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ON SOME TOPOLOGICAL PROPERTIES OF GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED

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Abstract: In this paper, we define new generalized difference sequence spaces $\ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u)$ and $\ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u)$, where $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ are sequences of Orlicz functions such that M_k and N_k are mutually complementary for each k. We also examine some topological properties and establish some inclusion relations between these spaces.

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1. Introduction

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrauss and Tzafriri [4] used the idea of Orlicz function to define the Orlicz sequence space

$$\ell_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is a Banach space with the norm

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$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 \leq \lambda \leq 1$.

Any Orlicz function M_k always has the integral representation

$$M_k(x) = \int_0^x p_k(t)dt,$$

where p_k, known as the kernel of M_k, is non-decreasing, is right continuous for $t > 0$, $p_k(0) = 0$, $p_k(t) > 0$ for $t > 0$ and $p_k(t) \to \infty$ as $t \to \infty$.

Given an Orlicz function M_k with kernel $p_k(t)$, define

$$q_k(s) = \sup \{t : p_k(t) \leq s,\ s \geq 0\}.$$

Then $q_k(s)$ possesses the same properties as $p_k(t)$ and the function N_k, defined as

$$N_k(x) = \int_0^x q_k(s)ds,$$

is an Orlicz function. The functions M_k and N_k are called mutually complementary Orlicz functions [1].

The difference sequence spaces were introduced by Kızmaz [2] and the concept was generalized by Et and Çolak [7]. Later, Et and Esi [6] extended the difference sequence spaces to the sequence spaces

$$X(\Delta_v^m) = \{x = (x_k) : (\Delta_v^m x_k) \in X\}$$

for $X = \ell_\infty$, c or c_0, where $v = (v_k)$ be any fixed sequence of non-zero complex numbers and $(\Delta_v^m x_k) = \left(\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}\right)$, $\Delta_v^m x = \sum_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for all $k \in \mathbb{N}$.

The sequence spaces $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} |v_i x_i| + \|\Delta_v^m x\|_\infty.$$

Definition 1. Let λ be a sequence space. Then λ is called

(i) *Solid* (or *normal*), if $(\alpha_k x_k) \in \lambda$ whenever $(x_k) \in \lambda$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$.

(ii) *Monotone*, if provided λ contains the canonical preimages of all its stepspaces.

(iii) *Perfect*, if $\lambda = \lambda^{\alpha\alpha}$, see [9].

Proposition 2. *λ is perfect $\Rightarrow$ λ is normal $\Rightarrow$ λ is monotone, see* [9].

A Banach sequence space (λ, S) is called a BK-space, if the topology S of λ is finer than the co-ordinatewise convergence topology, or equivalently, the projection maps $P_i : \lambda \to K$, $P_i(x) = x_i$, $i \geq 1$ are continuous, where K is the scalar field $\mathbb{R}$ (the set of all reals) or $\mathbb{C}$ (the complex plane). For $x = (x_1, ..., x_n, ...)$ and $n \in \mathbb{N}$ (the set of natural numbers), we write the n^{th} section of x as $x^{(n)} = (x_1, ..., x_n, 0, 0, ...)$. If $\{x^{(n)}\}$ tends to x in (λ, S) for each $x \in \lambda$, we say that (λ, S) is an $AK-$space. The norm $\|.\|_\lambda$ generating the topology S of λ is said to be monotone if $\|x\|_\lambda \leq \|y\|_\lambda$ for $x = \{x_i\}$, $y = \{y_i\}$ $\in \lambda$ with $|x_i| \leq |y_i|$, for all $i \geq 1$, see [8].

Definition 3. Any two Orlicz functions M_1 and M_2 are said to be *equivalent*, if there are positive constant α and β, and x_0 such that $M_1(\alpha x) \leq M_2(x) \leq M_1(\beta)$ for all x with $0 \leq x \leq x_0$, see [9].

2. Main Results

Definition 4. Let M_k and N_k be mutually complementary functions for each k and let $\lambda = \{\lambda_k\}$ be a sequence of strictly positive real numbers. Then we define the following sequence spaces:

$$\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u) = \{x = (x_k) : \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) < \infty, \text{ for some } \rho > 0\}$$

and

$$\ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u) = \{x = (x_k) : \sum_{k\geq 1} u_k N_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}.$$

Throughout the paper, we write $M_k(1) = 1$ and $N_k(1) = 1$ for all $k \in \mathbb{N}$.

