) - u^q(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1, \tag{1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{3}$$

where $q \ge 1$, $p \ge q+1$, $u_0 \in C^2([0,1])$, $u_0$ is decreasing on (0,1) and verifies

$$u_0^{'}(0) = 0, \quad u_0^{'}(1) = 0, \tag{4}$$

$$u_0^{''}(x) - u_0^q(x)u_0^{'}(x) + u_0^p(x) \ge 0, \quad 0 \le x \le 1, \tag{5}$$

$$u_0(x) > -\frac{p(p-1)}{q}u_0^{'}(x), \quad 0 < x < 1. \tag{6}$$

**Definition 1.** We say that the solution $u$ of (1)–(3) blows up in a finite time if there exists a finite time $T_b$ such that $\|u(.,t)\|_\infty < \infty \; for \; t \in [0,T_b)$ but

$$\lim_{t\to T_b} \|u(.,t)\|_\infty = \infty.$$

The time $T_b$ is called the blow-up time of the solution $u$.

The above problem arises in the theory of heat conduction. The heat transfer is the propagation of the heat from one place to another in a medium or between two different mediums. It is due to the movements of atoms and molecules in a material. Heat can be transferred between solids, liquids and gases or even in vaccum/space. Transfer of heat within a fluids is by conduction. The convection is the transfer of heat by the movement of fluids. The first equations is a heat equation including a nonlinear convection term $u^q u_x$ and a nonlinear source $u^p$. It is the term of convection which ensures the movement, generates instability and is also responsible of the turbulent appearance (here we will refer to it as intermittent since we are in one dimension) when it happens (see [14], [20], [23], [24], [28]).

Blow-up problems of some reaction-diffusion equations with a nonlinear convection term have been theoretically studied by many authors (see [3], [6], [7], [8], [9], [24], [25], [26] and the references cited therein). Local in time existence and uniqueness of the solution have been proved (see [4], [5], [22], [29] and the references cited therein). In this paper, we are interesting in the numerical study of the phenomenon of blow-up using a discrete form of (1)–(3). We give some assumptions under which the solution of a discrete form of (1)–(3) blows up in a finite time and estimate its numerical blow-up time. We also show that the numerical blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken (see [10], [22]).

The paper is structured as follows. In next Section 2, we present a discrete scheme of (1)–(3) and give some lemmas which will be used throughout the paper. In Section 3, under some conditions, we prove that the solution of the discrete form of (1)–(3) blows up in a finite time. In Section 4, we study the convergence of the numerical blow-up time. Finally, in last section, we give some numerical experiments.

## 2. Properties of the Discrete Scheme

In this section, we give some lemmas which will be used later. We start by the construction of the discrete scheme. Let $I$ be a positive integer and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution $u$ of (1)–(3) by the solution $U_h^{(n)} = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following discrete equations

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} - (U_i^{(n)})^q \delta^0 U_i^{(n)} + (U_i^{(n)})^p, \; 1 \leq i \leq I-1, \tag{7}$$

$$\delta_t U_0^{(n)} = \delta^2 U_0^{(n)} + (U_0^{(n)})^p, \tag{8}$$

$$\delta_t U_I^{(n)} = \delta^2 U_I^{(n)} + (U_I^{(n)})^p, \tag{9}$$

$$U_i^{(0)} = \varphi_i > 0, \quad 0 \leq i \leq I, \tag{10}$$

where

$$n \geq 0, \;\; q \geq 1, \;\; p \geq q+1,$$

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n}, \quad 0 \leq i \leq I,$$

$$\delta^2 U_i^{(n)} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_0^{(n)} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2}, \quad \delta^2 U_I^{(n)} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2},$$

$$\delta^0 U_i^{(n)} = \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}, \quad 1 \leq i \leq I-1,$$

$$\delta^0 U_0^{(n)} = 0, \qquad \delta^0 U_I^{(n)} = 0,$$

$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h}, \quad 0 \leq i \leq I-1,$$

$$\delta^+ \varphi_i \leq 0, \quad 0 \leq i \leq I-1,$$

$$\varphi_i^{p-1} > -\frac{p(p-1)}{q} h(\delta^0 \varphi_i)\varphi_{i-1}^{p-2}, \quad 1 \leq i \leq I-1.$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time $t$ approaches the blow-up time $T_b$ , we need to adapt the size of the time step. We choose

$$\Delta t_n = \min(\frac{h^2}{2}, \tau \|U_h^{(n)}\|_\infty^{1-p}) \text{ with } \tau \in (0,1).$$

Let us notice that the restriction on the time step ensures the nonnegativity of the discrete solution when this one is decreasing. To facilitate our discussion, we need to define the notion of numerical blow-up.

**Definition 2.** We say that the solution $U_h^{(n)}$, $n \geq 0$, of the discrete problem (7)–(10) blows up in a finite time, if

$$\lim_{n\to\infty} \|U_h^{(n)}\|_\infty = \infty,$$

and

$$T_h^{\Delta t} = \lim_{n\to\infty} \sum_{i=0}^{n-1} \triangle t_i < \infty.$$

The number $T_h^{\Delta t}$ is called the numerical blow-up time of the discrete solution.

The following lemma is a discrete form of the maximum principle.

**Lemma 3.** *Let $a_h^{(n)}$, $b_h^{(n)}$ and $V_h^{(n)}$ be three sequences such that $a_h^{(n)} \le 0$, $b_h^{(n)} \delta^0 V_h^{(n)} \le 0$ and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + b_i^{(n)} \delta^0 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \ge 0, \ 0 \le i \le I, \ n \ge 0, \tag{11}$$

$$V_i^{(0)} \ge 0. \tag{12}$$

*Then, we have*

$$V_i^{(n)} \ge 0 \quad \textit{for} \quad 0 \le i \le I, \ n > 0 \quad \textit{if} \quad \Delta t_n \le \frac{h^2}{2}.$$

*Proof.* A straightforward computation shows that for $1 \le i \le I - 1$,

$$V_i^{(n+1)} \ge (1 - 2\frac{\Delta t_n}{h^2}) V_i^{(n)} + \frac{\Delta t_n}{h^2}(V_{i+1}^{(n)} + V_{i-1}^{(n)})$$

$$-\Delta t_n b_i^{(n)} \delta^0 V_i^{(n)} - \Delta t_n a_i^{(n)} V_i^{(n)},$$

$$V_0^{(n+1)} \ge (1 - 2\frac{\Delta t_n}{h^2}) V_0^{(n)} + \frac{2\Delta t_n}{h^2} V_1^{(n)} - \Delta t_n a_0^{(n)} V_0^{(n)},$$

$$V_I^{(n+1)} \ge (1 - 2\frac{\Delta t_n}{h^2}) V_I^{(n)} + \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} - \Delta t_n a_I^{(n)} V_I^{(n)}.$$

If $V_h^{(n)} \ge 0$, then using an argument of recursion, we easily see that $V_h^{(n+1)} \ge 0$, because $1 - \frac{2\Delta t_n}{h^2} \ge 0$, $b_h^{(n)} \delta^0 V_h^{(n)} \le 0$ and $a_h^{(n)} \le 0$. This ends the proof. □

We need the following result about the operator $\delta_t$.

**Lemma 4.** *Let $U_h^{(n)}$, $n \ge 0$, a sequence such that $U_h^{(n)} > 0$. Then,*

$$\delta_t (U_i^{(n)})^p \ge p (U_i^{(n)})^{p-1} \delta_t U_i^{(n)}, \quad 0 \le i \le I.$$

*Proof.* Using Taylor's expansion, we get

$$\delta_t (U_i^{(n)})^p = p (U_i^{(n)})^{p-1} \delta_t U_i^{(n)} + \frac{p(p-1)}{2} \Delta t_n (\delta_t U_i^{(n)})^2 (\theta_i^{(n)})^{p-2}, \ 0 \le i \le I,$$

where $\theta_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_i^{(n+1)}$, $0 \le i \le I$.

Which leads us to the desired result thanks to $U_h^{(n)} > 0$. □

A direct consequence of Lemma 3 is the following comparison lemma.

