

ALTERNATIVE TO THE HYPERCOMPLEX  
FOURIER TRANSFORM DEFINITION

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**Abstract:** In this paper we present a construction of the quaternionic hypercomplex version of Fourier Series and an introduction of a simple quaternionic expansion of Fourier Transform in hypercomplex exponentials is provided. By some examples, we show how the representation is written by using the proposed expansion.

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**Key Words:** hypercomplex functions, quaternions, hypercomplex Fourier transform

1. Introduction and Motivation

The hypercomplex numbers are mostly concerned with quaternions and octonions. The involved algebras may be considered as extensions of the ordinary bi-dimensional complex algebra, according to [10], [2], the operation is non-commutative for quaternions.

The quaternions numbers were discovered in 1843 by Willian R. Hamilton (see [5]) and in that same year John T. Graves, a Hamilton's friend, found an 8-dimensional algebra whose property on non-associativity in a multiplication table holds. Two years later, in 1845, after some contributions on this subject by Arthur Cayley, the octonions have also been named as "Cayley numbers".

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Among many possible methods for estimating complex-valued functions, Fourier Series are specifically attractive because the uniform convergence of the Fourier series (as more terms are added) is guaranteed for continuous and bounded functions. Furthermore, the Fourier coefficients are designed to minimize the square of the error from the actual function. Finally, it is relatively simple to deal with complex exponentials and these frequently occur in physical phenomena.

The Fourier Transform is very important also in theoretical mathematics area. The Fourier Transform of time function is already a complex-valued function of frequency, whose absolute value represents the amount of that frequency in the original function, and whose complex argument is the phase offset of the basic sinusoid in that frequency.

In this paper, based on the previous work (Pendeza et al., see in [8]), a simple quaternionic expansion of Fourier transform in hypercomplex exponentials is proposed. Through some examples, we show how the representation is written by using the proposed expansion, in order to approach practical problems in a near future.

## 2. Hypercomplex Fourier Series

The quaternions form the 4-dimensional normed division algebra on  $\mathcal{R}$ . The quaternions algebra denoted by  $\mathcal{Q}$ , is an alternative division algebra. The quaternions set  $\mathcal{Q}$  is defined by  $\mathcal{Q} = \{a, b, c, d\} \in \mathcal{R}$ , where

$$(a, b, c, d) = (a', b', c', d') \iff a = a', b = b', c = c', d = d'.$$

The quaternions do not form a ring due to the non-commutativity of the multiplication. Also they do not form a group due to a nonassociativity of multiplication. They form a *Moufang Loop*, a Loop with identity element (N. Jacobson, [11]).

Let us consider a quaternionic number given by

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

The quaternion unities  $\mathbf{1}$ ,  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  form an orthonormal base of the 4-dimensional algebra. Similarly to [9], we start our contribution by considering  $f$  as a function defined on interval  $[-L, L]$ ,  $L > 0$  and, outside of this interval, a set as  $f(x) = f(x + 2L)$ , that is,  $f(x)$  is  $2L$ -periodical. If  $f$  and  $f'$  are piecewise continuous then the series of function given in (1),

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right], \tag{1}$$

is convergent and the limit is  $\tilde{f}(x) = \frac{\lim_{a \rightarrow x^+} f(x) + \lim_{a \rightarrow x^-} f(x)}{2}$ .

The coefficients of Fourier of  $f$ ,  $a_0$ ,  $a_n$  and  $b_n$  are given by:

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx,$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left( \frac{n\pi x}{L} \right) dx, \tag{2}$$

and

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \left( \frac{n\pi x}{L} \right) dx. \tag{3}$$

The trigonometric series presented in (1) with this choice of coefficients is the Fourier series of  $f$ .