Theorem 5. *Let $\mathcal{M} = (M_k)$ and $\mathcal{N} = (N_k)$ be two sequences of Orlicz functions. Then $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ and $\ell_{\mathcal{N}}^\lambda(\Delta_v^m, u)$ are linear spaces over the field of complex numbers.*

Proof. Let $x, y \in \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ and $a, b \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho_1}\right) < \infty$$

and

$$\sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m y_k|}{\lambda_k \rho_2}\right) < \infty.$$

Define $\rho_3 = \max(2|a|\rho_1, 2|b|\rho_2)$. Since M_k are non-decreasing and convex functions and Δ^m is linear, we have

$$\sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m (ax_k + by_k)|}{\lambda_k \rho_3}\right)$$
$$\leq \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho_1}\right) + \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m y_k|}{\lambda_k \rho_2}\right) < \infty.$$

This proves that $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ is a linear space. The proof for $\ell_{\mathcal{N}}^\lambda(\Delta_v^m, u)$ is similar. □

The proofs of the following theorems are easy and thus omitted.

Theorem 6. *The sequence space $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ is a normed space with norm*

$$\|x\|_\lambda^\Delta = \sum_{i=1}^m |x_i| + \inf\{\rho > 0 : \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) \leq 1\}.$$

Theorem 7. *The sequence space $\ell_{\mathcal{N}}^\lambda(\Delta_v^m, u)$ is a normed space with norm*

$$\|x\|_\Delta^\lambda = \sum_{i=1}^m |x_i| + \inf\{\rho > 0 : \sum_{k\geq 1} u_k N_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho}\right) \leq 1\}.$$

Theorem 8. *The spaces $\left(\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u), \|.\|_\lambda^\Delta\right)$ and $\left(\ell_{\mathcal{N}}^\lambda(\Delta_v^m, u), \|.\|_\Delta^\lambda\right)$ are Banach spaces.*

Theorem 9. *The sequence spaces* $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ *equipped with the norm* $\|.\|_\lambda^\Delta$ *and* $\ell_{\mathcal{N}}^\lambda(\Delta_v^m, u)$ *equipped with the norm* $\|.\|_\Delta^\lambda$ *are* BK*-spaces.*

Proof. The space $\left(\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u), \|.\|_\lambda^\Delta\right)$ is a Banach space by Theorem 8. Now let

$$\|x^n - x\|_\lambda^\Delta \to 0$$

as $n \to \infty$. Then

$$|x_k^n - x_k| \to 0$$

as $n \to \infty$ for each $k \leq m$ and

$$\inf\{\rho > 0 : \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k^n - \Delta_v^m x_k|}{\lambda_k \rho}\right) \leq 1\} \to 0$$

as $n \to \infty$ for all $k \in \mathbb{N}$. If $u_k M_k \left(\frac{|\Delta_v^m x_k^n - \Delta_v^m x_k|}{\lambda_k \|x\|_\lambda^\Delta}\right) \leq 1$, then $\frac{|\Delta_v^m x_k^n - \Delta_v^m x_k|}{\lambda_k \|x\|_\lambda^\Delta} \leq 1$ for all k. Therefore we also obtain

$$|\Delta_v^m x_k^n - \Delta_v^m x_k| \leq \lambda_k \|x^n - x\|_\lambda^\Delta .$$

Since $\|x^n - x\|_\lambda^\Delta \to 0$, then $|\Delta_v^m x_k^n - \Delta_v^m x_k| \to 0$ and

$$\left|\sum_{i=0}^m (-1)^i \binom{m}{v} v_{k+i} \left(x_{k+i}^n - x_{k+i}\right)\right| \to 0$$

as $n \to \infty$ for all $k \in \mathbb{N}$. On the other hand, we may write

$$\begin{aligned} \left|v_{k+m} \left(x_{k+m}^n - x_{k+m}\right)\right| \leq & \left|\sum_{i=0}^m (-1)^i \binom{m}{v} v_{k+i} \left(x_{k+i}^n - x_{k+i}\right)\right| \\ & + \left|\binom{m}{0} v_k \left(x_k^n - x_k\right)\right| + ... \\ & + \left|\binom{m}{m-1} v_{k+m-1} \left(x_{k+m-1}^n - x_{k+m-1}\right)\right| . \end{aligned}$$

Then $|x_k^n - x_k| \to 0$ as $n \to \infty$ for all $k \in \mathbb{N}$. Hence $\left(\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u), \|.\|_\lambda^\Delta\right)$ is a BK-space.