**Lemma 5.** *Let $a_h^{(n)}$, $V_h^{(n)}$ and $W_h^{(n)}$ be three sequences such that $a_h^{(n)} \leq 0$, $\delta^0 V_h^{(n)} \leq 0$, $\delta^0 W_h^{(n)} \leq 0$ and*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + (V_i^{(n)})^q \delta^0 V_i^{(n)} + a_i^{(n)} V_i^{(n)} < \delta_t W_i^{(n)}$$

$$-\delta^2 W_i^{(n)} + (W_i^{(n)})^q \delta^0 W_i^{(n)} + a_i^{(n)} W_i^{(n)},\ 0 \leq i \leq I,\ q \geq 1,\ n \geq 0, \tag{13}$$

$$V_i^{(0)} < W_i^{(0)}, \quad 0 \leq i \leq I. \tag{14}$$

*Then, we have*

$$V_i^{(n)} < W_i^{(n)} \quad \textit{for} \quad 0 \leq i \leq I,\ n > 0 \quad \textit{if} \quad \Delta t_n \leq \frac{h^2}{2}.$$

*Proof.* Define the sequence $Z_h^{(n)} = W_h^{(n)} - V_h^{(n)}$. A straightforward calculation gives

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + (W_i^{(n)})^q \delta^0 W_i^{(n)} - (V_i^{(n)})^q \delta^0 V_i^{(n)} + a_i^{(n)} Z_i^{(n)} > 0,\ 0 \leq i \leq I,$$

which is equivalent to

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + (W_i^{(n)})^q \delta^0 Z_i^{(n)} + ((q\mu_i^{(n)})^{q-1} \delta^0 V_i^{(n)} + a_i^{(n)}) Z_i^{(n)} > 0,\ 0 \leq i \leq I,$$

where $\mu_i^{(n)}$ is an intermediate value between $V_i^{(n)}$ and $W_i^{(n)}$, $0 \leq i \leq I$.

Knowing that $Z_h^{(0)} > 0$, from Lemma 3, we have $Z_h^{(n)} > 0$, which implies that $V_i^{(n)} < W_i^{(n)}$, $0 \leq i \leq I$ and the proof is complete. □

The following lemma shows the decreasing in space of the discrete solution.

**Lemma 6.** *Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (7)–(10). Then*

$$U_{i+1}^{(n)} < U_i^{(n)},\ 0 \leq i \leq I - 1. \tag{15}$$

*Proof.* Define the vector $Z_h^{(n)}$ such that $Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)}$, $0 \leq i \leq I-1$. We have

$$Z_i^{(n)} = U_i^{(n)} - U_{i+1}^{(n)},\ 1 \leq i \leq I-2,$$

$$Z_0^{(n)} = U_0^{(n)} - U_1^{(n)},$$

$$Z_{I-1}^{(n)} = U_{I-1}^{(n)} - U_I^{(n)}.$$

A straightforward computations reveals that for $1 \leq i \leq I-2$,

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + (U_i^{(n)})^q \delta^0 U_i^{(n)} - (U_{i+1}^{(n)})^q \delta^0 U_{i+1}^{(n)} - p(\beta_i^{(n)})^{p-1} Z_0^{(n)} = 0,$$

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} - (U_1^{(n)})^q \delta^0 U_1^{(n)} - p(\beta_0^{(n)})^{p-1} Z_0^{(n)} = 0,$$

$$\delta_t Z_{I-1}^{(n)} - \delta^2 Z_{I-1}^{(n)} + (U_{I-1}^{(n)})^q \delta^0 U_{I-1}^{(n)} - p(\beta_{I-1}^{(n)})^{p-1} Z_{I-1}^{(n)} = 0,$$

which are equivalent to for $1 \leq i \leq I-2$,

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + (U_i^{(n)})^q \delta^0 Z_i^{(n)} + (q(\mu_i^{(n)})^{q-1} \delta^0 U_{i+1}^{(n)} - p(\beta_i^{(n)})^{p-1}) Z_i^{(n)} = 0,$$

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} + (U_1^{(n)})^q \delta^0 Z_0^{(n)} - p(\beta_0^{(n)})^{p-1} Z_0^{(n)} = 0,$$

$$\delta_t Z_{I-1}^{(n)} - \delta^2 Z_{I-1}^{(n)} + (U_{I-1}^{(n)})^q \delta^0 Z_{I-1}^{(n)} - p(\beta_{I-1}^{(n)})^{p-1} Z_{I-1}^{(n)} = 0,$$

where $\beta_i^{(n)}$, $0 \leq i \leq I-1$, and $\mu_i^{(n)}$, $1 \leq i \leq I-2$, are intermediate values between $U_{i+1}^{(n)}$ and $U_i^{(n)}$.

Knowing that $Z_h^{(0)} > 0$, from Lemma 3, we have $Z_h^{(n)} > 0$, which implies that $U_{i+1}^{(n)} < U_i^{(n)}$, $0 \leq i \leq I-1$, and we obtain the desired result. □

The lemma below reveals the positivity of the discrete solution.

**Lemma 7.** *Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (7)–(10). Then*

$$U_i^{(n)} > 0 \quad \text{for} \quad 0 \leq i \leq I \quad \text{if} \quad \Delta t_n \leq \frac{h^2}{2}. \tag{16}$$

*Proof.* A routine calculation reveals that for $1 \leq i \leq I-1$,

$$U_i^{(n+1)} = (1 - \frac{\Delta t_n}{h^2})U_i^{(n)} + \frac{2\Delta t_n}{h^2}(U_{i+1}^{(n)} + U_{i-1}^{(n)}) - \Delta t_n (U_i^{(n)})^q \delta^0 U_i^{(n)} + \Delta t_n (U_i^{(n)})^p,$$

$$U_0^{(n+1)} = (1 - \frac{2\Delta t_n}{h^2})U_0^{(n)} + \frac{2\Delta t_n}{h^2}U_1^{(n)} + \Delta t_n (U_0^{(n)})^p,$$

$$U_I^{(n+1)} = (1 - \frac{2\Delta t_n}{h^2})U_I^{(n)} + \frac{2\Delta t_n}{h^2}U_{I-1}^{(n)} + \Delta t_n (U_I^{(n)})^p.$$

If $U_h^{(n)} > 0$, then using an argument of recursion, we easily see that $U_h^{(n+1)} > 0$, because $1 - \frac{2\Delta t_n}{h^2} \geq 0$ and $\delta^0 U_h^{(n)} < 0$. □

The following lemma gives the increasing in time of the discrete solution.

**Lemma 8.** *Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (7)–(10). Then*

$$\delta_t U_i^{(n)} > 0, \quad 0 \leq i \leq I.$$

*Proof.* Consider the vector $Z_h^{(n)}$ such that $Z_i^{(n)}=\delta_t U_i^{(n)}$, $0 \leq i \leq I$. A straightforward calculation gives for $1 \leq i \leq I-1$,

$$\delta_t Z_i^{(n)} = \delta^2 Z_i^{(n)} - \delta_t (U_i^{(n)})^q \delta^0 U_i^{(n)} - (U_i^{(n)})^q \delta^0 Z_i^{(n)} + \delta_t (U_i^{(n)})^p,$$

$$\delta_t Z_0^{(n)} = \delta^2 Z_0^{(n)} + \delta_t (U_0^{(n)})^p,$$

$$\delta_t Z_I^{(n)} = \delta^2 Z_I^{(n)} + \delta_t (U_I^{(n)})^p.$$

Using Lemma 4 and the fact that $-\delta^0 U_h^{(n)} \geq 0$, we finally have for $1 \leq i \leq I-1$,

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + (U_i^{(n)})^q \delta^0 Z_i^{(n)} + (q(U_i^{(n)})^{q-1} \delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1}) Z_i^{(n)} \geq 0,$$

$$\delta_t Z_0^{(n)} - \delta^2 Z_0^{(n)} - p(U_0^{(n)})^{p-1} Z_0^{(n)} \geq 0,$$

$$\delta_t Z_I^{(n)} - \delta^2 Z_I^{(n)} - p(U_I^{(n)})^{p-1} Z_I^{(n)} \geq 0,$$

Knowing that $Z_h^{(0)} > 0$, from Lemma 3, we have $Z_h^{(n)} > 0$, which implies that $\delta_t U_i^{(n)} > 0$, $0 \leq i \leq I$. We have the wished result. □

The following lemma is a discrete generalization of the condition (6).

