THE POSITIVITY OF THE HYPERGEOMETRIC TRANSLATION OPERATORS ASSOCIATED TO THE CHEREDNIK OPERATORS AND THE HECKMAN-OPDAM THEORY ATTACHED TO THE ROOT SYSTEM OF TYPE BC_2

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Abstract: We consider the hypergeometric translation operator associated to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type BC_2. We prove in this paper that these operators are positivity preserving and allows positive integral representations. In particular we deduce that the product formulas of the Opdam-Cherednik kernel and the Heckman-Opdam hypergeometric function are positive integral transforms, and we obtain best estimates of these kernels.

1. Introduction

In [1] I. Cherednik introduced a family of differential-difference operators that nowadays bear his name. These operators play a crucial role in the theory of Heckman-Opdam's hypergeometric functions, which generalizes the theory of Harish-Chandra's spherical functions on Riemannian symmetric spaces (see [2], [3], [6]).

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To study in [8], [9] a harmonic analysis associated with the Cherednik operators and the Heckman-Opdam theory, the author has introduced the hypergeometric translation operators. In many situations to solve problems of this harmonic analysis we need the positivity of these operators, and the product formulas of the Opdam-Cherednik kernel and the Heckman-Opdam hypergeometric function are given by integrals with positive measures. These properties are not yet proved in the general case, under some conditions on the root system and the multiplicity function (see [10]).

This paper is a contribution towards these questions in the case of the Cherednik operators and the Heckman-Opdam theory attached to the root system of type BC_2.

We prove in this paper the positivity of the hypergeometric translation operators $\mathcal{T}_x, \mathcal{T}_x^W, x \in \mathbb{R}^2$, associated respectively to the Cherednik operators and the Heckman-Opdam theory, and we deduce that for all $x, u \in \mathbb{R}^2$, there exist positive measures $m_{x,u}, m_{x,u}^W$ on $\mathbb{R}^2$ with compact support and of norm equal to 1, such that:

- For all C^∞-function g on $\mathbb{R}^2$ we have

$$\mathcal{T}_x(g)(u) = \int_{\mathbb{R}^2} g(z) dm_{x,u}(z). \tag{1}$$

- For all C^∞-function g on $\mathbb{R}^2$ invariant by the Weyl group W, we have

$$\mathcal{T}_x^W(g)(u) = \int_{\mathbb{R}^2} g(z) dm_{x,u}^W(z). \tag{2}$$

From the relations (1), (2) we deduce the following product formulas for the Opdam-Cherednik kernel $G_\lambda(x), \lambda \in \mathbb{C}^2$, and the Heckman-Opdam hypergeometric function $F_\lambda(x), \lambda \in \mathbb{C}^2$:

$$\forall\, x, u \in \mathbb{R}^2, G_\lambda(x) G_\lambda(u) = \int_{\mathbb{R}^2} G_\lambda(z) dm_{x,u}(z), \tag{3}$$

$$\forall\, x, u \in \mathbb{R}^2, F_\lambda(x) F_\lambda(u) = \int_{\mathbb{R}^2} F_\lambda(z) dm_{x,u}^W(z). \tag{4}$$

These formulas imply the following estimates for the functions $G_\lambda(x)$ and $F_\lambda(x)$:

$$\forall\, \lambda \in \mathbb{R}^2,\ \sup_{x \in \mathbb{R}^2} |G_\lambda(x)| = 1, \tag{5}$$

$$\forall\, \lambda \in \mathbb{R}^2,\ \sup_{x \in \mathbb{R}^2} |F_\lambda(x)| = 1. \tag{6}$$

Our proof of the positivity of the hypergeometric translation operator $\mathcal{T}_x, x \in \mathbb{R}^2$, uses essentially the properties of the heat kernel $p_t(x, y), t > 0$, associated with the Cherednik operators and the positivity of the transmutation operators

associated with the Cherednik operators proved in [12]. We deduce the positivity of the hypergeometric translation operator $\mathcal{T}_x^W$, $x \in \mathbb{R}^2$, from the positivity of the operator $\mathcal{T}_x$ and the relation between $\mathcal{T}_x$ and $\mathcal{T}_x^W$.

We remark that M. Rösler has considered in [4] the non-compact Grassmann manifolds $SO_0(p,q)/(SO(p) \times SO(q), SU(p,q)/SU(p) \times U(q)$ and $S_p(p,q)/S_p(p) \times S_p(q)$. Their real rank is q and their spherical functions can be identified with the hypergeometric functions of type BC_q with multiplicities depending on p. She gives an explicit product formula for these spherical functions. By using the interpolation program in these cases, she obtains a product formula for three classes of hypergeometric functions of type BC_q. This product formula for $q = 2$ is a particular case of the product formula (4) (see [9], Section 6).

2. The Cherednik operators on $\mathbb{R}^2$ and their eigenfunctions (See [2], [3], [6], [7])

We consider $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$ and inner product $\langle ., . \rangle$ for which this basis is orthonormal. We extend this inner product to a complex bilinear form on $\mathbb{C}^2$.

2.1. The root system of type BC_2 and the multiplicity function

The root system of type BC_2 can be identified with the set $\mathcal{R}$ given by

$$\mathcal{R} = \{\pm e_1, \pm e_2, \pm 2e_1, \pm 2e_2\} \cup \{\pm e_1 \pm e_2\}, \tag{2.1}$$

which can also be written in the form

$$\mathcal{R} = \{\pm \alpha_i, i = 1, 2, ..., 6\},$$

with

$$\alpha_1 = e_1, \alpha_2 = e_2, \alpha_3 = 2e_1, \alpha_4 = 2e_2, \alpha_5 = (e_1 - e_2), \alpha_6 = (e_1 + e_2). \tag{2.2}$$

We denote by $\mathcal{R}_+$ the set of positive roots

$$\mathcal{R}_+ = \{\alpha_i, i = 1, 2, ..., 6\}, \tag{2.3}$$

and by $\mathcal{R}_+^o$ the set of positive indivisible roots i.e., the roots $\alpha \in \mathcal{R}_+$ such that $\frac{\alpha}{2} \notin \mathcal{R}_+$. Then we have

$$\mathcal{R}_+^0 = \{\alpha_1, \alpha_2, \alpha_5, \alpha_6\}. \tag{2.4}$$

For $\alpha \in \mathcal{R}$, we consider

$$r_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha, \text{ with } \check{\alpha} = \frac{2\alpha}{\|\alpha\|^2}, \tag{2.5}$$

the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^2$ orthogonal to α. The reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(2)$, called the Weyl group associated with $\mathcal{R}$. In this case W is isomorphic to the hyperoctahedral group which is generated by permutations and sign changes of the $e_i, i = 1, 2,$.

A function $k : \mathcal{R} \to]0, +\infty[$ on the root system $\mathcal{R}$ is called a multiplicity function if it is invariant under the action of the Weyl group W. In this case it can be written in the form $k = (k_1, k_2, k_3)$, where k_1, k_2 are the values on the roots α_1, α_2 and α_3, α_4, and k_3 is the value on the roots α_5, α_6.

The positive Weyl chamber denoted by $\mathfrak{a}^+$ is given by

$$\mathfrak{a}^+ = \{x \in \mathbb{R}^2 \; ; \forall \, \alpha \in \mathcal{R}_+, \langle \alpha, x \rangle > 0\}, \tag{2.6}$$

it can also be written in the form

$$\mathfrak{a}^+ = \{(x_1, x_2) \in \mathbb{R}^2 \; ; x_1 > x_2 > 0\}, \tag{2.7}$$

we denote by $\overline{\mathfrak{a}^+}$ its closure. Let also $\mathbb{R}^2_{reg}$ be the subset of regular elements in $\mathbb{R}^2$. i.e. those elements which belong to no hyperplane $H_\alpha = \{x \in \mathbb{R}^2; \langle \alpha, x \rangle = 0\}, \alpha \in \mathcal{R}$.

Let $\mathcal{A}_k$ denote the weight function

$$\forall \, x \in \mathbb{R}^2, \mathcal{A}_k(x) = \prod_{\alpha \in \mathcal{R}_+} |\sinh\langle \frac{\alpha}{2}, x\rangle|^{2k(\alpha)}. \tag{2.8}$$

2.2. The Cherednik operators on $\mathbb{R}^2$

The Cherednik operators $T_j, j = 1, 2$, on $\mathbb{R}^2$ associated with the Weyl group W and the multiplicity function k are defined for f of class C^1 on $\mathbb{R}^2$ and $x \in \mathbb{R}_{reg} = \mathbb{R}^2 \backslash \bigcup_{\alpha \in \mathcal{R}} H_\alpha$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j f(x), \tag{2.9}$$

with

$$\rho_j = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j, \quad \text{and} \quad \alpha^j = \langle \alpha, e_j \rangle. \tag{2.10}$$

They form a commutative system of differential-difference operators. The operators T_1, T_2 can also be written in the following form

$$
\begin{aligned}
T_1 gf(x) \quad &= \frac{\partial}{\partial x_1} f(x) + k_1 \frac{\{f(x) - f(r_{\alpha_1} x)\}}{1 - e^{-\langle \alpha_1, x \rangle}} + 2k_2 \frac{\{f(x) - f(r_{\alpha_3} x)\}}{1 - e^{-\langle \alpha_3, x \rangle}} \\
&+ k_3 \left[\frac{f(x) - f(r_{\alpha_5} x)}{1 - e^{-\langle \alpha_5, x \rangle}} + \frac{f(x) - f(r_{\alpha_6} x)}{1 - e^{-\langle \alpha_6, x \rangle}} \right] - (\frac{1}{2} k_1 + k_2 + k_3) f(x),
\end{aligned}
$$
(2.11)

$$
\begin{aligned}
T_2 f(x) \quad &= \frac{\partial}{\partial x_2} f(x) + k_1 \frac{\{f(x) - f(r_{\alpha_2} x)\}}{1 - e^{-\langle \alpha_2, x \rangle}} + 2k_2 \frac{\{f(x) - f(r_{\alpha_4} x)\}}{1 - e^{-\langle \alpha_4, x \rangle}} \\
&+ k_3 \left[- \left(\frac{f(x) - f(r_{\alpha_5} x)}{1 - e^{\langle \alpha_5, x \rangle}} \right) + \left(\frac{f(x) - f(r_{\alpha_6} x)}{1 - e^{-\langle \alpha_6, x \rangle}} \right) \right] - (\frac{1}{2} k_1 + k_2) f(x).
\end{aligned}
$$
(2.12)

2.3. The eigenfunctions of the Cherednik operators

We denote by $G_\lambda, \lambda \in \mathbb{C}^2$, the unique solution of the system

$$
\begin{cases}
T_j G_\lambda(x) &= -i\lambda_j G_\lambda(x), \quad x \in \mathbb{R}^2, \; j = 1, 2, \\
G_\lambda(0) &= 1.
\end{cases}
$$
(2.13)

It is called the Opdam-Cherednik kernel.