The proof is similar for $\ell_{\mathcal{N}}^\lambda(\Delta_v^m, u)$. □

Theorem 10. *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. Then* $\ell_\lambda^{\mathcal{M}}(\Delta_v^{m-1}, u) \subset \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$. *In general* $\ell_\lambda^{\mathcal{M}}(\Delta_v^i, u) \subset \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$, *for* $i = 1, 2, ..., m-1$.

Proof. Let $x \in \ell_\lambda^{\mathcal{M}}(\Delta_v^{m-1}, u)$. Then

$$\sum_{k\geq 1} u_k M_k \left(\frac{\left|\Delta_v^{m-1} x_k\right|}{\lambda_k 2\rho} \right) < \infty.$$

Since M_k are non-decreasing and convex functions,

$$\sum_{k\geq 1} u_k M_k \left(\frac{\left|\Delta_v^{m} x_k\right|}{\lambda_k 2\rho} \right) = \sum_{k\geq 1} u_k M_k \left(\frac{\left|\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}\right|}{\lambda_k 2\rho} \right)$$
$$\leq \sum_{k\geq 1} u_k M_k \left(\frac{\left|\Delta_v^{m-1} x_k\right|}{\lambda_k \rho} \right) + \sum_{k\geq 1} u_k M_k \left(\frac{\left|\Delta_v^{m-1} x_{k+1}\right|}{\lambda_k \rho} \right) < \infty.$$

Thus $\ell_\lambda^{\mathcal{M}}(\Delta_v^{m-1}, u) \subset \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$. This completes the proof. □

Theorem 11. *Let $\mathcal{N} = (N_k)$ be a sequence of Orlicz functions. Then* $\ell_{\mathcal{N}}^{\lambda}(\Delta_v^{m-1}, u) \subset \ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u)$. *In general* $\ell_{\mathcal{N}}^{\lambda}(\Delta_v^i, u) \subset \ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u)$, *for* $i = 1, 2, ..., m-1$.

Proof. Let $x \in \ell_{\mathcal{N}}^{\lambda}(\Delta_v^{m-1}, u)$. Then

$$\sum_{k\geq 1} u_k N_k \left(\frac{\left|\Delta_v^{m-1} x_k\right|}{\lambda_k \rho} \right) < \infty.$$

Since N_k are non-decreasing and convex functions

$$\sum_{k\geq 1} u_k N_k \left(\frac{\lambda_k \left|\Delta_v^{m} x_k\right|}{2\rho} \right) = \sum_{k\geq 1} u_k N_k \left(\frac{\lambda_k \left|\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1}\right|}{2\rho} \right)$$
$$\leq \sum_{k\geq 1} u_k N_k \left(\frac{\lambda_k \left|\Delta_v^{m-1} x_k\right|}{\rho} \right) + \sum_{k\geq 1} u_k N_k \left(\frac{\lambda_k \left|\Delta_v^{m-1} x_{k+1}\right|}{\rho} \right) < \infty.$$

Thus $\ell_{\mathcal{N}}^{\lambda}(\Delta_v^{m-1}, u) \subset \ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u)$. This completes the proof. □

Theorem 12. *Let $\mathcal{M} = (M_k)$ and $\mathcal{T} = (T_k)$ be any two sequence of Orlicz functions. Then we have:*

(i) $\ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) \cap \ell_{\lambda}^{\mathcal{T}}(\Delta_v^m, u) \subset \ell_{\lambda}^{\mathcal{M}+\mathcal{T}}(\Delta_v^m, u)$,
(ii) *If* $\mathcal{M}$ *and* $\mathcal{T}$ *are equivalent, then* $\ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) = \ell_{\lambda}^{\mathcal{T}}(\Delta_v^m, u)$.

Proof. **(i)** Let $x \in \ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) \cap \ell_{\lambda}^{\mathcal{T}}(\Delta_v^m, u)$. Then

$$\sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) < \infty$$

and

$$\sum_{k\geq 1} u_k T_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) < \infty.$$

We have

$$(M_k + T_k)\left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) \leq \left[M_k\left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right)\right] + \left[T_k\left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right)\right].$$

Applying $\sum\limits_{k\geq 1}$ and multiplying u_k both side of this inequality, we get,

$$\sum_{k\geq 1} u_k \left(M_k + T_k\right)\left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right)$$
$$\leq \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) + \sum_{k\geq 1} u_k T_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right).$$

This completes the proof.