**Lemma 9.** *Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (7)–(10). Then*

$$(U_i^{(n)})^{p-1} > -\frac{p(p-1)}{q} h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2} \quad \textit{for} \quad 1 \leq i \leq I-1.$$

*Proof.* Consider the vectors $Z_h^{(n)}$, $K_h^{(n)}$ and $V_h^{(n)}$ such that $Z_i^{(n)} = K_i^{(n)} - V_i^{(n)}$ with $K_i^{(n)} = (U_i^{(n)})^{p-1}$ and $V_i^{(n)} = -\frac{p(p-1)}{q} h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2}$. We have for $1 \leq i \leq I-1$,

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + (K_i^{(n)})^q \delta^0 K_i^{(n)} - (V_i^{(n)})^q \delta^0 V_i^{(n)} - ((K_i^{(n)})^p - (V_i^{(n)})^p) = 0,$$

which is equivalent to

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + (K_i^{(n)})^q \delta^0 Z_i^{(n)} + (q(\mu_i^{(n)})^{q-1} \delta^0 V_i^{(n)} - p(\beta_i^{(n)})^{p-1}) Z_i^{(n)} = 0,$$

where $\mu_i^{(n)}$ and $\beta_i^{(n)}$ are intermediate values between $V_i^{(n)}$ and $K_i^{(n)}$. Knowing that $Z_h^{(0)} > 0$, from Lemma 3, we have $Z_h^{(n)} > 0$, which implies that $V_i^{(n)} < K_i^{(n)}$, $1 \leq i \leq I-1$, and we obtain the desired result. □

Now, let us give a property of the operator $\delta^2$ stated in the following lemma.

**Lemma 10.** *Let $U_h^{(n)}$, $n \geq 0$, be a sequence such that $U_h^{(n)} > 0$. Then, we have*

$$\delta^2 (U_i^{(n)})^p \geq p(U_i^{(n)})^{p-1} \delta^2 (U_i^{(n)}) \quad \textit{for} \quad 0 \leq i \leq I.$$

*Proof.* Applying Taylor's expansion, we obtain

$$\delta^2 (U_i^{(n)})^p = p(U_i^{(n)})^{p-1} \delta^2 U_i^{(n)} + (U_{i-1}^{(n)} - U_i^{(n)})^2 \frac{p(p-1)}{2h^2} (\theta_i^{(n)})^{p-2}$$

$$+(U_{i+1}^{(n)} - U_i^{(n)})^2 \frac{p(p-1)}{2h^2} (\xi_i^{(n)})^{p-2} \quad \textit{if} \quad 1 \leq i \leq I-1,$$

$$\delta^2 (U_0^{(n)})^p = p(U_0^{(n)})^{p-1} \delta^2 U_0^{(n)} + (U_1^{(n)} - U_0^{(n)})^2 \frac{p(p-1)}{2h^2} (\theta_0^{(n)})^{p-2},$$

$$\delta^2 (U_I^{(n)})^p = p(U_I^{(n)})^{p-1}\delta^2 U_I^{(n)} + (U_{I-1}^{(n)} - U_I^{(n)})^2 \frac{p(p-1)}{2h^2}(\theta_I^{(n)})^{p-2},$$

where $p \geq 2$,
$\theta_0^{(n)}$ is an intermediate value between $U_0^{(n)}$ and $U_1^{(n)}$,
$\theta_i^{(n)}$ is an intermediate value between $U_{i-1}^{(n)}$ and $U_i^{(n)}$, $1 \leq i \leq I-1$,
$\theta_I^{(n)}$ is an intermediate value between $U_{I-1}^{(n)}$ and $U_I^{(n)}$,
$\xi_i^{(n)}$ is an intermediate value between $U_i^{(n)}$ and $U_{i+1}^{(n)}$, $1 \leq i \leq I-1$.

The result follows taking into account the fact that $U_h^{(n)} > 0$. □

**Lemma 11.** *Let $U_h^{(n)}$, $n \geq 0$, be the solution of the discrete problem (7)–(10). Then, we have for $1 \leq i \leq I-1$,*

$$-U_i^{(n)}\delta^0 (U_i^{(n)})^p \geq -p(U_i^{(n)})^p \delta^0 U_i^{(n)} - p(p-1)h(\delta^0 U_i^{(n)})^2 U_i^{(n)} (U_{i-1}^{(n)})^{p-2}.$$

*Proof.* Using Taylor's expansion, we get for $1 \leq i \leq I-1$ and $p = 2$,

$$\delta^0 (U_i^{(n)})^p = p(U_{i-1}^{(n)})^{p-1}\delta^0 U_i^{(n)} + (U_{i+1}^{(n)} - U_{i-1}^{(n)})^2 \frac{p(p-1)}{4h}(U_{i-1}^{(n)})^{p-2},$$

for $1 \leq i \leq I-1$ and $p \geq 3$,

$$\delta^0 (U_i^{(n)})^p = p(U_{i-1}^{(n)})^{p-1}\delta^0 U_i^{(n)} + (U_{i+1}^{(n)} - U_{i-1}^{(n)})^2 \frac{p(p-1)}{4h}(U_{i-1}^{(n)})^{p-2} +$$

$$(U_{i+1}^{(n)} - U_{i-1}^{(n)})^3 \frac{p(p-1)(p-2)}{12h}(\zeta_i^{(n)})^{p-3},$$

where $\zeta_i^{(n)} \in (U_{i+1}^{(n)}, U_{i-1}^{(n)})$.

From Lemma 6 and the fact that $U_h^{(n)} > 0$, we obtain the desired result. □

## 3. Discrete Blow-Up Solutions

In this section under some assumptions, we show that the solution $U_h^{(n)}$ of the discrete problem (7)–(10) blows up in a finite time and estimate its numerical blow-up time.

**Theorem 12.** *Let $U_h^{(n)}$ be the solution of the discrete problem (7)–(10). Suppose that there exists a positive integer $\lambda \in (0,1)$ such that*

$$\delta^2\varphi_i - \varphi_i^q\delta^0\varphi_i + \varphi_i^p \geq \lambda\varphi_i^p,\ 0 \leq i \leq I. \tag{17}$$

*Then, the solution $U_h^{(n)}$ blows up in a finite time $T_h^{\Delta t}$ and we have the following estimate*

$$T_h^{\Delta t} \leq \frac{\tau(1+\tau')^{p-1}}{\|\varphi_h\|_\infty^{p-1}((1+\tau')^{p-1}-1)},$$

*where*

$$\tau' = \lambda\min\{\frac{h^2\|\varphi_h\|_\infty^{p-1}}{2},\tau\},\quad 0<\tau<1.$$

*Proof.* Introduce the vector $J_h^{(n)}$ defined as follows

$$J_i^{(n)} = \delta_t U_i^{(n)} - \lambda(U_i^{(n)})^p,\ \ 0 \leq i \leq I,\quad n \geq 0. \tag{18}$$

A straightforward calculation gives for $1 \leq i \leq I-1$,

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + (U_i^{(n)})^q\delta^0 J_i^{(n)} + (q(U_i^{(n)})^{q-1}\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1})J_i^{(n)}$$

$$\geq (1-\lambda)\delta_t(U_i^{(n)})^p + \lambda\delta^2(U_i^{(n)})^p - \lambda(U_i^{(n)})^q\delta^0(U_i^{(n)})^p$$

$$-\lambda q(U_i^{(n)})^{q-1}(U_i^{(n)})^p\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1}\delta_t U_i^{(n)} + \lambda p(U_i^{(n)})^{p-1}(U_i^{(n)})^p,$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1}J_0^{(n)} = (1-\lambda)\delta_t(U_0^{(n)})^p$$

$$+\lambda\delta^2(U_0^{(n)})^p - p(U_0^{(n)})^{p-1}\delta_t U_0^{(n)} + \lambda p(U_0^{(n)})^{p-1}(U_0^{(n)})^p,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1}J_I^{(n)} = (1-\lambda)\delta_t(U_I^{(n)})^p$$

$$+\lambda\delta^2(U_I^{(n)})^p - p(U_I^{(n)})^{p-1}\delta_t U_I^{(n)} + \lambda p(U_I^{(n)})^{p-1}(U_I^{(n)})^p.$$

From Lemma 4, $\delta_t(U_i^{(n)})^p \geq p(U_i^{(n)})^{p-1}\delta_t U_i^{(n)}$, $0 \leq i \leq I$ and the fact that $0<\lambda<1$, we have $(1-\lambda)\delta_t(U_i^{(n)})^p \geq (1-\lambda)p(U_i^{(n)})^{p-1}\delta_t U_i^{(n)}$, $0 \leq i \leq I$.