We consider the function $F_\lambda, \lambda \in \mathbb{C}^2$, defined by

$$
\forall \, x \in \mathbb{R}^2, F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx).
$$
(2.14)

This function is the unique W-invariant function on $\mathbb{R}^2$, which satisfies the partial differential equation

$$
\begin{cases}
p(T) F_\lambda(x) &= p(-i\lambda) F_\lambda(x), \quad x \in \mathbb{R}^2, \\
F_\lambda(0) &= 1,
\end{cases}
$$
(2.15)

for all W-invariant polynomial p on $\mathbb{R}^2$ and $p(T) = p(T_1, T_2)$. It is called the Heckman-Opdam hypergeometric function.

The functions G_λ and F_λ possess the following properties (see [6]):

i) For all $x \in \mathbb{R}^2$ the function $\lambda \to G_\lambda(x)$ is entire on $\mathbb{C}^2$.

ii) We have

$$
\forall \, x \in \mathbb{R}^2, \, \forall \, \lambda \in \mathbb{C}^2, \; \overline{G_\lambda(x)} = G_{-\overline{\lambda}}(x).
$$
(2.16)

iii) We have
$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{C}^2,\ |G_\lambda(x)| \le G_{iIm(\lambda)}(x). \tag{2.17}$$

iv) We have
$$\forall\, x \in \mathbb{R}^2\,,\forall\, \lambda \in \mathbb{R}^2, |G_\lambda(x)| \le |W|^{1/2}. \tag{2.18}$$
$$\forall\, x \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{R}^2, |F_\lambda(x)| \le |W|^{1/2}. \tag{2.19}$$

v) For $x \in \mathbb{R}^2$, we denote by x^+ the only point in the orbit Wx which lies in $\overline{\mathfrak{a}^+}$. Then we have
$$\forall\, x \in \mathbb{R}^2, G_0(x) \asymp \prod_{\substack{\alpha \in \mathcal{R}_+^0 \\ \langle \alpha, x \rangle \geq 0}} (1 + \langle \alpha, x \rangle) e^{-\langle \rho, x^+ \rangle}, \tag{2.20}$$

with ρ given by (2.10).

vi) The function F_0 satisfies the estimate
$$\forall\, x \in \overline{\mathfrak{a}_+}, F_0(x) \asymp e^{-\langle \rho, x \rangle} \prod_{\alpha \in \mathcal{R}_+^0} (1 + \langle \alpha, x \rangle). \tag{2.21}$$

vii) Let p and q be polynomials of degree m and n. Then there exists a positive constant M such that for all $x \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}^2$, we have
$$|p(\frac{\partial}{\partial \lambda})q(\frac{\partial}{\partial x})G_\lambda(x)| \le M(1 + \|\lambda\|)^n (1 + \|x\|)^m F_0(x) e^{\max_{w \in W}\, Im\langle w\lambda, x \rangle}. \tag{2.22}$$

viii) The function $G_\lambda, \lambda \in \mathbb{C}^2$, admits the following Laplace type representation
$$\forall\, x \in \mathbb{R}^2, G_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle \lambda, y \rangle} d\mu_x(y), \tag{2.23}$$

where μ_x is the positive measure on $\mathbb{R}^2$ with support in $B(0, \|x\|)$ the closed ball of center 0 and radius $\|x\|$, (see [12] p.197).

ix) From (2.14), (2.23) we deduce that the function $F_\lambda, \lambda \in \mathbb{C}^2$, possesses the Laplace type representation
$$\forall\, x \in \mathbb{R}^2, F_\lambda(x) = \int_{\mathbb{R}^2} e^{-i\langle \lambda, y \rangle} d\mu_x^W(y), \tag{2.24}$$

where μ_x^W is the positive measure with support in $B(0, \|x\|)$ given by
$$\mu_x^W = \frac{1}{|W|} \sum_{w \in W} \mu_{wx}. \tag{2.25}$$

3. The transmutation operators associated with the Cherednik operators (See [7])

Notations. We use the following denotations:

- $\mathcal{E}(\mathbb{R}^2)$ the space of C^∞-functions on $\mathbb{R}^2$. Its topology is defined by the seminorms

$$q_{n,K}(\varphi) = \sup_{\substack{|\mu| \leq n \\ x \in K}} |D^\mu \varphi(x)|,$$

where K is a compact subset of $\mathbb{R}^2, n \in \mathbb{N}$, and

$$D^\mu = \frac{\partial^{|\mu|}}{\partial^{\mu_1} x_1 \partial^{\mu_2} x_2}, \mu = (\mu_1, \mu_2) \in \mathbb{N}^2, |\mu| = \mu_1 + \mu_2.$$

- $\mathcal{D}(\mathbb{R}^2)$ the space of C^∞-functions on $\mathbb{R}^2$ with compact support. We have

$$\mathcal{D}(\mathbb{R}^2) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R}^2),$$

where $\mathcal{D}_a(\mathbb{R}^2)$ is the space of C^∞-functions on $\mathbb{R}^2$ with support in the closed ball $B(0,a)$ of center 0 and radius a. The topology of $\mathcal{D}_a(\mathbb{R}^2)$ is defined by the semi-norms

$$p_n(\psi) = \sup_{\substack{|\mu| \leq n \\ x \in B(0,a)}} |D^\mu \psi(x)|, \quad n \in \mathbb{N}.$$

The space $\mathcal{D}(\mathbb{R}^2)$ is equipped with the inductive limit topology.

- $\mathcal{S}(\mathbb{R}^2)$ the classical Schwartz space on $\mathbb{R}^2$. Its topology is defined by the seminorms

$$Q_{\ell,n}(f) = \sup_{\substack{|\mu| \leq n \\ x \in \mathbb{R}^2}} (1 + \|x\|^2)^\ell |D^\mu f(x)|, \quad n, \ell \in \mathbb{N}.$$

- $\mathcal{S}_2(\mathbb{R}^2)$ the generalized Schwartz space of C^∞-functions on $\mathbb{R}^2$ such that for all $\ell, n \in \mathbb{N}$, we have

$$P_{\ell,n}(f) = \sup_{\substack{|\mu| \leq n \\ x \in \mathbb{R}^2}} (1 + \|x\|^2)^\ell (F_0(x))^{-1} |D^\mu f(x)| < +\infty,$$

where $F_0(x)$ is the Opdam-Cherednik kernel corresponding to the eigenvalue zero.

It is topologized by means of the seminorms $P_{\ell,n}, \ell, n \in \mathbb{N}$.

By using for $x \in \mathbb{R}^2$, the positive measure μ_x given by (2.23), we define the transmutation operator called also the trigonometric Dunkl intertwining operator V_k on $\mathcal{E}(\mathbb{R}^2)$ by

$$\forall\, x \in \mathbb{R}^2,\, V_k(g)(x) = \int_{\mathbb{R}^2} g(y)d\mu_x(y). \tag{3.1}$$

The operator V_k is the unique linear topological isomorphism from $\mathcal{E}(\mathbb{R}^2)$ onto itself satisfying the transmutation relations

$$\forall\, x \in \mathbb{R}^2,\, T_j V_k(g)(x) = V_k(\frac{\partial}{\partial y_j}g)(x),\; j = 1,2, \tag{3.2}$$

and the condition

$$V_k(g)(0) = g(0), \tag{3.3}$$

(see [7], p. 309-310).

The dual ^tV_k of the operator V_k is defined by the following duality relation

$$\int_{\mathbb{R}^2} {}^tV_k(f)(y)g(y)dy = \int_{\mathbb{R}^2} V_k(g)(x)f(x)\mathcal{A}_k(x)dx, \tag{3.4}$$

with f in $\mathcal{D}(\mathbb{R}^2)$ and g in $\mathcal{E}(\mathbb{R}^2)$.