Now, we will detail some properties considering quaternion  $q \in \mathcal{Q}$  given by

$$q = u_1 + u_2i + u_3j + u_4k = u_1 + \vec{u}.$$

The equation of De Moivre for quaternions is given by

$$e^q = e^{u_1} \left\{ \cos \left( \sqrt{u_2^2 + u_3^2 + u_4^2} \right) + \vec{u} \left( \frac{\sin(\sqrt{u_2^2 + u_3^2 + u_4^2})}{\sqrt{u_2^2 + u_3^2 + u_4^2}} \right) \right\}. \tag{4}$$

Then, using (4) we can obtain:

$$\frac{e^{iy} + e^{-iy}}{2} = \frac{e^0(\cos|y| + iy\frac{\sin|y|}{|y|}) + e^0(\cos|-y| - iy\frac{\sin|-y|}{|-y|})}{2}.$$

Since  $\cos|-y| = \cos|y| = \cos(y)$  and  $\sin|-y| = \sin|y|$ , we have

$$\frac{e^{iy} + e^{-iy}}{2} = \cos(y).$$

Following the same arguments, we can obtain

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} = \frac{e^{jy} + e^{-jy}}{2} = \frac{e^{ky} + e^{-ky}}{2}, \tag{5}$$

and

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} = \frac{e^{jy} - e^{-jy}}{2j} = \frac{e^{ky} - e^{-ky}}{2k}. \quad (6)$$

From (5) and (6), we write

$$\begin{aligned} \cos y &= \frac{(e^{iy} + e^{jy} + e^{ky}) + (e^{-iy} + e^{-jy} + e^{-ky})}{6}, \\ \sin y &= \frac{1}{3} \left[ \frac{e^{iy} - e^{-iy}}{2i} + \frac{e^{jy} - e^{-jy}}{2j} + \frac{e^{ky} - e^{-ky}}{2k} \right]. \end{aligned}$$

Consequently,

$$\begin{aligned} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) &= a_n \left[ \frac{(e^{i(\frac{n\pi x}{L})} + e^{j(\frac{n\pi x}{L})} + e^{k(\frac{n\pi x}{L})})}{6} \right. \\ &\quad \left. + \frac{(e^{-i(\frac{n\pi x}{L})} + e^{-j(\frac{n\pi x}{L})} + e^{-k(\frac{n\pi x}{L})})}{6} \right] + \frac{b_n}{3} \left[ \frac{e^{i(\frac{n\pi x}{L})} - e^{-i(\frac{n\pi x}{L})}}{2i} \right. \\ &\quad \left. + \frac{e^{j(\frac{n\pi x}{L})} - e^{-j(\frac{n\pi x}{L})}}{2j} + \frac{e^{k(\frac{n\pi x}{L})} - e^{-k(\frac{n\pi x}{L})}}{2k} \right] \\ &= \frac{1}{3} \left[ \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2i}\right)}_{c_n^1} e^{\frac{in\pi x}{L}} + \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2i}\right)}_{c_{-n}^1} e^{\frac{-in\pi x}{L}} + \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2j}\right)}_{c_n^2} e^{\frac{jn\pi x}{L}} \right. \\ &\quad \left. + \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2j}\right)}_{c_{-n}^2} e^{\frac{-jn\pi x}{L}} + \underbrace{\left(\frac{a_n}{2} + \frac{b_n}{2k}\right)}_{c_n^3} e^{\frac{kn\pi x}{L}} + \underbrace{\left(\frac{a_n}{2} - \frac{b_n}{2k}\right)}_{c_{-n}^3} e^{\frac{-(k)n\pi x}{L}} \right]. \end{aligned}$$

By considering that

$$c_n^1 = \frac{a_n}{2} + \frac{b_n}{2i} = \frac{1}{2}(a_n - ib_n),$$

and using (2) and (3), we have

$$c_n^1 = \frac{1}{2L} \int_{-L}^L f(x) e^{-i(\frac{n\pi x}{L})} dx. \quad (7)$$

Again, using the same argument we can obtain

$$c_{-n}^1 = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\left(\frac{-n}{L}\pi x\right)} dx. \tag{8}$$