Using also Lemma 10, $\delta^2(U_i^{(n)})^p \geq p(U_i^{(n)})^{p-1}\delta^2 U_i^{(n)}$, $0 \leq i \leq I$, we get for $1 \leq i \leq I-1$,

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + (U_i^{(n)})^q \delta^0 J_i^{(n)} + (q(U_i^{(n)})^{q-1}\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1})J_i^{(n)}$$

$$\geq \lambda p(U_i^{(n)})^{p-1}(-\delta_t U_i^{(n)} + \delta^2 U_i^{(n)} + (U_i^{(n)})^p) - \lambda(U_i^{(n)})^q \delta^0 (U_i^{(n)})^p$$

$$-\lambda q(U_i^{(n)})^{q-1}(U_i^{(n)})^p \delta^0 U_i^{(n)},$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1} J_0^{(n)} \geq \lambda p(U_0^{(n)})^{p-1}(-\delta_t U_0^{(n)} + \delta^2 U_0^{(n)} + (U_0^{(n)})^p),$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1} J_I^{(n)} \geq \lambda p(U_I^{(n)})^{p-1}(-\delta_t U_I^{(n)} + \delta^2 U_I^{(n)} + (U_I^{(n)})^p).$$

Using (7)–(9), we have these equalities

$$-\delta_t U_i^{(n)} + \delta^2 U_i^{(n)} + (U_i^{(n)})^p = (U_i^{(n)})^q \delta^0 U_i^{(n)}, \ 1 \leq i \leq I-1,$$

$$-\delta_t U_0^{(n)} + \delta^2 U_0^{(n)} + (U_0^{(n)})^p = 0,$$

$$-\delta_t U_I^{(n)} + \delta^2 U_I^{(n)} + (U_I^{(n)})^p = 0.$$

We arrive at, for $1 \leq i \leq I-1$,

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + (U_i^{(n)})^q \delta^0 J_i^{(n)} + (q(U_i^{(n)})^{q-1}\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1})J_i^{(n)}$$

$$\geq \lambda(U_i^{(n)})^{q-1}(p(U_i^{(n)})^p \delta^0 U_i^{(n)} - U_i^{(n)} \delta^0 (U_i^{(n)})^p - q(U_i^{(n)})^p \delta^0 U_i^{(n)}),$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1} J_0^{(n)} \geq 0,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1} J_I^{(n)} \geq 0.$$

Using Lemma 11, for $1 \leq i \leq I-1$,

$$-U_i^{(n)} \delta^0 (U_i^{(n)})^p \geq -p(U_i^{(n)})^p \delta^0 U_i^{(n)} - p(p-1)h(\delta^0 U_i^{(n)})^2 U_i^{(n)} (U_{i-1}^{(n)})^{p-2},$$

we have

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + (U_i^{(n)})^q \delta^0 J_i^{(n)} + (q(U_i^{(n)})^{q-1}\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1})J_i^{(n)} \geq$$

$$-\lambda (U_i^{(n)})^q \delta^0 U_i^{(n)} (q(U_i^{(n)})^{p-1} + p(p-1)h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2}),\ 1 \leq i \leq I-1,$$

$$\delta_t J_0^{(n)} - \delta^2 J_0^{(n)} - p(U_0^{(n)})^{p-1} J_0^{(n)} \geq 0,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} - p(U_I^{(n)})^{p-1} J_I^{(n)} \geq 0.$$

From Lemma 9, $(U_i^{(n)})^{p-1} > -\frac{p(p-1)}{q} h(\delta^0 U_i^{(n)})(U_{i-1}^{(n)})^{p-2},\ 1 \leq i \leq I-1$, and thanks to $-\lambda (U_h^{(n)})^q \delta^0 U_h^{(n)} \geq 0$, we get

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} + (U_i^{(n)})^q \delta^0 J_i^{(n)} + (q(U_i^{(n)})^{q-1}\delta^0 U_i^{(n)} - p(U_i^{(n)})^{p-1})J_i^{(n)} \geq 0, \quad 1 \leq i \leq I-1,$$

$$\delta_t J_0^{(n)} - \frac{(2J_1^{(n)} - 2J_0^{(n)})}{h^2} - p(U_0^{(n)})^{p-1} J_0^{(n)} \geq 0,$$

$$\delta_t J_I^{(n)} - \frac{(2J_{I-1}^{(n)} - 2J_I^{(n)})}{h^2} - p(U_I^{(n)})^{p-1} J_I \geq 0.$$

From (17), we observe that

$$J_i^{(0)} = \delta^2 \varphi_i - \varphi_i^q \delta^0 \varphi_i + \varphi_i^p - \lambda \varphi_i^p \geq 0,\ 0 \leq i \leq I.$$

We deduce from Lemma 3 that $J_h^{(n)} \geq 0$ for $n \geq 0$, which implies that

$$\delta_t U_i^{(n)} \geq \lambda (U_i^{(n)})^p, \quad 0 \leq i \leq I, \tag{19}$$

which is equivalent to

$$U_i^{(n+1)} \geq U_i^{(n)} + \lambda \Delta t_n (U_i^{(n)})^p,\ 0 \leq i \leq I,$$

Therefore,

$$U_i^{(n+1)} \geq U_i^{(n)}(1 + \lambda \Delta t_n (U_i^{(n)})^{p-1}),\ 0 \leq i \leq I, \tag{20}$$

which implies that

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty (1 + \lambda \Delta t_n \|U_h^{(n)}\|_\infty^{p-1}).$$

From Lemma 8, $\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty$. By induction, we obtain

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty = \|\varphi_h\|_\infty.$$

Then, we have

$$\|U_h^{(n)}\|_\infty^{p-1} \geq \|\varphi_h\|_\infty^{p-1},$$

and with $\lambda \Delta t_n \|U_h^{(n)}\|_\infty^{p-1} \geq \tau^{'}$, we arrive at

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty (1 + \tau^{'}).$$

By induction, we get

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty (1 + \tau^{'})^n, \ n \geq 0,$$

which leads us to

$$\|U_h^{(n)}\|_\infty \geq \|\varphi_h\|_\infty (1 + \tau^{'})^n, \ n \geq 0.$$

Since the term on the right hand side of the above inequality tends to infinity as $n$ approaches infinity, we conclude that $\|U_h^{(n)}\|_\infty$ tends to infinity. Now, let us estimate the numerical blow-up time. It is not hard to see that

$$\sum_{n=0}^{+\infty} \Delta t_n \leq \sum_{n=0}^{+\infty} \frac{\tau}{\|U_h^{(n)}\|_\infty^{p-1}} \leq \frac{\tau}{\|\varphi_h\|_\infty^{p-1}} \sum_{n=0}^{+\infty} (\frac{1}{(1+\tau^{'})^{p-1}})^n.$$

Using the fact that the series $\frac{\tau}{\|\varphi_h\|_\infty^{p-1}} \sum_{n=0}^{+\infty} (\frac{1}{(1+\tau^{'})^{p-1}})^n$ converges towards

$$\frac{\tau(1+\tau^{'})^{p-1}}{\|\varphi_h\|_\infty^{p-1}((1+\tau^{'})^{p-1} - 1)},$$

we deduce that

$$T_h^{\Delta t} = \sum_{n=0}^{+\infty} \Delta t_n \leq \frac{\tau(1+\tau^{'})^{p-1}}{\|\varphi_h\|_\infty^{p-1}((1+\tau^{'})^{p-1} - 1)},$$

and we conclude the proof. □

**Remark 13.** Using Taylor's expansion, we get

$$(1+\tau^{'})^{p-1} = 1 + (p-1)\tau^{'} + o(\tau^{'}),$$

which implies that

$$\frac{\tau}{(1+\tau')^{p-1}-1} = \frac{\tau}{\tau'}\frac{1}{(p-1+o(1))} \leq \frac{2\tau}{\tau'(p-1)}.$$

If we take $\tau = \frac{h^2}{2}$, we have

$$\frac{\tau^{'}}{\tau} = \lambda \min\{\frac{\|\varphi_h\|_\infty^{p-1}}{2}, 1\},$$

and therefore

$$\frac{\tau}{\tau'} = \frac{1}{\lambda}\min\{2\|\varphi_h\|_\infty^{1-p}, 1\}.$$

Then

$$\frac{\tau}{(1+\tau')^{p-1}-1} \leq \frac{2\tau}{\tau'(p-1)} = \frac{2}{\lambda(p-1)}\min\{2\|\varphi_h\|_\infty^{1-p}, 1\}.$$

We conclude that $\frac{\tau}{(1+\tau^{'})^{p-1}-1}$ is bounded.