This operator is given by

$$\forall\, y \in \mathbb{R}^2, {}^tV_k(f)(y) = \int_{\mathbb{R}^2} f(x)d\nu_y(x), \tag{3.5}$$

where $\nu_y, y \in \mathbb{R}^2$, is the positive measure on $\mathbb{R}^2$ given by the relation (4.29) of [12] p.195, and verifying

$$\nu_y(K) < +\infty,\text{ for every compact } K \subset \mathbb{R}^2. \tag{3.6}$$

The operator ^tV_k is a linear topological isomorphism from
- $\mathcal{D}(\mathbb{R}^2)$ onto itself,
- $\mathcal{S}_2(\mathbb{R}^2)$ onto $\mathcal{S}(\mathbb{R}^2)$,
satisfying the transmutation relations

$$\forall\, y \in \mathbb{R}^2,\, {}^tV_k((T_j + S_j)f)(y) = \frac{\partial}{\partial y_j}{}^tV_k(f)(y), \quad j = 1,2, \tag{3.7}$$

where S_j is the operator on $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) given by

$$\forall\, x \in \mathbb{R}^2,\, S_j(h)(x) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha)\alpha^j h(r_\alpha x). \tag{3.8}$$

The operator ^tV_k possesses also the following property : for all f in $\mathcal{D}(\mathbb{R}^2)$ we have

$$\text{supp} f \subset B(0,a) \Leftrightarrow \text{supp}^tV_k(f) \subset B(0,a), \tag{3.9}$$

where $B(0,a)$ is the closed ball of center 0 and radius $a > 0$.

4. The hypergeometric Fourier transform associated with the Cherednik operators (See [3], [6])

Notations:

For $a > 0$ we denote by $PW(\mathbb{C}^2)_a$ the space of functions h which are entire on $\mathbb{C}^2$ and satisfying

$$\forall\, m \in \mathbb{N}, \mathfrak{s}_m(f) = \sup_{\lambda \in \mathbb{C}^2} (1 + \|\lambda\|^2)^m |h(\lambda)| e^{-a\|Im(\lambda)\|} < +\infty.$$

Its topology is given by the seminorms $s_m, m \in \mathbb{N}$.

We consider the space $PW(\mathbb{C}^2)$ of entire functions on $\mathbb{C}^2$ which are rapidly decreasing and of exponential type. We have

$$PW(\mathbb{C}^2) = \cup_{a>0} PW(\mathbb{C}^2)_a.$$

It is equipped with the inductive limit topology.

Definition 4.1. The hypergeometric Fourier transform $\mathcal{H}$ is defined for f in $\mathcal{D}(\mathbb{R}^2)$ by

$$\forall\, \lambda \in \mathbb{C}^2, \mathcal{H}(f)(\lambda) = \int_{\mathbb{R}^2} f(x) G_\lambda(x) \mathcal{A}_k(x) dx. \tag{4.1}$$

Theorem 4.2. *The transform $\mathcal{H}$ is a topological isomorphism from*

i) $\mathcal{D}(\mathbb{R}^2)$ *onto* $PW(\mathbb{C}^2)$.

ii) $\mathcal{S}_2(\mathbb{R}^2)$ *onto* $\mathcal{S}(\mathbb{R}^2)$.

The inverse transform $\mathcal{H}^{-1}$ is given by

$$\forall\, x \in \mathbb{R}^2, \mathcal{H}^{-1}(h)(x) = \int_{\mathbb{R}^2} h(\lambda) G_\lambda(-x) \mathcal{C}_k(\lambda) d\lambda, \tag{4.2}$$

where

$$\forall\, \lambda \in \mathbb{R}^2, \mathcal{C}_k(\lambda) = c|C_k(\lambda)|^{-2} \prod_{\alpha \in \mathcal{R}_+} \left(1 - \frac{k(\alpha)}{i\langle\lambda, \breve{\alpha}\rangle - \frac{1}{2}k(\frac{\alpha}{2})}\right), \tag{4.3}$$

with c a positive constant chosen in such a way that $\mathcal{C}_k(-\rho) = 1$, and for $\lambda \in \mathbb{R}^2$:

$$(C_k(\lambda))^{-1} = \prod_{\alpha \in \mathcal{R}_+} \frac{\Gamma(i\langle\lambda, \breve{\alpha}\rangle + k(\alpha) + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(i\langle\lambda, \alpha\rangle + \frac{1}{2}k(\frac{\alpha}{2}))} \tag{4.4}$$

with the convention that $k(\frac{\alpha}{2}) = 0$ if $\frac{\alpha}{2} \notin \mathcal{R}_+$.

We have

$$\forall\, \lambda \in \mathbb{R}^2, |C_k(\lambda)|^2 = C_k(\lambda)C_k(-\lambda) = C_k(\lambda)\overline{C_k(\lambda)}. \tag{4.5}$$

Remark 4.3. The function $\mathcal{C}_k(\lambda)$ is continuous on $\mathbb{R}^2$ and satisfies the following estimate

$$\forall \, \lambda \in \mathbb{R}^2, |\mathcal{C}_k(\lambda)| \leq \text{ const } (1 + \|\lambda\|^2)^b \qquad (4.6)$$

for some $b > 0$.

5. The hypergeometric translation operator and its dual and the hypergeometric convolution product associated with the Cherednik operators

5.1. The hypergeometric translation operator and its dual

The hypergeometric translation operator $\mathcal{T}_x, x \in \mathbb{R}^2$, is defined on $\mathcal{E}(\mathbb{R}^2)$ by

$$\forall \, y \in \mathbb{R}^2, \mathcal{T}_x(f)(y) = (V_k)_x(V_k)_y[V_k^{-1}(f)(x + y)], \qquad (5.1)$$

(see [8], p. 33).

In the following we give some properties of the operator $\mathcal{T}_x, x \in \mathbb{R}^2$:

1. For all $x \in \mathbb{R}^2$, the operator $\mathcal{T}_x$ is continuous from $\mathcal{E}(\mathbb{R}^2)$ into itself.

2. For all f in $\mathcal{E}(\mathbb{R}^2)$ and $x, y \in \mathbb{R}^2$, we have

$$\mathcal{T}_x(f)(0) = f(x) \quad \text{ and } \mathcal{T}_x(f)(y) = \mathcal{T}_y(f)(x). \qquad (5.2)$$

3. For all $x, y \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}^2$, we have the product formula

$$\mathcal{T}_x(G_\lambda)(y) = G_\lambda(x)G_\lambda(y), \qquad (5.3)$$

where $G_\lambda(x)$ is the Opdam-Cherednik kernel given by (2.13).

For each $x \in \mathbb{R}^2$, the dual of the hypergeometric translation operator $\mathcal{T}_x$ is the operator ${}^t\mathcal{T}_x$ defined on $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$), by

$$\forall \, y \in \mathbb{R}^2, {}^t\mathcal{T}_x(f)(y) = (V_k)_x({}^tV_k^{-1})_y[{}^tV_k(f)(y - x)], \qquad (5.4)$$

(see [8], p. 34-39).

It satisfies the following properties:

1. For all $x \in \mathbb{R}^2$, the operator ${}^t\mathcal{T}_x$ is continuous from
   - $\mathcal{D}(\mathbb{R}^2)$ into itself.
   - $\mathcal{S}_2(\mathbb{R}^2)$ into itself.

2. The operator ${}^t\mathcal{T}_x, x \in \mathbb{R}^2$, is related to the operator $\mathcal{T}_x, x \in \mathbb{R}^2$, by the following two relations

   i) For g in $\mathcal{E}(\mathbb{R}^2)$ and f in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) we have

$$\int_{\mathbb{R}^2} \mathcal{T}_x(g)(z)f(z)\mathcal{A}_k(z)dz = \int_{\mathbb{R}^2} g(y)\,{}^t\mathcal{T}_x(f)(y)\mathcal{A}_k(y)dy. \qquad (5.5)$$

   ii) For f in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) we have

$$\forall\, x,y \in \mathbb{R}^2, {}^t\mathcal{T}_x(f)(y) = \mathcal{T}_y(\breve{f})(-x), \qquad (5.6)$$

where $\breve{f}$ is the function given by

$$\forall\, x \in \mathbb{R}^2, \breve{f}(x) = f(-x).$$

3. For all f in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) and $x \in \mathbb{R}^2$, we have

$$\forall\, \lambda \in \mathbb{R}^2, \mathcal{H}({}^t\mathcal{T}_x(f))(\lambda) = G_\lambda(x)\mathcal{H}(f)(\lambda). \qquad (5.7)$$

4. For all f in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) and $x,y \in \mathbb{R}^2$, we have

$$ {}^t\mathcal{T}_x(f)(y) = \int_{\mathbb{R}^2} G_\lambda(x)G_\lambda(-y)\mathcal{H}(f)(\lambda)\mathcal{C}_k(\lambda)d\lambda. \qquad (5.8)$$

5. For all f in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) and $x,y \in \mathbb{R}^2$, we have

$$ {}^t\mathcal{T}_x(f)(y) = {}^t\mathcal{T}_{-y}(f)(-x) \qquad (5.9)$$

6. For all f in $\mathcal{D}(\mathbb{R}^2)$ with support in the closed ball $B(0,a)$ of center 0 and radius $a > 0$, and $x \in \mathbb{R}^2$, we have

$$\operatorname{supp}{}^t\mathcal{T}_x(f) \subset B(0, a + \|x\|). \qquad (5.10)$$

5.2. The hypergeometric convolution product

By using the operator ${}^t\mathcal{T}_x, x \in \mathbb{R}^2$, we define the hypergeometric convolution product $f *_{\mathcal{H}} g$ of the functions f, g in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) by

$$\forall\, y \in \mathbb{R}^2, f *_{\mathcal{H}} g(y) = \int_{\mathbb{R}^2} {}^t\mathcal{T}_x(f)(y)g(x)\mathcal{A}_k(x)dx, \qquad (5.11)$$

(see [8], p. 26 and p. 39-41).