(ii) The proof follows from Definition 3. □

Theorem 13. *Let* $\mathcal{N} = (N_k)$ *and* $\mathcal{T} = (T_k)$ *be any two sequence of Orlicz functions. Then we have:*
(i) $\ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u) \cap \ell_{\mathcal{T}}^{\lambda}(\Delta_v^m, u) \subset \ell_{\lambda}^{\mathcal{N}+\mathcal{T}}(\Delta_v^m, u)$,
(ii) *If* $\mathcal{M}$ *and* $\mathcal{T}$ *are equivalent, then* $\ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u) = \ell_{\mathcal{T}}^{\lambda}(\Delta_v^m, u)$.

Proof. **(i)** Let $x \in \ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) \cap \ell_{\lambda}^{\mathcal{T}}(\Delta_v^m, u)$. Then

$$\sum_{k\geq 1} u_k N_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho}\right) < \infty$$

and

$$\sum_{k\geq 1} u_k T_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho}\right) < \infty.$$

We have

$$(N_k + T_k)\left(\frac{\lambda_k\left|\Delta_v^m x_k\right|}{\rho}\right) \le \left[N_k\left(\frac{\lambda_k\left|\Delta_v^m x_k\right|}{\rho}\right)\right] + \left[T_k\left(\frac{\lambda_k\left|\Delta_v^m x_k\right|}{\rho}\right)\right].$$

Applying $\sum\limits_{k\ge 1}$ and multiplying u_k both side of this inequality, we get,

$$\sum_{k\ge 1} u_k\left(N_k + T_k\right)\left(\frac{\lambda_k\left|\Delta_v^m x_k\right|}{\rho}\right)$$
$$\le \sum_{k\ge 1} u_k N_k\left(\frac{\lambda_k\left|\Delta_v^m x_k\right|}{\rho}\right) + \sum_{k\ge 1} u_k T_k\left(\frac{\lambda_k\left|\Delta_v^m x_k\right|}{\rho}\right).$$

This completes the proof.

(ii) The proof follows from Definition 3. □

Theorem 14. *If μ is a normal sequence space containing λ, then $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ is a proper subspace of μ. In addition, if μ is equipped with the monotone norm (quasi-norm) $\|.\|_\mu$, the inclusion map $I : \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u) \to \mu(\Delta_v^m, u)$ is continuous with $\|I\| \le \|\{\lambda_k\}\|_\mu$.*

Proof. Let $x \in \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$. Since $\sum\limits_{k\ge 1} u_k M_k\left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) < \infty$ for some $\rho > 0$, then there exists a constant $K > 0$ such that $\frac{|\Delta_v^m x_k|}{\lambda_k \rho} \le K$ for all $k \in \mathbb{N}$. Since μ is a normal sequence space containing λ, we have $(\Delta_v^m x_k) \in \mu$ and so that $x \in \mu(\Delta_v^m)$. Hence $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u) \subset \mu(\Delta_v^m, u)$.

Further, since $M_k(1) = 1$ for all $k \in \mathbb{N}$, then

$$\sum_{k\ge 1} u_k M_k\left(\frac{\left|\Delta_v^m x_k\right|}{\lambda_k \left\|x\right\|_\lambda^\Delta}\right) \le 1,$$

and so that

$$\left|\Delta_v^m x_k\right| \le \lambda_k \left\|x\right\|_\lambda^\Delta, \text{ for all } k \in \mathbb{N}.$$

As $\|.\|_\mu$ is monotone, $\|Ix\|_\mu = \|(\Delta_v^m x_k)\|_\mu \le \|\{\lambda_k\}\|_\mu \|x\|_\lambda^\Delta$ and hence $\|I\| \le \|\{\lambda_k\}\|_\mu$. □

Theorem 15. *If η is a normal sequence space containing $\{\frac{1}{\lambda_k}\} \equiv \lambda^{-1}$, then $\ell_{\mathcal{N}}^{\lambda}(\Delta^m)$ is a proper subspace of η. If the norm (quasi-norm) $\|.\|_\eta$ on η is monotone, then the inclusion map $J : \ell_{\mathcal{N}}^{\lambda}(\Delta_v^m, u) \to \eta(\Delta_v^m, u)$ is continuous with $\|J\| \le \left\|\{\lambda_k^{-1}\}\right\|_\eta$.*

The proof is similar to Theorem 10 and therefore we omit it.