**Remark 14.** From

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty (1+\tau^{'}),$$

we get

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(k)}\|_\infty (1+\tau^{'})^{n-k} \quad \text{for} \quad n \geq k,$$

which implies that

$$\sum_{n=k}^{+\infty} \Delta t_n \leq \frac{\tau}{\|U_h^{(k)}\|_\infty^{p-1}} \sum_{n=k}^{+\infty} (\frac{1}{(1+\tau')^{p-1}})^{n-k}.$$

We deduce that

$$T_h^{\Delta t} - t_k \leq \frac{\tau}{\|U_h^{(k)}\|_\infty^{p-1}} \frac{(1+\tau^{'})^{p-1}}{(1+\tau')^{p-1}-1} \quad \text{with} \quad \Delta t_k = \sum_{j=0}^{k-1} \Delta t_j.$$

In the sequel, we take $\tau = \frac{h^2}{2}$.

## 4. Convergence of the Numerical Blow-Up Time

In this section, under some assumptions, we show that the discrete solution blows up in a finite time and its numerical blow-up time converges to the real one when the mesh size goes to zero. In order to obtain the convergence of the numerical blow-up time, we firstly prove the following theorem about the convergence of the discrete scheme.

**Theorem 15.** *Assume that the continuous problem (1)-(3) has a solution* $u \in C^{4,2}([0,1] \times [0,T])$ *and the initial condition at (10) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \text{ as } h \to 0. \tag{21}$$

*Then, for h sufficiently small, the discrete problem (7)–(10) has a solution* $U_h^{(n)}$, $0 \leq n \leq J$, *and we have the following relation*

$$\max_{0 \leq n \leq J} \|U_h^{(n)} - u_h(t_n)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2) \text{ as } h \to 0. \tag{22}$$

*where* $J$ *is such that* $\sum_{j=0}^{J-1} \Delta t_j \leq T$ *and* $t_n = \sum_{j=0}^{n-1} \Delta t_j$.

*Proof.* For each $h$, the discrete problem (7)–(10) has a solution $U_h^{(n)}$. Let $N \leq J$, the greatest value of $n$ such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < 1 \text{ for } n < N. \tag{23}$$

We know that $N \geq 1$ because of (21). Due to the fact $u \in C^{4,2}([0,1] \times [0,T])$, there exists a positive constant $K$ such that $\|u\|_\infty \leq K$. Using the triangle inequality, we have

$$\|U_h^{(n)}\|_\infty \leq \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \leq K + 1, \; n < N. \tag{24}$$

Since $u \in C^{4,2}([0,1] \times [0,T])$. Applying Taylor's expansion, we obtain

$$\delta_t u(x_i, t_n) - \delta^2 u(x_i, t_n) + u^q(x_i, t_n)\delta^0 u(x_i, t_n) - u^p(x_i, t_n) = \frac{h^2}{6} u^q(x_i, t_n) u_{xxx}(\widetilde{x}_i, t_n) + \frac{\Delta t_n}{2} u_{tt}(x_i, \widetilde{t}_n), \; 1 \leq i \leq I-1,$$

$$\delta_t u(x_0, t_n) - \delta^2 u(x_0, t_n) - u^p(x_0, t_n) = -\frac{h^2}{12} u_{xxxx}(\widetilde{x}_0, t_n) + \frac{\Delta t_n}{2} u_{tt}(x_0, \widetilde{t}_n),$$

$$\delta_t u(x_I,t_n) - \delta^2 u(x_I,t_n) - u^p(x_I,t_n) = -\frac{h^2}{12}u_{xxxx}(\widetilde{x}_I,t_n) + \frac{\Delta t_n}{2}u_{tt}(x_I,\widetilde{t}_n),$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization, for $n < N$,

$$\begin{aligned}\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} + u^q(x_i,t_n)\delta^0 e_i^{(n)} + (q(\mu_i^{(n)})^{q-1}\delta^0 u(x_i,t_n) - p(\beta_i^{(n)})^{p-1})e_i^{(n)} \\ = -\frac{h^2}{6}u^q(x_i,t_n)u_{xxx}(\widetilde{x}_i,t_n) - \frac{\Delta t_n}{2}u_{tt}(x_i,\widetilde{t}_n),\ 1 \le i \le I-1,\end{aligned}$$

$$\delta_t e_0^{(n)} - \frac{(2e_1^{(n)} - 2e_0^{(n)})}{h^2} - p(\beta_0^{(n)})^{p-1}e_0^{(n)} = \frac{h^2}{12}u_{xxxx}(\widetilde{x}_0,t_n) - \frac{\Delta t_n}{2}u_{tt}(x_0,\widetilde{t}_n),$$

$$\delta_t e_I^{(n)} - \frac{(2e_{I-1}^{(n)} - 2e_I^{(n)})}{h^2} - p(\beta_I^{(n)})^{p-1}e_I^{(n)} = \frac{h^2}{12}u_{xxxx}(\widetilde{x}_I,t_n) - \frac{\Delta t_n}{2}u_{tt}(x_I,\widetilde{t}_n),$$

where $\beta_i^{(n)}$ and $\mu_i^{(n)}$ are intermediate values between $U_i^{(n)}$ and $u(x_i,t_n)$, for $i \in \{0,...,I\}$. Since $u_{xxx}(x,t)$, $u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded and $\Delta t_n = O(h^2)$, then there exists a positive constant $Q > 0$ such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} + u^q(x_i,t_n)\delta^0 e_i^{(n)} \le c_i^{(n)}e_i^{(n)} + Qh^2,\ 1 \le i \le I-1, \tag{25}$$

$$\delta_t e_0^{(n)} - \frac{(2e_1^{(n)} - 2e_0^{(n)})}{h^2} \le c_0^{(n)}e_0^{(n)} + Qh^2, \tag{26}$$

$$\delta_t e_I^{(n)} - \frac{(2e_{I-1}^{(n)} - 2e_I^{(n)})}{h^2} \le c_I^{(n)}e_I^{(n)} + Qh^2, \tag{27}$$

where

$$c_i^{(n)} = p(\beta_i^{(n)})^{p-1} - q(\mu_i^{(n)})^{q-1}\delta^0 u(x_i,t_n),\ 1 \le i \le I-1,$$

$$c_0^{(n)} = p(\beta_0^{(n)})^{p-1},\ c_I^{(n)} = p(\beta_I^{(n)})^{p-1}.$$

Set $L = \max_{0 \le i \le I}\{c_i^{(n)}\}$ and introduce the vector $Z_h^{(n)}$ defined as follows

$$Z_i^{(n)} = e^{(L+1)t_n}(\|\varphi_h - u_h(0)\| + Qh^2),\ 0 \le i \le I,\ n < N.$$

A straightforward computations reveals that

$$\delta_t Z_i^{(n)} - \delta^2 Z_i^{(n)} + u^q(x_i, t_n)\delta^0 Z_i^{(n)} > c_i^{(n)} Z_i^{(n)} + Qh^2,\ 1 \le i \le I-1, \tag{28}$$

$$\delta_t Z_0^{(n)} - \frac{(2Z_1^{(n)} - 2Z_0^{(n)})}{h^2} > c_0^{(n)} Z_0^{(n)} + Qh^2, \tag{29}$$

$$\delta_t Z_I^{(n)} - \frac{(2Z_{I-1}^{(n)} - 2Z_I^{(n)})}{h^2} > c_I^{(n)} Z_I^{(n)} + Qh^2, \tag{30}$$

$$Z_i^{(0)} > e_i^{(0)}, \quad 0 \le i \le I. \tag{31}$$

It follows from Lemma 5 that

$$Z_i^{(n)} > e_i^{(n)}, \quad 0 \le i \le I.$$

By the same way, we also prove that

$$Z_i^{(n)} > -e_i^{(n)}, \quad 0 \le i \le I,$$

which implies that

$$Z_i^{(n)} > |e_i^{(n)}|, \quad 0 \le i \le I.$$

We deduce that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \le e^{(L+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Qh^2),\ n < N. \tag{32}$$

Now, let us show that $N = J$. Suppose that $N < J$. If we replace $n$ by $N$ in (32), and taking into account the inequality (23), we obtain

$$1 \le \|U_h^{(N)} - u_h(t_N)\|_\infty \le e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + Qh^2). \tag{33}$$