   The hypergeometric convolution product $*_{\mathcal{H}}$ satisfies the following properties:

1. It is commutative and associative.

2. For all f, g in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) the function $f *_{\mathcal{H}} g$ belongs to $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$).

3. For all f, g in $\mathcal{D}(\mathbb{R}^2)$ (resp.$\mathcal{S}_2(\mathbb{R}^2)$), we have

$$\forall \lambda \in \mathbb{R}^2, \mathcal{H}(f *_{\mathcal{H}} g)(\lambda) = \mathcal{H}(f)(\lambda).\mathcal{H}(g)(\lambda). \tag{5.12}$$

4. For all f, g in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$), we have

$${}^t V_k(f *_{\mathcal{H}} g) = {}^t V_k(f) * {}^t V_k(g),$$

where $*$ is the classical convolution product on $\mathbb{R}^2$.

6. The Gauss and heat kernels associated with the Cherednik operators (See [6], [8])

Definition 6.1. Let $t > 0$. The Gauss kernel $p_t(x, y)$ associated with the Cherednik operators, is defined for $x, y \in \mathbb{R}^2$, by

$$p_t(x, y) = \int_{\mathbb{R}^2} e^{-t(\|\lambda\|^2 + \|\rho\|^2)} G_\lambda(x) G_\lambda(-y) \mathcal{C}_k(\lambda) d\lambda. \tag{6.1}$$

Notations. We denote by:
- H_k the heat operator associated with the Cherednik operators given by

$$H_k = \mathcal{L}_k - \frac{\partial}{\partial t} - \|\rho\|^2, \tag{6.2}$$

where $\mathcal{L}_k$ is the Heckman-Opdam Laplacian defined for all f of class C^2 on $\mathbb{R}^2$ by

$$\mathcal{L}_k f = (T_1^2 + T_2^2) f. \tag{6.3}$$

It has the following form : For all $x \in \mathbb{R}^2_{reg}$,

$$\begin{aligned} \mathcal{L}_k f(x) &= \Delta f(x) + \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \coth(\tfrac{\langle \alpha, x \rangle}{2}) \langle \nabla f(x), \alpha \rangle \\ &+ \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha) \|\alpha\|^2}{4 \sinh^2 \frac{\langle \alpha, x \rangle}{2}} \{f(r_\alpha x) - f(x)\} + \|\rho\|^2 f(x), \end{aligned} \tag{6.4}$$

where Δ and ∇ are respectively the Laplacian and the gradient on $\mathbb{R}^2$.
- $E_t, t > 0$, the fundamental solution of the operator H_k given by
$$\forall x \in \mathbb{R}^2, E_t(x) = p_t(x, 0). \tag{6.5}$$

It is called also the heat kernel.

Proposition 6.1.

i) For all $t > 0$, the function E_t belongs to $\mathcal{S}_2(\mathbb{R}^2)$.

ii) For all $t > 0$ and $x \in \mathbb{R}^2$, we have

$$E_t(x) \asymp t^{-\gamma-1} \left[\prod_{\alpha \in \mathcal{R}_+^o} M_\alpha(t, x) \right] e^{-t\|\rho\|^2 - \langle \rho, x^+ \rangle - \frac{\|x\|^2}{4t}}, \qquad (6.6)$$

where x^+ denotes the unique conjugate of x in $\overline{\mathfrak{a}_+}$,

$$\gamma = \sum_{\alpha \in \mathcal{R}_+} k(\alpha) \qquad (6.7)$$

and

$$M_\alpha(t, x) = 2^{-\gamma-1}[(1 + |\langle \alpha, x \rangle|)(1 + 2t + |\langle \alpha, x \rangle|)^{k(\alpha)+k(2\alpha)-1}]. \qquad (6.8)$$

iii) For all $t > 0$, we have

$$\forall \lambda \in \mathbb{R}^2, \mathcal{H}(E_t)(\lambda) = e^{-t(\|\lambda\|^2 + \|\rho\|^2)}. \qquad (6.9)$$

iv) The function $(x, t) \to E_t(x)$ is strictly positive on $\mathbb{R}^2 \times]0, +\infty[$.

v) For all $t > 0$, we have

$$\int_{\mathbb{R}^2} E_t(x) \mathcal{A}_k(x) dx = 1. \qquad (6.10)$$

vi) The function $(x, t) \longrightarrow E_t(x)$ satisfies

$$H_k E_t(x) = 0, \quad \text{on } \mathbb{R}^2 \times]0, +\infty[. \qquad (6.11)$$

Proposition 6.2.

i) For all $t > 0$ and $x \in \mathbb{R}^2$, the function $y \to p_t(x, y)$ belongs to $\mathcal{S}_2(\mathbb{R}^2)$.

ii) For all $t > 0$ and $x, y \in \mathbb{R}^2$, we have

$$p_t(x, y) = {}^t\mathcal{T}_x(E_t)(y). \qquad (6.12)$$

iii) The function $(x, y, t) \longrightarrow p_t(x, y)$ is strictly positive on $\mathbb{R}^2 \times \mathbb{R}^2 \times]0, +\infty[$.

iv) *For all $t > 0$ and $x \in \mathbb{R}^2$, we have*

$$\int_{\mathbb{R}^2} p_t(x, y) \mathcal{A}_k(y) dy = 1. \tag{6.13}$$

v) *For all $y \in \mathbb{R}^d$, the function $(x, t) \to p_t(x, y)$ satisfies*

$$H_k p_t(x, y) = 0, \ \ on \ \mathbb{R}^2 \times]0, +\infty[. \tag{6.14}$$

Définition 6.3. The heat semigroup $\{P_t\}_{t\geq 0}$ associated with the Cherednik operators, is defined for f in $\mathcal{S}_2(\mathbb{R}^2)$ by

$$\forall\, x \in \mathbb{R}^2, P_t f(x) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} p_t(x, y) f(y) \mathcal{A}_k(y) dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0. \end{cases} \tag{6.15}$$

(See [6]).

Remark 6.4. The function $P_t f(x)$ can also be written in the following form

$$\forall\, x \in \mathbb{R}^2 \ , P_t f(x) = E_t *_{\mathcal{H}} \breve{f}(-x), \tag{6.16}$$

where $*_{\mathcal{H}}$ is the hypergeometric convolution product given by the relation (5.11), and $\breve{f}$ is the function defined by

$$\forall\, x \in \mathbb{R}^2, \breve{f}(x) = f(x).$$

We consider the Cauchy problem: Given a continuous bounded function on $\mathbb{R}^2$. Find a function $u(x, t)$ of class C^2 on $\mathbb{R}^2 \times]0, +\infty[$, such that

$$\begin{cases} H_k u(x, t) & = 0, \ \text{on } \mathbb{R}^2 \times]0, +\infty[, \\ u(x, 0) & = f(x). \end{cases} \tag{6.17}$$

Proposition 6.5. (see [6])

i) $\{P_t\}_{t\geq 0}$ *is a strongly continuous semigroup on $\mathcal{S}_2(\mathbb{R}^2)$.*

ii) *Let f be a continuous bounded function on $\mathbb{R}^2$. Then the function $u(x, t) = P_t f(x)$ solves the Cauchy problem (6.17).*

7. Positivity of the function ${}^t\mathcal{T}_x(p_t(u,.))(y)$

Proposition 7.1. *Let $t > 0$ and $x \in \mathbb{R}^2$. We have*

$$\forall\, u, y \in \mathbb{R}^2, {}^t\mathcal{T}_x(p_t(u,.))(y) = \int_{\mathbb{R}^2} e^{-t(\|\lambda\|^2 + \|\rho\|^2)} G_\lambda(x) G_\lambda(u) G_\lambda(-y) \mathcal{C}_k(\lambda) d\lambda.$$
(7.1)

Proof. From Proposition 6.1 i), for all $u \in \mathbb{R}^2$, the function $z \to p_t(u,z)$ belongs to $\mathcal{S}_2(\mathbb{R}^2)$. Then by using (5.8) we obtain

$$\forall\, u, z \in \mathbb{R}^2, {}^t\mathcal{T}_x(p_t(u,.))(y) = \int_{\mathbb{R}^2} G_\lambda(x) G_\lambda(-y) \mathcal{H}(p_t(u,.))(\lambda) \mathcal{C}_k(\lambda) d\lambda. \quad (7.2)$$

On the other hand from (6.12), (5.7), we have

$$\forall\, u, \lambda \in \mathbb{R}^2, \mathcal{H}(p_t(u,.))(\lambda) = \mathcal{H}({}^t\mathcal{T}_u(E_t))(\lambda) = G_\lambda(u)\mathcal{H}(E_t)(\lambda),$$

thus from (6.9) we obtain

$$\forall\, u, \lambda \in \mathbb{R}^2, \mathcal{H}(p_t(u,.))(\lambda) = e^{-t(\|\lambda\|^2 + \|\rho\|^2)} G_\lambda(u). \quad (7.3)$$

We deduce (7.1) from (7.2), (7.3). $\qquad\square$

Corollary 7.2. *Let $t > 0$ and $x \in \mathbb{R}^2$. We have*

$$\forall\, u, y \in \mathbb{R}^2, {}^t\mathcal{T}_x(p_t(u,.))(y) = e^{-t\|\rho\|^2} \int_{\mathbb{R}^2} U_{t,x}(-y,z) d\breve{\mu}_u(z), \quad (7.4)$$

where μ_u is the measure given by (2.23) and $\breve{\mu}_u$ is the measure on $\mathbb{R}^2$ defined by

$$\int_{\mathbb{R}^2} f(z) d\breve{\mu}_u(z) = \int_{\mathbb{R}^2} f(-z) d\mu_u(z), \quad f \in \mathcal{E}(\mathbb{R}^2), \quad (7.5)$$

and

$$U_{t,x}(y,z) = \int_{\mathbb{R}^2} e^{-t\|\lambda\|^2} G_\lambda(x) G_\lambda(y) e^{i\langle \lambda, z\rangle} \mathcal{C}_k(\lambda) d\lambda. \quad (7.6)$$

Proof. We deduce (7.4) from (7.1), (2.18), (2.23), (7.5), the positivity of the measure $\breve{\mu}_x$ and Fubini's theorem. $\qquad\square$

In this section we prove first that for $t > 0$ and $x \in \mathbb{R}^2$, the function $(y, z) \to U_{t,x}(y, z)$ is positive on $\mathbb{R}^2 \times \mathbb{R}^2$, and next we deduce the positivity of the function ${}^t\mathcal{T}_x(p_t(u,.))(y)$.