Extending the used ideas, we conclude that

$$c_n^2 = \frac{1}{2L} \int_{-L}^L f(x) e^{-j\left(\frac{n}{L}\pi x\right)} dx, \tag{9}$$

since  $(j)^2 = -1$ ,

$$c_n^3 = \frac{1}{2L} \int_{-L}^L f(x) e^{-k\left(\frac{n}{L}\pi x\right)} dx, \tag{10}$$

since  $(k)^2 = -1$ . From (7), (8), (9) and (10) we can write

$$\begin{aligned} \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \\ = \frac{1}{3} \sum_{-\infty, n \neq 0}^{\infty} \left[ c_n^1 e^{\frac{in\pi x}{L}} + c_n^2 e^{\frac{jn\pi x}{L}} + c_n^3 e^{\frac{kn\pi x}{L}} \right]. \end{aligned}$$

Since  $f : R \rightarrow R$  is periodic, with period  $2L$ ,  $f$  and  $f'$  are piecewise continuous, we have that the Fourier series of  $f$  presented in (1) can be written as follows:

$$\tilde{f}(x) = c_0 + \frac{1}{3} \left\{ \sum_{-\infty, n \neq 0}^{\infty} \left( c_n^1 e^{\frac{in\pi x}{L}} + c_n^2 e^{\frac{jn\pi x}{L}} + c_n^3 e^{\frac{kn\pi x}{L}} \right) \right\}, \tag{11}$$

by considering  $c_0 = \frac{a_0}{2}$ .

The series (11) is the Hypercomplex Fourier Series of  $f$ .

### 3. The Hypercomplex Fourier Transform

In this section we will consider the definition of Hypercomplex Fourier Series (11) to deduce a model for Hypercomplex Fourier Transform. From equation (11), consider  $f$  defined in the interval  $(-L, L)$  with  $f$  and  $f'$  are piecewise continuous. Denoting  $\alpha_n = \frac{n\pi}{L}$ , we obtain

$$\tilde{f}(x) = \frac{1}{2L} \int_{-L}^L f(u) du + \frac{1}{6L} \sum_{-\infty, n \neq 0}^{\infty} \left\{ \left[ \int_{-L}^L f(u) e^{-i(\alpha_n u)} du \right] e^{i\alpha_n x} \right.$$

$$+ \left[ \int_{-L}^L f(u)e^{-j(\alpha_n u)} du \right] e^{j\alpha_n x} + \left[ \int_{-L}^L f(u)e^{-k(\alpha_n u)} du \right] e^{k\alpha_n x} \Big\},$$

considering  $\Delta\alpha = \alpha_{n+1} - \alpha_n$ ,

$$= \frac{\Delta\alpha}{2\pi} \int_{-L}^L f(u)du + \frac{1}{6\pi} \sum_{-\infty, n \neq 0}^{\infty} \left\{ \left[ \int_{-L}^L f(u)e^{-i(\alpha_n u)} du \right] e^{i\alpha_n x} \right. \\ \left. + \left[ \int_{-L}^L f(u)e^{-j(\alpha_n u)} du \right] e^{j\alpha_n x} + \left[ \int_{-L}^L f(u)e^{-k(\alpha_n u)} du \right] e^{k\alpha_n x} \right\} \Delta\alpha,$$

calculating the limit  $\Delta\alpha \rightarrow 0$  and assuming  $\int_{-\infty}^{\infty} |f(u)|du < \infty$

$$= \lim_{\Delta\alpha \rightarrow 0} \frac{1}{6\pi} \sum_{-\infty, n \neq 0}^{\infty} \left\{ \left[ \int_{-L}^L f(u)e^{-i(\alpha_n u)} du \right] e^{i\alpha_n x} \right. \\ \left. + \left[ \int_{-L}^L f(u)e^{-j(\alpha_n u)} du \right] e^{j\alpha_n x} + \left[ \int_{-L}^L f(u)e^{-k(\alpha_n u)} du \right] e^{k\alpha_n x} \right\} \Delta\alpha.$$

From the definition of the Riemann integral, we obtain

$$= \frac{1}{6\pi} \int_{-\infty}^{\infty} \left\{ \left[ \int_{-\infty}^{\infty} f(u)e^{-i(\alpha u)} du \right] e^{i\alpha x} + \left[ \int_{-\infty}^{\infty} f(u)e^{-j(\alpha u)} du \right] e^{j\alpha x} \right. \\ \left. + \left[ \int_{-\infty}^{\infty} f(u)e^{-k(\alpha u)} du \right] e^{k\alpha x} \right\} d\alpha.$$