Since $e^{(L+1)T}(\|\varphi_h - u_h(0)\|_\infty + Qh^2) \to 0$ as $h \to 0$, we deduce from (33) that $1 \le 0$, which is impossible. Consequently $N = J$, and we conclude the proof. □

**Theorem 16.** *Suppose that the solution $u$ of the continuous problem (1)–(3) blows up in a finite time $T_b$ such that $u \in C^{4,2}([0,1] \times [0,T_b),\mathbb{R})$ and the initial condition at* (10) *satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \quad as \quad h \to 0. \tag{34}$$

*Under the assumptions of Theorem 12, the discrete problem (7)–(10) has a solution $U_h^{(n)}$ which blows up in a finite time $T_h^{\Delta t}$ and the following relation holds*

$$\lim_{h\to 0} T_h^{\Delta t} = T_b. \tag{35}$$

*Proof.* Remark 13 allows us to say that $\frac{\tau}{(1+\tau')^{p-1}-1}$ is bounded. Letting $0<\varepsilon<\frac{T_b}{2}$. Then, there exists a constant $R>0$ such that

$$\frac{\tau y^{1-p}}{(1+\tau')^{p-1}-1} < \frac{\varepsilon}{2} \quad \text{for} \quad y \in [R,\infty). \tag{36}$$

Since $u$ blows up at the time $T_b$. There exists $T_1 \in (T_b - \frac{\varepsilon}{2}, T_b)$ and $h_0(\varepsilon) > 0$ such that

$$\|u(.,t_n)\|_\infty \geq 2R \quad \text{for} \quad t_n \in [T_1,T_b), \quad h \leq h_0(\varepsilon).$$

Let $T_2 = \frac{T_1+T_b}{2}$ and $k$ be a positive integer such that

$$t_k = \sum_{n=0}^{k-1} \Delta t_n \in [T_1,T_2] \quad \text{for} \quad h \leq h_0(\varepsilon).$$

We have

$$0 < \|u_h(t_n)\|_\infty < \infty \quad \text{for} \quad n \leq k,\ h \leq h_0(\varepsilon).$$

It follows from Theorem 15 that the discrete problem (7)–(10) has a solution $U_h^{(n)}$, which verifies

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < R \quad \text{for} \quad n \leq k,\ h \leq h_0(\varepsilon),$$

which implies

$$\|U_h^{(n)}\|_\infty \geq \|u_h(t_n)\|_\infty - \|U_h^{(n)} - u_h(t_n)\|_\infty \geq R,\ h \leq h_0(\varepsilon).$$

From Theorem 12, $U_h^{(n)}$ blows up at the time $T_h^{\Delta t}$. It follows from Remark 14 and (36) that

$$|T_h^{\Delta_t} - t_k| \leq \frac{\tau(1+\tau')^{p-1}\|U_h^{(k)}\|_\infty^{1-p}}{(1+\tau')^{p-1}-1} < \frac{\varepsilon}{2},$$

because, we have $\|U_h^{(k)}\|_\infty \geq R$ for $h \leq h_0(\varepsilon)$. We deduce that for $h \leq h_0(\varepsilon)$,

$$|T_h^{\Delta_t} - T_b| \leq |T_h^{\Delta_t} - t_k| + |t_k - T_b| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

and the proof is complete. □

## 5. Numerical Experiments

In this section, we present some numerical approximations to the blow-up time of the problem (1)–(3). We use the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} - (U_i^{(n)})^q\left(\frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h}\right) + (U_i^{(n)})^p, \quad 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (U_I^{(n)})^p,$$

$$U_i^{(0)} > 0, \quad 0 \leq i \leq I,$$

where $n \geq 0$, $q \geq 1$, $p \geq q+1$, $\Delta t_n = \min(\frac{h^2}{2}, \tau\|U_h^{(n)}\|_\infty^{1-p})$ with $\tau = const \in (0,1)$.

Also we use the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} - (U_i^{(n)})^q\left(\frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h}\right) + (U_i^{(n)})^p, \quad 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + (U_I^{(n)})^p,$$

$$U_i^{(0)} > 0, \quad 0 \leq i \leq I,$$

where $n \geq 0$, $q \geq 1$, $p \geq q+1$, $\Delta t_n = \tau \|U_h^{(n)}\|_\infty^{1-p}$ with $\tau = const \in (0,1)$.

In Tables 1-6, in rows, we present the numerical blow-up times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. In Tables 7-9, we compare the numerical blow-up times when $q$ goes to $p$ ($q < p$). The numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = |T^{n+1} - T^n| \leq 10^{-16}$. The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

**First case:** $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $q = 2$, $p = 3$ and $\tau = \frac{h^2}{2}$.

**Table 1:** Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| $I$ | $T^n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.501465 | 7993 | - | - |
| 32 | 0.500366 | 30528 | - | - |
| 64 | 0.500092 | 116410 | 1 | 2.00 |
| 128 | 0.500023 | 442904 | 7 | 2.00 |
| 256 | 0.500006 | 1680740 | 50 | 2.00 |
| 512 | 0.500001 | 6359533 | 378 | 2.00 |
| 1024 | 0.500000 | 23984472 | 2839 | 2.00 |

**Table 2 :** Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

| $I$ | $T^n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.501465 | 7993 | - | - |
| 32 | 0.500366 | 30528 | - | - |
| 64 | 0.500092 | 116410 | 2 | 2.00 |
| 128 | 0.500023 | 442904 | 13 | 2.00 |
| 256 | 0.500006 | 1680740 | 94 | 2.00 |
| 512 | 0.500001 | 6359533 | 713 | 2.00 |
| 1024 | 0.500000 | 23984472 | 14739 | 2.00 |

**Second case:** $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $q=2$, $p=3$ and $\tau = \frac{h^2}{2}$.

**Table 3 :** Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| $I$ | $T^n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.934333 | 8348 | - | - |
| 32 | 0.931817 | 31818 | - | - |
| 64 | 0.931188 | 120039 | 1 | 1.99 |
| 128 | 0.931031 | 457418 | 7 | 1.99 |
| 256 | 0.930991 | 1738790 | 52 | 1.99 |
| 512 | 0.930981 | 6591727 | 392 | 2.00 |
| 1024 | 0.930979 | 24913253 | 2947 | 2.00 |

**Table 4 :** Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

| $I$ | $T^n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 0.933252 | 8277 | - | - |
| 32 | 0.931546 | 31667 | - | - |
| 64 | 0.931120 | 119708 | 2 | 2.00 |
| 128 | 0.931014 | 456095 | 13 | 2.00 |
| 256 | 0.930987 | 1733502 | 96 | 2.00 |
| 512 | 0.930980 | 6570571 | 720 | 2.00 |
| 1024 | 0.930979 | 24828622 | 5490 | 2.00 |

**Third case:** $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $q=2$, $p=3$ and $\tau = \frac{h^2}{2}$.

**Table 5 :** Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

| $I$ | $T^n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 1.1480319 | 8284 | - | - |
| 32 | 1.477444 | 31675 | - | - |
| 64 | 1.476725 | 121572 | 1 | 1.99 |
| 128 | 1.476545 | 463547 | 7 | 1.99 |
| 256 | 1.476500 | 1763309 | 53 | 1.99 |
| 512 | 1.476489 | 6689804 | 398 | 2.00 |
| 1024 | 1.476486 | 25305558 | 2997 | 2.00 |

**Table 6 :** Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

| $I$ | $T^n$ | $n$ | $CPU\ time$ | $s$ |
|---|---|---|---|---|
| 16 | 1.479194 | 8181 | - | - |
| 32 | 1.477160 | 30949 | - | - |
| 64 | 1.476654 | 118612 | 2 | 2.00 |
| 128 | 1.476528 | 451706 | 13 | 2.00 |
| 256 | 1.476496 | 1715944 | 94 | 2.00 |
| 512 | 1.476488 | 6500343 | 712 | 2.00 |
| 1024 | 1.476486 | 24547716 | 5436 | 2.00 |