Proposition 7.3. *Let $t > 0$ and $x \in \mathbb{R}^2$. The function $U_{t,x}(y, z)$ is of class C^∞ on $\mathbb{R}^2 \times \mathbb{R}^2$ with respect to the variables y and z and satisfies the equation*

$$\forall\, y, z \in \mathbb{R}^2, (T_j + \frac{\partial}{\partial z_j})U_{t,x}(y, z) = 0, \quad j = 1, 2. \tag{7.7}$$

Proof. We obtain the results by derivation under the integral sign with respect to the variables $y_j, z_j, j = 1, 2$, in the relation (7.6), by using (2.22), (2.19), and by applying the relation (2.13). □

Proposition 7.4

i) Let $t > 0$ and $x \in \mathbb{R}^2$. There exists a positive function $M_0(t)$ such that

$$\forall\, y, z \in \mathbb{R}^2, |U_{t,x}(y, z)| \leq M_0(t)F_0(y), \tag{7.8}$$

where $F_0(y)$ is the Heckman-Opdam hypergeometric function corresponding to the eigenvalue zero.

ii) Let $t > 0$ and $x \in \mathbb{R}^2$. We have

$$\forall\, y \in \mathbb{R}^2, \lim_{\|z\| \to +\infty} U_{t,x}(y, z) = 0. \tag{7.9}$$

iii) Let $t > 0$ and $x \in \mathbb{R}^2$. The function $(y, z) \to U_{t,x}(y, z)$ is bounded on $\mathbb{R}^2 \times \mathbb{R}^2$ and we have

$$\lim_{\|(y,z)\| \to +\infty} U_{t,x}(y, z) = 0. \tag{7.10}$$

Proof.

i) We deduce (7.8) from (7.6), (2.22).

ii) By using (7.6) and the fact that from (2.18) the function $e^{-t\|\lambda\|^2}\, G_\lambda(x)$ $G_\lambda(y)$ is for all $x, y \in \mathbb{R}^2$, integrable with respect to the Lebesgue measure on $\mathbb{R}^2$, we deduce (7.9) from Riemann-Lebesgue lemma.

iii) - The relations (7.6), (2.18) imply that the function $(y, z) \to U_{t,x}(y, z)$ is bounded on $\mathbb{R}^2 \times \mathbb{R}^2$.

- Let y be in $\mathbb{R}^2_{reg}$. There exists $w \in W$ such that $y = wy^+$ with $y^+ \in \mathfrak{a}_+$. Then from the i) and (2.21) there exists a positive function $M_1(t)$ such that

$$\forall y \in \mathbb{R}^2_{reg}, \ \forall z \in \mathbb{R}^2, \ |U_{t,x}(y, z)| \leq M_1(t)(\prod_{\alpha \in \mathcal{R}_0^+} (1 + \langle \alpha, y^+ \rangle))e^{-\langle \alpha, y^+ \rangle}.$$

Thus the function $U_{t,x}(y, z)$ tends to zero when y goes to infinity, uniformly with respect to the variable z. We deduce (7.10) from this result and the continuity of the function $U_{t,x}(y, z)$ on $\mathbb{R}^2$ with respect to the variable y.

$\square$

Proposition 7.5.

i) Let $t > 0$ and $x \in \mathbb{R}^2$. For all $y, z \in \mathbb{R}^2$ the function $U_{t,x}(y, z)$ is real.

ii) Let $t > 0$ and $x \in \mathbb{R}^2$. The function $U_{t,x}(y, z)$ is strictly positive on the set
$$Y = \{(y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \; ; y \in \mathbb{R}^2, z = 0\}.$$

Proof.

i) We obtain the result from the relations (7.6), (2.16), (4.3) and a change of variables.

ii) By using the relation
$$\forall \, \lambda \in \mathbb{R}^2, G_\lambda(0) = 1,$$

we deduce from (7.6), (6.1) and Proposition 6.2 iii), that
$$\forall \, y \in \mathbb{R}^2, U_{t,x}(y, 0) = e^{-t\|\rho\|^2} p_t(x, -y) > 0.$$

$\square$

Proposition 7.6. Let $t > 0$ and $x \in \mathbb{R}^2$. For all $\alpha \in \mathcal{R}_+$, and $y, z \in \mathbb{R}^2$, we have
$$\begin{aligned} U_{t,x}(r_{\alpha_i}y, z) - U_{t,x}(y, z) \quad &= -\langle \breve{\alpha}, y \rangle \langle \nabla U_{t,x}(y, z), \alpha \rangle \\ &+ \tfrac{1}{2}(\langle \breve{\alpha}, y \rangle)^2 \alpha^t D^2 U_{t,x}(\xi, z)\alpha, \end{aligned} \qquad (7.11)$$
with some ξ on the line segment between y and $r_\alpha y$.

Proof. We obtain (7.11) from the relation (2.5) and Taylor's formula. $\square$

Proposition 7.7. The Weyl chambers attached to the root system of type BC_2 are the following
$$\mathfrak{a}^+ = \{x \in \mathbb{R}^2; \langle \alpha, x \rangle > 0, i = 1, 2, ..., 6\}, \qquad (7.12)$$

$$\mathfrak{a}^- = -\mathfrak{a}^+, \qquad (7.13)$$

$$a_1^+ = \{x \in \mathbb{R}^2 \; ; \langle \alpha_i, x \rangle > 0, i = 1, 2, 3, 4, 6; \langle \alpha_5, x \rangle < 0\}, \qquad (7.14)$$

$$\mathfrak{a}_1^- = -\mathfrak{a}_1^+. \qquad (7.15)$$

Proof. We determine the Weyl chambers corresponding to the six roots of $\mathcal{R}_+$, and next by applying the relations

$$\begin{cases} 2\alpha_1 & = \alpha_3 \\ 2\alpha_2 & = \alpha_4 \\ \alpha_1 - \alpha_2 & = \alpha_5 \\ \alpha_1 + \alpha_2 & = \alpha_6 \end{cases},$$

we obtain the Weyl chambers (7.12), (7.13), (7.14), (7.15), the others are empty. $\qquad\square$

Notations:

We denote by $C_\ell, \ell = 1, 2$, the Weyl chambers $\mathfrak{a}^+, \mathfrak{a}_1^+$ and by $C_\ell, \ell = 3, 4$, the Weyl chambers $\mathfrak{a}^-, \mathfrak{a}_1^-$. Then we have

$$\mathbb{R}^2 = \bigcup_{\ell=1}^{4} \overline{C}_\ell,$$

where $\overline{C}_\ell$ is the closure of C_ℓ.

Theorem 7.8. *For all $t > 0$, and $x \in \mathbb{R}^2$ we have*

$$\forall\, y, z \in \mathbb{R}^2\ , U_{t,x}(y, z) \geq 0. \tag{7.16}$$

Proof. The proof is made up in two steps.

In the first step we obtain some results concerning the positivity of the function $U_{t,x}(y, z)$ given by (7.6) on each of the set $\overline{C}_\ell \times \mathbb{R}^2, \ell = 1, 2, 3, 4$.

In the second step we use the fact that $\mathbb{R}^2 \times \mathbb{R}^2 = (\cup_{\ell=1}^{4}\overline{C}_\ell) \times \mathbb{R}^2$ and the result of the first step to deduce the positivity of the function $U_{t,x}(y, z)$ on $\mathbb{R}^2 \times \mathbb{R}^2$.