Or, equivalently, we have

$$\frac{1}{6\pi} \int_{-\infty}^{\infty} \left\langle \left( \int_{-\infty}^{\infty} f(u)e^{-i(\alpha u)} du, \int_{-\infty}^{\infty} f(u)e^{-j(\alpha u)} du, \right. \right. \\ \left. \left. \int_{-\infty}^{\infty} f(u)e^{-k(\alpha u)} du, \right) (e^{i\alpha x}, e^{j\alpha x}, e^{k\alpha x}) \right\rangle d\alpha, \tag{12}$$

denoting

$$F_i(\alpha) = \int_{-\infty}^{\infty} f(u)e^{-i(\alpha u)} du, \quad F_j(\alpha) = \int_{-\infty}^{\infty} f(u)e^{-j(\alpha u)} du, \\ F_k(\alpha) = \int_{-\infty}^{\infty} f(u)e^{-k(\alpha u)} du.$$

Thus (12) can be written as

$$\tilde{f}(x) = \frac{1}{6\pi} \int_{-\infty}^{\infty} \langle (F_i(\alpha), F_j(\alpha), F_k(\alpha)), (e^{i\alpha x}, e^{j\alpha x}, e^{k\alpha x}) \rangle d\alpha. \tag{13}$$

We define transformed as Hypercomplex,

$$F_{\mathbb{H}}(\alpha) = \mathfrak{T}_{\mathbb{H}}(f(x)) = (F_i(\alpha), F_j(\alpha), F_k(\alpha)).$$

The inverse transform is calculated by the equation

$$\mathfrak{T}_{\mathbb{H}}^{-1} = \frac{1}{6\pi} \int_{-\infty}^{\infty} \langle F_{\mathbb{H}}(\alpha), (e^{i\alpha x}, e^{j\alpha x}, e^{k\alpha x}) \rangle d\alpha,$$

We present below a numerical example for the Hypercomplex Fourier Transform calculation.

**Example 1.** Consider the function

$$f(x) = \begin{cases} 1, & \text{for } -\pi < x < 0 \\ -1, & \text{for } 0 < x < \pi \end{cases}.$$

Computing the Hypercomplex Fourier Transform we obtain

$$\begin{aligned} F_{\mathbb{H}}(\alpha) &= \mathfrak{T}_{\mathbb{H}}(f(x)) \\ &= 2 \left( i \frac{\cos(\alpha\pi) - 1}{\alpha}, j \frac{\cos(\alpha\pi) - 1}{\alpha}, k \frac{\cos(\alpha\pi) - 1}{\alpha} \right). \end{aligned}$$

**Example 2.** The Dirac delta (see [8]) can be poorly thought as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

and which is also constrained to satisfy the identity

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$

The delta function is a tempered distribution and, therefore, it has a well-defined Fourier transform.

$$\mathfrak{T}(\delta(\xi)) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} \delta(x) dx = 1.$$

Thus, calculating

$$\begin{aligned} F_{\mathbb{H}}(\alpha) &= \mathfrak{F}_{\mathbb{H}}(\delta(\xi)) \\ &= (F_i(\alpha), F_j(\alpha), F_k(\alpha)) \\ &= (1, 1, 1), \end{aligned}$$

we obtain consequently,

$$\begin{aligned} \mathfrak{F}_{\mathbb{H}}^{-1}(1, 1, 1) &= \frac{1}{6\pi} \int_{-\infty}^{\infty} \langle (1, 1, 1), (e^{i\alpha x}, e^{j\alpha x}, e^{k\alpha x}) \rangle d\alpha \\ &= \delta(\xi). \end{aligned}$$

#### 4. Concluding Remarks

In order to generalize the Fourier transform to its quaternionic form, first we described Fourier series and we introduced the hypercomplex model by considering results from [8, 9], which unable to obtain a formulation for hypercomplex Fourier transform, which depends on a product internal. Furthermore, since not few models of Theoretical Physics may be analyzed through the geometry and algebra of hypercomplex, it will be our concern to concentrate the next steps in making all possible applications of our results in the context of unified physical theories for higher dimensional space-times.

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