**Comparison of the numerical blow-up times when $q$ goes to $p$ ($q < p$)**

**Table 7 :** $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $p = 4$ and $\tau = \frac{h^2}{2}$.

| | Explicit | | | Implicit | | |
|---|---|---|---|---|---|---|
| | $q=1$ | $q=2$ | $q=3$ | $q=1$ | $q=2$ | $q=3$ |
| $I$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ |
| 16 | 0.041830 | 0.041830 | 0.041830 | 0.041830 | 0.041830 | 0.041830 |
| 32 | 0.041707 | 0.041707 | 0.041707 | 0.041707 | 0.041707 | 0.041707 |
| 64 | 0.041677 | 0.041677 | 0.041677 | 0.041677 | 0.041677 | 0.041677 |
| 128 | 0.041669 | 0.041669 | 0.041669 | 0.041669 | 0.041669 | 0.041669 |
| 256 | 0.041667 | 0.041667 | 0.041667 | 0.041667 | 0.041667 | 0.041667 |
| 512 | 0.041667 | 0.041667 | 0.041667 | 0.041667 | 0.041667 | 0.041667 |
| 1024 | 0.041667 | 0.041667 | 0.041667 | 0.041669 | 0.041669 | 0.041669 |

**Table 8 :** $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $p=4$ and $\tau = \frac{h^2}{2}$.

| | Explicit | | | Implicit | | |
|---|---|---|---|---|---|---|
| | $q=1$ | $q=2$ | $q=3$ | $q=1$ | $q=2$ | $q=3$ |
| $I$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ |
| 16 | 0.437081 | 0.434179 | 0.424625 | 0.436323 | 0.433243 | 0.423016 |
| 32 | 0.435522 | 0.432479 | 0.423296 | 0.435333 | 0.432245 | 0.423016 |
| 64 | 0.435133 | 0.432054 | 0.422849 | 0.435085 | 0.431996 | 0.422849 |
| 128 | 0.435133 | 0.431948 | 0.422733 | 0.435023 | 0.431933 | 0.422715 |
| 256 | 0.435011 | 0.431921 | 0.422703 | 0.435008 | 0.431918 | 0.422695 |
| 512 | 0.435005 | 0.431915 | 0.422696 | 0.435004 | 0.431914 | 0.422696 |
| 1024 | 0.435003 | 0.431913 | 0.422694 | 0.435003 | 0.431913 | 0.422694 |

**Table 9 :** $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $p=4$ and $\tau = \frac{h^2}{2}$.

| | Explicit | | | Implicit | | |
|---|---|---|---|---|---|---|
| | $q=1$ | $q=2$ | $q=3$ | $q=1$ | $q=2$ | $q=3$ |
| $I$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ | $T^n$ |
| 16 | 1.504081 | 1.687438 | 1.808074 | 1.502383 | 1.685276 | 1.805737 |
| 32 | 1.500052 | 1.684238 | 1.805591 | 1.499623 | 1.683688 | 1.804992 |
| 64 | 1.499043 | 1.683437 | 1.804970 | 1.498936 | 1.683299 | 1.804819 |
| 128 | 1.498791 | 1.683237 | 1.804815 | 1.498764 | 1.683202 | 1.804777 |
| 256 | 1.498728 | 1.683187 | 1.804776 | 1.498721 | 1.683178 | 1.804766 |
| 512 | 1.498712 | 1.683174 | 1.804766 | 1.498710 | 1.683172 | 1.804764 |
| 1024 | 1.498708 | 1.683171 | 1.804764 | 1.498708 | 1.683170 | 1.804763 |

**Remark 17.** We observe that, the solution of our problem blows up in a

finite time for all $p \geq q+1$ such that $q \geq 1$.

**First case:** the initial data is the constant $(\frac{1}{2})^{3-p}$.

When $q$ approaches $p$ $(q < p)$, the blow-up is global. In this case, the convection term, which is null, has no turbulence effect on the blow-up created by the reaction term. (No turbulent blow-up).

**Second case:** the initial data is $(\frac{1}{2})^{6-p} + (1-(ih)^2)^2$. When $q$ approaches p $(q < p)$, the blow-up is local with a blow-up time more and more small. The convection term accelerates the blow-up created by the reaction term. (No turbulent blow-up).

**Third case:** the initial data is $(\frac{1}{2})^{30-p} + (1-(ih)^2)^2$.

When $q$ approaches $p$ $(q < p)$, the blow-up is local with a blow-up time more and more greater. The convection term, responsible of the turbulence, delays the blow-up created by the reaction term. (Turbulent blow-up).

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where $I = 16$, $q = 2$ and $p = 3$. In Figures 1 and 2 we can appreciate that the discrete solution blows up globally in a finite time where the initial data is a constant. In Figures 3, 4, 5 and 6, we see that the blow-up is local when the initial data is not a constant. Figures 7, 8, 9, 10, 11 and 12 show the effect of the convection term on the evolution of the solution. In Figures 13, 14, 15, 16, 17 and 18, we observe that the solution of our problem blows up in a finite time, when the initial data is $(\frac{1}{2})^{3-p}$, $(\frac{1}{2})^{6-p} + (1-(ih)^2)^2$ or $(\frac{1}{2})^{30-p} + (1-(ih)^2)^2$ with $i \in \{0, ..., I\}$.

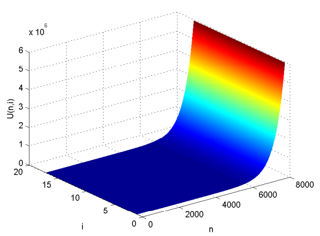


Figure 1: Evolution of the discrete solution (explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $q = 2$, $p = 3$.

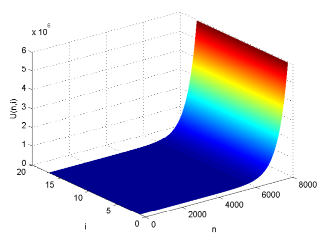


Figure 2: Evolution of the discrete solution (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $q = 2$, $p = 3$.

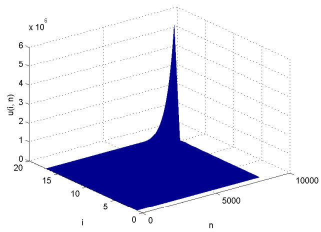


Figure 3: Evolution of the discrete solution with(explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

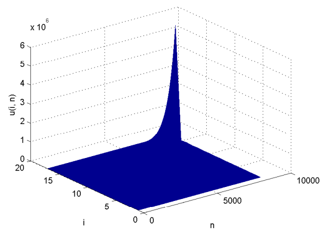


Figure 4: Evolution of the discrete solution (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

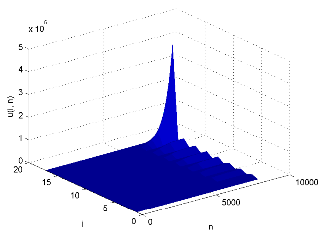


Figure 5: Evolution of the discrete solution with(explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

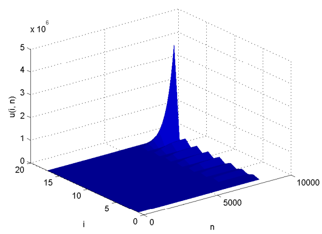


Figure 6: Evolution of the discrete solution (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

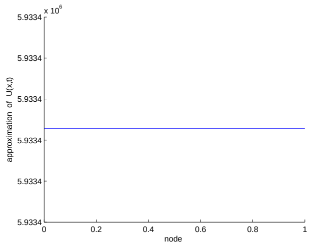


Figure 7: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $q = 2$, $p = 3$.

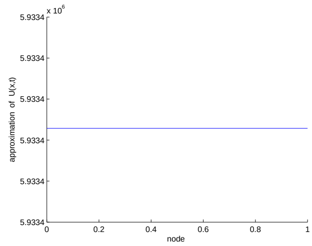


Figure 8: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $q = 2$, $p = 3$.

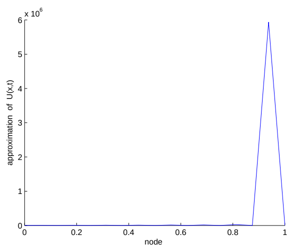


Figure 9: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

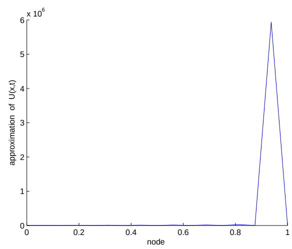


Figure 10: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

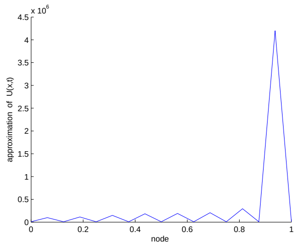


Figure 11: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $q=2$, $p=3$.