1st Step

We consider the set Y_ℓ defined by

$$Y_\ell = \{(y, z) \in \mathbb{R}^2 \times \mathbb{R}^2\ ;\quad y \in \overline{C}_\ell, z \in \mathbb{R}^2\}.$$

We denote by

$$\mathcal{V}_{t,x}^\ell(y, z) = U_{t,x}(y, z)1_{Y_\ell}(y, z),$$

where 1_{Y_ℓ} is the characteristic function of the set Y_ℓ. From Proposition 7.5 ii) the function $\mathcal{V}_{t,x}^\ell(y, z)$ is strictly positive on the set Y. We shall prove that it is positive on the set $Y_\ell \backslash Y$. If not we suppose by using Proposition 7.5 i) and Proposition 7.4 iii) that it attains a strictly negative absolute minimum at $(y^\ell, z^\ell) \in Y_\ell \backslash Y$, i.e.

$$\mathcal{V}_{t,x}^{\ell}(y^{\ell}, z^{\ell}) = \inf_{(y,z)\in Y_{\ell}} \mathcal{V}_{t,x}^{\ell}(y, z) < 0. \tag{7.17}$$

There are two possibilities: The point (y^{ℓ}, z^{ℓ}) is in the open subset $(Y_{\ell}\backslash Y)^{0}$ of the set $Y_{\ell}\backslash Y$, or in the set

$$Y_{\ell}^{0} = \{(y, z) \in \mathbb{R}^{2} \times \mathbb{R}^{2} ; \quad y \in \partial \overline{C}_{\ell}, z \in \mathbb{R}^{2}\}. \tag{7.18}$$

We suppose that $(y^{\ell}, z^{\ell}) \in (Y_{\ell}\backslash Y)^{0}$. As the point (y^{ℓ}, z^{ℓ}) is an absolute minimum, then we have

$$\frac{\partial}{\partial y_{j}}\mathcal{V}_{t,x}^{\ell}(y^{\ell}, z^{\ell}) = \frac{\partial}{\partial z_{j}}\mathcal{V}_{t,x}^{\ell}(y^{\ell}, z^{\ell}) = 0, \quad j = 1, 2. \tag{7.19}$$

By using the fact that

$$\forall \, \alpha \in \mathcal{R}_{+}, (r_{\alpha}y^{\ell}, z^{\ell}) \notin Y_{\ell},$$

and by applying the relations (7.7), (2.11), (2.12), (7.19), we get

$$\left\{k_{1}(\frac{1}{1 - e^{-\langle \alpha_{1}, y^{\ell}\rangle}} - \frac{1}{2}) + 2k_{2}\Big(\frac{1}{1 - e^{-\langle \alpha_{3}, y^{\ell}\rangle}} - \frac{1}{2}\Big).\right.$$

$$\left. + k_{3}\Big[\Big(\frac{1}{1 - e^{-\langle \alpha_{5}, y^{\ell}\rangle}} - \frac{1}{2}\Big) + \Big(\frac{1}{1 - e^{-\langle \alpha_{6}, y^{\ell}\rangle}} - \frac{1}{2}\Big)\Big]\right\}\mathcal{V}_{t}^{\ell}(y^{\ell}, z^{\ell}) = 0, \tag{7.20}$$

$$\left\{k_{1}\Big(\frac{1}{1 - e^{-\langle \alpha_{2}, y^{\ell}\rangle}} - \frac{1}{2}\Big) + 2k_{2}\Big(\frac{1}{1 - e^{-\langle \alpha_{4}, y^{\ell}\rangle}} - \frac{1}{2}\Big)\right.$$

$$\left. + k_{3}\Big[-\Big(\frac{1}{1 - e^{-\langle \alpha_{3}, y^{\ell}\rangle}} - \frac{1}{2}\Big) + \Big(\frac{1}{1 - e^{-\langle \alpha_{6}, y^{\ell}\rangle}} - \frac{1}{2}\Big)\Big]\right\}\mathcal{V}_{t}^{\ell}(y^{\ell}, z^{\ell}) = 0, \tag{7.21}$$

Using the fact that from (7.17) the function $\mathcal{V}_{t,x}^{\ell}(y^{\ell}, z^{\ell})$ is different from zero and that $k_{3} > 0$, the equations (7.20), (7.21) can also be written in the form

$$\frac{k_{1}}{k_{3}}X_{1}^{\ell} + \frac{2k_{2}}{k_{3}}X_{3}^{\ell} + X_{5}^{\ell} + X_{6}^{\ell} = 0, \tag{7.22}$$

$$\frac{k_{1}}{k_{3}}X_{2}^{\ell} + \frac{2k_{2}}{k_{3}}X_{4}^{\ell} - X_{5}^{\ell} + X_{6}^{\ell} = 0, \tag{7.23}$$

with

$$X_{i}^{\ell} = \frac{1 + e^{-\langle \alpha_{i}, y^{\ell}\rangle}}{1 - e^{-\langle \alpha_{i}, y^{\ell}\rangle}} , \quad i = 1, 2, ..., 6. \tag{7.24}$$

Then the $X_{i}^{\ell}, i = 1, 2, ..., 6$, are solutions of the system (S) of linear equations on $\mathbb{R}^{4}$:

$$(S) \begin{cases} \frac{k_{1}}{k_{3}}X_{1} + 2\frac{k_{2}}{k_{3}}X_{3} + X_{5} + X_{6} = 0, \\ \frac{k_{1}}{k_{3}}X_{2} + 2\frac{k_{2}}{k_{3}}X_{4} - X_{5} + X_{6} = 0. \end{cases} \tag{7.25}$$

On the other hand from (7.24) we obtain

$$e^{-\langle \alpha_i, y^\ell \rangle} = \frac{X_i^\ell - 1}{X_i^\ell + 1}, \quad i = 1, 2, ..., 6. \tag{7.26}$$

We consider the function f defined on $\mathbb{R} \backslash \{-1\}$ by

$$f(y) = \frac{y - 1}{y + 1},$$

we have

$$f(y) \leq 0 \Leftrightarrow y \in]-1, 1], \tag{7.27}$$

$$0 < f(y) < 1 \Leftrightarrow y \in]1, +\infty[, \tag{7.28}$$

$$f(y) > 1 \Leftrightarrow y \in]-\infty, -1[. \tag{7.29}$$

From the relation (7.26), we have

$$e^{-\langle \alpha_i, x^\ell \rangle} = f(X_i^\ell), \quad i = 1, 2, ..., 6. \tag{7.30}$$

As the first member of (7.30) is strictly positive, then from (7.27) the X_i^ℓ, $i = 1, 2, ..., 6$, are not in the interval $] - 1, 1[$. They are in the interval $] - \infty, -1[\cup]1, +\infty[$. We consider two cases.

1st Case:

1. If $x^\ell \in C_\ell, \ell = 1$.
    From the relations (7.12), we have $\langle \alpha_i, x^\ell \rangle > 0$ for $i = 1, 2, ..., 6$. Then by using (7.30), (7.28) we obtain

    $$X_i^\ell \in]1, +\infty[, \quad i = 1, 2, ..., 6. \tag{7.31}$$

    By applying (7.31) we get

    $$\frac{k_1}{k_3} X_1^\ell + \frac{2k_2}{k_3} X_3^\ell + X_5^\ell + X_6^\ell > \frac{k_1}{k_3} + 2\frac{k_2}{k_3} + 2 > 0.$$

    Thus from (7.22) we obtain an absurdity, and then the $X_i^\ell, i = 1, 2, ..., 6$, are not solutions of the system (S) given by (7.25).

2. If $x^\ell \in C_\ell, \ell = 2$.
    From the relation (7.14),(7.15) we have $\langle \alpha_i, x^\ell \rangle > 0$ for $i = 1, 2, 3, 4, 6$ and $\langle \alpha_5, x^\ell \rangle < 0$. Then by using (7.30), (7.28), (7.29), we obtain

    $$X_i^\ell \in]1, +\infty[, i = 1, 2, 3, 4, 6 \text{ and } X_5^\ell \in]-\infty, -1[. \tag{7.32}$$

    By applying (7.32) we get

$$\frac{k_1}{k_3}X_2^\ell + 2\frac{k_2}{k_3}X_4^\ell - X_5^\ell + X_6^\ell > \frac{k_1}{k_3} + 2\frac{k_2}{k_3} + 2 > 0.$$

Thus from (7.23) we obtain an absurdity, and then the $X_i^\ell, i = 1, 2, ..., 6$, are not solution of the system (S) given by (7.25).

2nd Case: If $x^\ell \in C_\ell, \ell = 3, 4$.
The same proof as for the first case shows that when $y^\ell \in C_\ell, \ell = 3, 4$, we obtain an absurdity, and then the $X_i^\ell, i = 1, 2, ..., 6$, are not solutions of the system (S) given by (7.25).

From the first and second cases we deduce that our supposition that the function $\mathcal{V}_{t,x}^\ell(y, z)$ attains a strictly negative absolute minimum at (y^ℓ, z^ℓ) in $(Y_\ell \backslash Y)^0$ is absurd. Then the point (y^ℓ, z^ℓ) does not belong to $(Y_\ell \backslash Y)^0$, and it is in the set Y_ℓ^0.

2nd Step
From Proposition 7.5 ii) the function $U_{t,x}(y, z)$ is strictly positive on the set Y.

We shall prove that the function $U_{t,x}(y, z)$ is positive on the set $\mathbb{R}^2 \times \mathbb{R}^2 \backslash Y$. If not we suppose by using Proposition 7.5.i) and Proposition 7.4 iii) that it attains a strictly negative absolute minimum at $(y_0, z_0) \in \mathbb{R}^2 \times \mathbb{R}^2 \backslash Y$.

From the first step and the relation $\mathbb{R}^2 = \cup_{\ell=1}^4 \overline{C}_\ell$, the point (y_0, z_0) is in the set

$$Y^0 = \bigcup_{\ell=1}^4 Y_\ell^0,$$

with Y_ℓ^0 given by (7.18). We have

$$Y^0 = \{(y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 ; \forall \, \alpha \in \mathcal{R}_+, \, \langle \alpha, y \rangle = 0, z \in \mathbb{R}^2\},$$

then

$$\forall \, \alpha \in \mathcal{R}_+, \, \langle \alpha, y_0 \rangle = 0.$$

We shall prove in the following that the point (y_0, z_0) is not in the set Y^0.