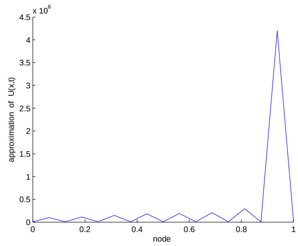


Figure 12: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $q=2$, $p=3$.

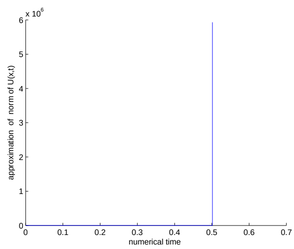


Figure 13: Evolution of norm of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $q=2$, $p=3$.

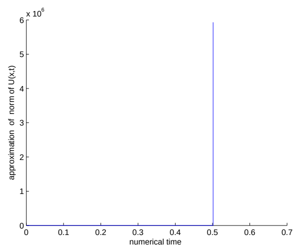


Figure 14: Evolution of norm of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{3-p}$, $q=2$, $p=3$.

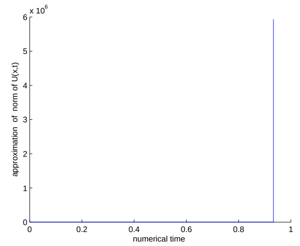


Figure 15: Evolution of norm of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

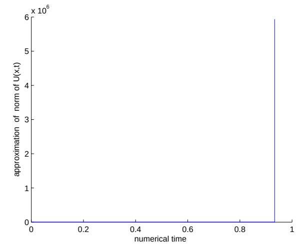


Figure 16: Evolution of norm of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{6-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

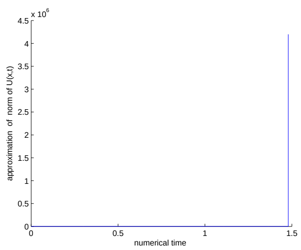


Figure 17: Evolution of norm of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

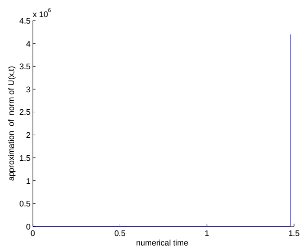


Figure 18: Evolution of norm of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = (\frac{1}{2})^{30-p} + (1-(ih)^2)^2$, $q = 2$, $p = 3$.

## References


[1] L.M. Abia, J.C. López-Marcos and J. Martinez, On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, *Appl. Numer. Math.*, **26** (1998), 399-414.

[2] J. Bec, K. Khanin, Burgers turbulence, *Physics Reports*, **447** (2007), 1-66.

[3] J. von Below, An existence result for semilinear parabolic network equations with dynamical node conditions, In: *Progress in Partial Differential Equations: Elliptic and Parabolic Problems*, Pitman Research Notes in Math. Ser. Longman Harlow Essex, # **266** (1992), 274-283.

[4] J. von Below, *Parabolic Network Equations*, Tubingen, 2 Ed. (1994).

[5] J. von Below, C. De Coster, A qualitative theory for parabolic problems under dynamical boundary conditions, *J. of Inequalities and Applications*, **5** (2000), 467-486.

[6] J. von Below, S. Nicaise, Dynamical interface transition in ramified media with diffusion, *Comm. Partial Differential Equations*, **21** (1996), 255-279.

[7] J. von Below, G. Pincet-Mailly, Blow up for reaction diffusion equations under dynamical boundary conditions, *Communications in Partial Differential Equations*, **28** (2003), 223-247.

[8] J. von Below, G. Pincet-Mailly, Blow-up for some nonlinear parabolic problems with convection under dynamical boundary conditions, *Discrete and Continuous Dynamical Systems*, Supplement Vol. (2007), 1031-1041.

[9] J. von Below, G. Pincet-Mailly, J-F. Rault, Growth order and blow-up points for the parabolic Burgers' equation under dynamical boundary conditions, *Discrete Contin. Dyn. Syst. Ser. S.*, **6**, No 3 (2013), 825-836.

[10] T.K. Boni, H. Nachid and D. Nabongo, Blow-up for discretization of a localized semilinear heat equation, *Annals of the Alexandru Ioan Cuza University-Mathematics*, **56**, No 1 (2010), 385-406.

[11] S. Chen, Global existence and blow-up of solutions for a parabolic equation with a gradient term, *Proc. Amer. Math. Soc.*, **129** (2001), 975-981.

[12] M. Chipot, F.B. Weissler, Some blow-up results for a nonlinear parabolic equation with a gradient term, *SIAM J. Math. Anal.*, **20** (1989), 886-907.

[13] M. Chlebik, M. Fila, P. Quittner, Blow-up of positive solutions of a semilinear parabolic equation with a gradient term, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **10** (2003), 525-537.

[14] A. Constantin, J. Escher, Global existence for fully parabolic boundary value problems, *NoDEA* , **13** (2006), 91-118.

[15] J. Ding, Blow-up solutions for a class of nonlinear parabolic equations with Dirichlet boundary conditions, *Nonlinear Anal.*, **52** (2003), 1645-1654.

[16] J. Ding, B.-Z. Guo, Global and blow-up solutions for nonlinear Parabolic equations with a gradient term, *Houston Journal of Mathematics*, **37**, No 4 (2011), 1265-1277.

[17] J. Ding, B.-Z. Guo, Global existence and blow-up solutions for quasilinear reaction-diffusion equations with a gradient term, *Applied Mathematics Letters*, **24** (2011), 936-942.

[18] M. Fila, Remarks on blow-up for a nonlinear parabolic equation with a gradient term, *Proc. Amer. Math. Soc.*, **111** (1991), 795-801.

[19] U. Frisch, Turbulence: *The Legacy of A.N. Kolmogorov*, Cambridge University Press (1995).

[20] G.R. Goldstein, Derivation and physical interpretation of general boundary conditions, *Adv. in Diff. Equ.*, **11** (2006), 457-480.

[21] O. Ladyzenskaya, V. Solonnikov, N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. of Math. Monographs, # **23** (1968).

[22] P.E. Mensah, M.M. Taha, A.K. Toure, Numerical approximation of the blow-up time for a semilinear parabolic equation with nonlinear boundary conditions, *Far East J. of Math. Sciences (FJMS)*, **60**, No 45 (2012), 125-167.

[23] K. N'Guessan, D. Nabongo and A.K. Toure, Blow-up for semidiscretizations of some semilinear parabolic equations with a convection term, *J. of Progressive Research in Mathematics (JPRM)*, **5**, No 2 (2015), 499-518.

[24] G. Pincet-Mailly, J-F. Rault, Nonlinear convection in reaction-diffusion equations under dynamical boundary conditions, *Electronic J. of Differential Equations*, **2013**, No 10 (2013), 1-14.

[25] J.-F. Rault, Phenomene d'explosion et existence globale pour quelques problmes paraboliques sous les conditions au bord dynamiques, *PhD Thesis*, Universite du Littoral Cǒte d'Opale (2010).

[26] J.-F. Rault, A bifurcation for a generalized Burgers' equation in dimension one, *Discrete Contin. Dyn. Syst. Ser. S.*, **5**, No 3 (2012), 683-706.

[27] P. Souplet, Finite time blow-up for a non-linear parabolic equation with a gradient term and applications, *Math. Methods Appl. Sci.*, **19** (1996), 1317-1333.

[28] P. Souplet, Recent results and open problems on parabolic equations with gradient nonlinearities, *Electronic J. of Differential Equations*, **2001**, No 20 (2001), 1-19.

[29] P. Souplet, F. Weissler, Self-similar subsolutions and blow-up for nonlinear parabolic equations, *J. of Mathematical Analysis and Applications*, **212** (1997), 60-74.

[30] P. Souplet, S. Tayachi, F.B. Weissler, Exact self-similar blow-up of solutions of a semilinear parabolic equation with a nonlinear gradient term, *Indiana Univ. Math. J.*, **45** (1996), 655-682.

[31] A.K. Toure, K. N'Guessan and D. Nabongo, Blow-up for Semidiscretizations of some reaction-diffusion equations with a nonlinear convection term, *Global J. of Pure and Applied Mathematics (GJPAM)*, **11**, No 6 (2015), 4273-4296.

[32] F. Weissler, Existence and nonexistence of global solutions for a semilinear heat equation, *Israel J. of Mathematics*, **38** (1981), 29-40.