As the point (y_0, z_0) is a strictly negative absolute minimum, then we have the following relations

$$U_{t,x}(y_0, z_0) = \inf_{(y,z) \in \mathbb{R}^2 \times \mathbb{R}^2} U_{t,x}(y, z) < 0, \tag{7.33}$$

and

$$\frac{\partial}{\partial y_1} U_{t,x}(y_0, z_0) = \frac{\partial}{\partial z_1} U_{t,x}(y_0, z_0) = 0 . \tag{7.34}$$

We write the relations (7.6), (7.7), (2.11) for y, z_0, and we get

$$\frac{\partial}{\partial y_1} U_{t,x}(y, z_0) + \frac{\partial}{\partial z_1} U_{t,x}(y, z_0) + k_1 \frac{\{U_{t,x}(y, z_0) - U_{t,x}(r_{\alpha_1} y, z_0)\}}{1 - e^{-\langle \alpha_1, y \rangle}}$$
$$+ 2k_2 \frac{\{U_{t,x}(y, z_0) - U_{t,x}(r_{\alpha_3} y, z_0)\}}{1 - e^{-\langle \alpha_3, y \rangle}}$$
$$+ k_3 \left[\frac{U_{t,x}(y, z_0) - U_{t,x}(r_{\alpha_5} y, z_0)}{1 - e^{-\langle \alpha_5, y \rangle}} + \frac{U_{t,x}(y, z_0) - U_{t,x}(r_{\alpha_6} y, z_0)}{1 - e^{-\langle \alpha_6, y \rangle}} \right]$$
$$= (\frac{1}{2} k_1 + k_2 + k_3) U_{t,x}(y, z_0). \tag{7.35}$$

Then by passing to the limit in (7.35), when $\langle \alpha, y \rangle$, for all $\alpha \in \mathcal{R}_+$, goes to $\langle \alpha, y_0 \rangle = 0$, and by using Proposition 7.6 and the relation (7.34), we obtain

$$(\frac{1}{2} k_1 + k_2 + k_3) U_{t,x}(y_0, z_0) = 0.$$

As $k_i > 0, i = 1, 2, 3$, then

$$U_{t,x}(y_0, z_0) = 0. \tag{7.36}$$

Thus (7.33) and (7.36) imply a contradiction, and the point (y_0, z_0) is not in the set Y^0.

Then the function $U_{t,x}(y, z)$ is positive on the set $\mathbb{R}^2 \times \mathbb{R}^2 \backslash Y$. We deduce the relation (7.16) from this result and the fact that the function $U_{t,x}(y, z)$ is positive on the set Y.   □

Theorem 7.9. *Let $t > 0$ and $x \in \mathbb{R}^2$. Then we have*

$$\forall , u, y \in \mathbb{R}^2, {}^t\mathcal{T}_x(p_t(u, .))(y) \geq 0. \tag{7.37}$$

Proof. We deduce (7.37) from the relation (7.4), the positivity of the measure $\breve{\mu}_x$ given by (7.5) and Theorem 7.8.   □

8. Positivity of the hypergeometric translation operator and of its dual

Theorem 8.1. *For all positive function g in $\mathcal{E}(\mathbb{R}^2)$, we have*

$$\forall x, u \in \mathbb{R}^2, \mathcal{T}_x(g)(u) \geq 0. \tag{8.1}$$

Proof. - For $x = 0$: From the relation (5.2) we have

$$\mathcal{T}_0(g)(u) = g(u) \geq 0.$$

- For $x \in \mathbb{R}^2 \backslash \{0\}$: We consider two cases.

1st Case: We suppose that the function g belongs to $\mathcal{D}(\mathbb{R}^2)$.

Let $t > 0$ and $u \in \mathbb{R}^2$. From Proposition 6.2 i) the function $z \to p_t(u, z)$ belongs to $\mathcal{S}_2(\mathbb{R}^2)$. On the other hand by using the properties of the operators $\mathcal{T}_x$ and ${}^t\mathcal{T}_x$ and the relations (5.6), (5.5), we obtain

$$\int_{\mathbb{R}^2} \mathcal{T}_x(g)(z) p_t(u, z) \mathcal{A}_k(z) dz = \int_{\mathbb{R}^2} g(y) \, {}^t\mathcal{T}_x(p_t(u, .))(y) \mathcal{A}_k(y) dy.$$

As from (7.37) the second member of this relation is positive, then we have

$$\int_{\mathbb{R}^2} \mathcal{T}_x(g)(z) p_t(u, z) \mathcal{A}_k(z) dz \geq 0.$$

By using (6.15) this relation can also be written in the form

$$P_t(\mathcal{T}_x(g))(u) \geq 0. \tag{8.2}$$

As the function $\mathcal{T}_x(g)$ belongs to $\mathcal{D}(\mathbb{R}^2)$, then from Proposition 6.5 ii) we obtain

$$\lim_{t \to 0} P_t(\mathcal{T}_x(g))(u) = \mathcal{T}_x(g)(u) \geq 0.$$

2nd Case: We suppose that the function g is in $\mathcal{E}(\mathbb{R}^2)$. Let φ be a positive function in $\mathcal{D}(\mathbb{R}^2)$ radial such that supp $\varphi \subset B(0, 2), \varphi(y) = 1$ for all $y \in B(0, 1)$, and for all $y \in \mathbb{R}^2$, $0 \leq \varphi(y) \leq 1$. We consider the increasing sequence $\{\varphi_n\}_{n \in \mathbb{N} \backslash \{0\}}$ defined by

$$\forall y \in \mathbb{R}^2, \varphi_n(y) = \varphi(\frac{y}{n}).$$

As the function $g\varphi_n$ belongs to $\mathcal{D}(\mathbb{R}^2)$, then from the first case, we have

$$\forall x, u \in \mathbb{R}^2, \mathcal{T}_x(g\varphi_n)(x) \geq 0. \tag{8.3}$$

We obtain (8.1) by using (8.3) and the fact that the sequence $\{g\varphi_n\}_{n \in \mathbb{N} \backslash \{0\}}$ converges to the function g in $\mathcal{E}(\mathbb{R}^2)$, and the continuity of the operator $\mathcal{T}_x$ from $\mathcal{E}(\mathbb{R}^2)$ into itself. $\qquad \square$

Corollary 8.2. *For all positive function f in $\mathcal{D}(\mathbb{R}^2)$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$), we have*

$$\forall x, y \in \mathbb{R}^2, {}^t\mathcal{T}_x(f)(y) \geq 0. \tag{8.4}$$

Proof. We deduce (8.4) from the relation (5.6) and Theorem 8.1. $\qquad \square$

Theorem 8.3. *There exists a σ-algebra $\mathfrak{m}$ in $\mathbb{R}^2$ which contains all Borel sets in $\mathbb{R}^2$, and for each $x, u \in \mathbb{R}^2$, there exists a unique positive measure $m_{x,u}$ on $\mathbb{R}^2$ such that for every g in $\mathcal{E}(\mathbb{R}^2)$, we have*

$$\mathcal{T}_x(g)(u) = \int_{\mathbb{R}^2} g(z)dm_{x,u}(z). \tag{8.5}$$

Proof.
First case: We suppose that the function g is in $\mathcal{D}(\mathbb{R}^d)$. We obtain the results from the relation (8.1) and Theorem 2.14, of [5], p.42.
Second case: We suppose that the function g belongs to $\mathcal{E}(\mathbb{R}^d)$. As we have $g = g^+ - g^-$, we suppose in the following that g is positive. We consider the sequence $\{g\varphi_n\}_{n\in\mathbb{N}}$, given in the proof of Theorem 8.1. From the relation (8.5), for all $n \in \mathbb{N}$ and $x, u \in \mathbb{R}^d$, we have

$$\mathcal{T}_x(g\varphi_n)(u) = \int_{\mathbb{R}^d} g(z)\varphi_n(z)dm_{x,u}(z). \tag{8.6}$$

By making a proof analogue to the proof given in the 2nd case of the proof of Theorem 8.1, we obtain

$$\forall\, x, u \in \mathbb{R}^d, \quad \lim_{n\to\infty} \mathcal{T}_x(g\varphi_n)(u) = \mathcal{T}_x(g)(u). \tag{8.7}$$

On the other hand by using the fact that the sequence $\{g\varphi_n\}_{n\in\mathbb{N}}$ is positive and increasing, then from the monotone convergence theorem, we have

$$\forall\, x, u \in \mathbb{R}^d, \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} g(z)\varphi_n(z)dm_{x,u}(z) = \int_{\mathbb{R}^d} g(z)dm_{x,u}(z). \tag{8.8}$$

We deduce (8.5) from the relations (8.6),(8.7),(8.8). $\qquad\square$

Remark 8.4. By using the relations (5.6), (8.5), we obtain for all f in $\mathcal{D}(\mathbb{R}^2)$ and $u, y \in \mathbb{R}^2$ the following integral representation of the function ${}^t\mathcal{T}_u(f)(y)$:

$$ {}^t\mathcal{T}_u(f)(y) = \int_{\mathbb{R}^2} f(z)d\breve{m}_{y,-u}(z). \tag{8.9}$$

where $\breve{m}_{y,-u}$ is the positive measure on $\mathbb{R}^2$ given by

$$\int_{\mathbb{R}^2} f(z)d\breve{m}_{y,-u}(z) = \int_{\mathbb{R}^2} f(-z)dm_{y,-u}(z). \tag{8.10}$$

Corollary 8.5. *The product formula (5.3) can also be written in the form*

$$\forall\, x, u \in \mathbb{R}^2,\ \forall\, \lambda \in \mathbb{C}^2, G_\lambda(x)G_\lambda(u) = \int_{\mathbb{R}^2} G_\lambda(z)dm_{x,u}(z). \tag{8.11}$$

Proof. We deduce the relation (8.11) from Theorem 8.3 and the fact that for all $\lambda \in \mathbb{C}^2$, the function $G_\lambda(x)$ belongs to $\mathcal{E}(\mathbb{R}^2)$. $\qquad\square$

Corollary 8.6. *For all* $x, u \in \mathbb{R}^2$, *we have*

$$\|m_{x,u}\| = m_{x,u}(1) = 1. \tag{8.12}$$

Proof.

- We prove first that

$$\forall x \in \mathbb{R}^2, \quad G_{-i\rho}(x) = 1. \tag{8.13}$$

From the relations (2.9), (2.13) the function $f(x) = G_{-i\rho}(x)$ is the unique solution satisfying $f(0) = 1$ of the system

$$\frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathcal{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} = 0, \quad j = 1, 2, x \in \mathbb{R}^2. \tag{8.14}$$

But, the constant function $f(x) = 1$ is also a solution of this system. Thus we obtain (8.13) from the unicity of the solution of the system (8.14).

- We obtain (8.12) by taking $\lambda = -i\rho$ in the relation (8.11) and by using (8.13). □

Corollary 8.7. *The Cherednik-Opdam kernel* $G_\lambda(x)$ *admits the following estimate*

$$\forall \, \lambda \in \mathbb{R}^2, \sup_{x \in \mathbb{R}^2} |G_\lambda(x)| = 1. \tag{8.15}$$

Proof. From the relations (8.11), (2.18), for all $\lambda \in \mathbb{R}^2$, we have

$$\left(\sup_{x \in \mathbb{R}^2} |G_\lambda(x)| \right) \left(\sup_{u \in \mathbb{R}^2} |G_\lambda(u)| \right) \leq \int_{\mathbb{R}^2} \left(\sup_{z \in \mathbb{R}^2} |G_\lambda(z)| \right) dm_{x,u}(z).$$

Thus by applying (8.12) we get

$$\sup_{x \in \mathbb{R}^2} |G_\lambda(x)| \leq \int_{\mathbb{R}^d} dm_{x,u}(z) = 1.$$

We deduce (8.15) from this relation and (2.13). □

Proposition 8.8. *For all* $x, u \in \mathbb{R}^2$, *we have*

$$supp\, m_{x,u} \subset \{z \in \mathbb{R}^2 \, ; \, |\|x\| - \|u\|| \leq \|z\| \leq \|x\| + \|u\|\}. \tag{8.16}$$

To prove this proposition we need the following lemma.

Lemma 8.9. *Let* $x, u \in \mathbb{R}^2$ *and* f *be in* $\mathcal{D}(\mathbb{R}^2)$. *if the support of* f *is contained in the closed ball* $B(0, |\|x\| - \|u\||)$, *we have*

$$\mathcal{T}_x(f)(u) = 0. \tag{8.17}$$

Proof. Set $\eta = |\|x\| - \|u\||$. Since by the relation

$$\mathcal{T}_x(f)(u) = \mathcal{T}_u(f)(x),$$

we may suppose that $\|x\| \leq \|u\|$. Let $y = \eta + \|x\|\sigma$, with σ is a point of the unit sphere S^1 of $\mathbb{R}^2$. □

From (5.6), (5.10), we deduce that

$$\mathcal{T}_x(f)(y) = 0.$$

By replacing η by its value in y, we obtain $y = \|u\|\sigma$. Thus,

$$\mathcal{T}_x(f)(u) = 0.$$

□

Proof of Proposition 8.8.

For all $x, u \in \mathbb{R}^2$, the relation (8.11) can also be written in the form

$$\forall \lambda \in \mathbb{C}^2, \mathcal{H}(m_{x,u})(\lambda) = G_\lambda(x)G_\lambda(u), \tag{8.18}$$

where $\mathcal{H}$ is the hypergeometric Fourier transform of the measure $m_{x,u}$ which is of compact support (see [8], p.32).

On the other hand from the relation (2.22), (2.19) there exists a positive constant M_0 such that

$$\forall \lambda \in \mathbb{C}^d, |G_\lambda(x)G_\lambda(u)| \leq M_0 e^{(\|x\|+\|u\|)\|Im\lambda\|},$$

thus (8.18) implies

$$\forall \lambda \in \mathbb{C}^2, |\mathcal{H}(m_{x,u})(\lambda)| \leq M_0 e^{(\|x\|+\|u\|)\|Im\lambda\|}.$$

From this relation and Theorem 2.4 of [8], we deduce that

$$\operatorname{supp} m_{x,u} \subset B(0, \|x\| + \|u\|). \tag{8.19}$$

We obtain (8.16) from Lemma 8.9 and (8.19). □

9. Positivity of the hypergeometric translation operator and its dual associated with the Heckman-Opdam theory

We have defined and studied in [9] the transmutation operator V_k^W and its dual ${}^tV_k^W$ associated with the Heckman-Opdam theory defined respectively on $\mathcal{E}(\mathbb{R}^2)^W$, $\mathcal{D}(\mathbb{R}^2)^W$ and $S_2(\mathbb{R}^2)^W$ (the subspaces respectively of $\mathcal{E}(\mathbb{R}^2)$, $\mathcal{D}(\mathbb{R}^2)$ and $S_2(\mathbb{R}^2)$ which are W-invariant). In this section we give the definition and the properties of the hypergeometric translation operator $\mathcal{T}_x^W$, $x \in \mathbb{R}^2$, and we

deduce from the results of the section 8, that it satisfies the analogue of the relation (8.1) and its consequences.

The hypergeometric translation operator $\mathcal{T}_x^W, x \in \mathbb{R}^2$, associated with the Heckman-Opdam theory, is defined on $\mathcal{E}(\mathbb{R}^2)^W$ by

$$\forall\, y \in \mathbb{R}^2, \mathcal{T}_x^W(f)(y) = (V_k^W)_x (V_k^W)_y [(V_k^W)^{-1}(x+y)]. \tag{9.1}$$

The dual $^t\mathcal{T}_x^W, x \in \mathbb{R}^2$, of the operator $\mathcal{T}_x^W, x \in \mathbb{R}^2$, is defined on $\mathcal{D}(\mathbb{R}^2)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^2)^W$) by

$$\forall\, y \in \mathbb{R}^2, {}^t\mathcal{T}_x^W(f)(y) = \mathcal{T}_y^W(f)(-x). \tag{9.2}$$

The main properties satisfied by the operator $\mathcal{T}_x^W, x \in \mathbb{R}^2$, are the following:

1. For all $x \in \mathbb{R}^2$, the operator $\mathcal{T}_x^W$ is continuous from
   - $\mathcal{D}(\mathbb{R}^2)^W$ into itself.
   - $\mathcal{S}_2(\mathbb{R}^2)^W$ into itself.

2. For all $x, y \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}^2$, we have the product formula

$$\mathcal{T}_x^W(F_\lambda)(y) = F_\lambda(x)F_\lambda(y) \tag{9.3}$$

   for the Heckman-Opdam hypergeometric function $F_\lambda(x)$ defined by (2.14).

3. For all f in $\mathcal{D}(\mathbb{R}^2)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^2)^W$) and $x, y \in \mathbb{R}^2$, we have

$$\mathcal{T}_x^W(f)(y) = \int_{\mathbb{R}^2} F_\lambda(x)F_\lambda(-y)\mathcal{H}^W(f)(\lambda)\mathcal{C}_k^W(\lambda)d\lambda, \tag{9.4}$$

   where

$$\mathcal{C}_k^W(\lambda) = c|C_k(\lambda)|^{-2},$$

with c a positive constant chosen in such a way that $\mathcal{C}_k^W(-\rho) = 1, |C_k(\lambda)|^{-2}$ the function given by (4.4), (4.5), and $\mathcal{H}^W$ the hypergeometric Fourier transform associated with the Heckman-Opdam theory defined on $\mathcal{D}(\mathbb{R}^2)^W$ (resp. $\mathcal{S}_2(\mathbb{R}^2)$) by

$$\forall\, \lambda \in \mathbb{R}^2, \mathcal{H}^W(f)(\lambda) = \int_{\mathbb{R}^2} f(x)F_\lambda(x)\mathcal{A}_k(x)dx. \tag{9.5}$$

By using the results of Section 8, we deduce the following results.

Theorem 9.1. *For all $x, u \in \mathbb{R}^2$ and g in $\mathcal{E}(\mathbb{R}^2)^W$ we have*

$$\mathcal{T}_x^W(g)(x) = \int_{\mathbb{R}^2} g(z)dm_{x,u}^W(z), \tag{9.6}$$

with

$$m_{x,u}^W = \frac{1}{|W|^2} \sum_{w,w' \in W} m_{wx,w'u},$$ (9.7)

where $m_{x,u}$ is the measure given by (8.5).

Proposition 9.2.

  i) *The product formula (9.3) can also be written in the form*

$$\forall\, x, u \in \mathbb{R}^2, \forall\, \lambda \in \mathbb{C}^2, F_\lambda(x) F_\lambda(u) = \int_{\mathbb{R}^2} F_\lambda(z) dm_{x,u}^W(z).$$ (9.8)

  ii) *For all $x, u \in \mathbb{R}^2$, we have*

$$\|m_{x,u}^W\| = m_{x,u}^W(1) = 1.$$ (9.9)

  iii) *For all $x, u \in \mathbb{R}^2$, we have*

$$\mathrm{supp}\, m_{x,u}^W \subset \{z \in \mathbb{R}^2\,; |\|x\| - \|u\|| \leq \|z\| \leq \|x\| + \|u\|\}.$$ (9.10)

Proposition 9.3. *The Heckman-Opdam hypergeometric function $F_\lambda(x)$ admits the following estimate*

$$\forall\, \lambda \in \mathbb{R}^2, \ \sup_{x \in \mathbb{R}^2} |F_\lambda(x)| \leq 1.$$ (9.11)

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