3. Interrelationship between the Spaces $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ and $\ell_{\mathcal{M}}^\lambda(\Delta_v^m, u)$

If $\lambda_k = 1$ for all $k \in \mathbb{N}$, then the sequence space $\ell_{\mathcal{M}}^\lambda(\Delta_v^m, u)$ reduces to the sequence space

$$\ell_{\mathcal{M}}(\Delta_v^m, u) = \{x = (x_k) : \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}.$$

Theorem 16. *If $\lambda = \{\lambda_k\}$ is a bounded sequence such that* $\inf \lambda_k > 0$, *then $\ell_{\mathcal{M}}^\lambda(\Delta_v^m, u) = \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u) = \ell_{\mathcal{M}}(\Delta_v^m, u)$.*

Proof. Let $x \in \ell_{\mathcal{M}}(\Delta_v^m, u)$. Then $\sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\rho}\right) < \infty$ for some $\rho > 0$. Since $\lambda = \{\lambda_k\}$ is bounded, we can write $a \leq \lambda_k \leq b$ for some $b > a \geq 0$. Define $\rho_1 = \rho b$. Since M_k is increasing, it follows that $\sum_{k\geq 1} u_k M_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho_1}\right) \leq \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\rho}\right) < \infty$. Hence $\ell_{\mathcal{M}}(\Delta_v^m, u) \subset \ell_{\mathcal{M}}^\lambda(\Delta_v^m, u)$. The other inclusion $\ell_{\mathcal{M}}^\lambda(\Delta_v^m, u) \subset \ell_{\mathcal{M}}(\Delta_v^m, u)$ follows from the inequality

$$\sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\rho/a}\right) \leq \sum_{k\geq 1} u_k M_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho}\right) < \infty.$$

Therefore, $\ell_{\mathcal{M}}^\lambda(\Delta_v^m, u) = \ell_{\mathcal{M}}(\Delta_v^m, u)$.

Similarly, one can prove $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u) = \ell_{\mathcal{M}}(\Delta_v^m, u)$. □

Theorem 17. *If $\{\lambda_k\} \in \ell_\infty$ with $a = \sup_{k\geq 1} \lambda_k \geq 1$ and $\{\lambda_k^{-1}\}$ is unbounded, then $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u)$ is properly contained in $\ell_{\mathcal{M}}^\lambda(\Delta_v^m, u)$ and the inclusion map $T : \ell_\lambda^{\mathcal{M}}(\Delta_v^m, u) \to \ell_{\mathcal{M}}^\lambda(\Delta_v^m, u)$ is continuous with $\|T\| \leq a^2$.*

Proof. For any $\rho > 0$ and $\rho' = \rho a^2$, we have

$$\sum_{k\geq 1} u_k M_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho'}\right) \leq \sum_{k\geq 1} u_k M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho}\right) < \infty$$

for $x = \{x_k\}$. Hence $\ell_\lambda^{\mathcal{M}}(\Delta_v^m, u) \subset \ell_{\mathcal{M}}^\lambda(\Delta_v^m, u)$.

We now show that the containment $\ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) \subset \ell_{\mathcal{M}}^{\lambda}(\Delta_v^m, u)$ is proper. From the unboundedness of the sequence $\{\lambda_k^{-1}\}$, choose a sequence $\{k_n\}$ of positive integers such that $\lambda_{k_n}^{-1} \geq n$. Define $\Delta_v^m x = \{\Delta_v^m x_k\}$ as follows:

$$\Delta_v^m x_k = \begin{cases} 1/n, & k = k_n, \quad n = 1, 2, ... \\ 0, & \text{otherwise} \end{cases}.$$

Then $x \in \ell_{\mathcal{M}}^{\lambda}(\Delta_v^m, u)$; but $x \notin \ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u)$.

To prove the continuity of the inclusion map T, let us first consider the case obtained for $a = 1$. For $x \in \ell_{\lambda}^{\mathcal{M}}(\Delta_v^m)$, we write

$$A_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) = \left\{ \rho > 0 : \sum_{k \geq 1} M_k \left(\frac{|\Delta_v^m x_k|}{\lambda_k \rho} \right) \leq 1 \right\}$$

and

$$B_{\mathcal{M}}^{\lambda}(\Delta_v^m, u) = \left\{ \rho > 0 : \sum_{k \geq 1} M_k \left(\frac{\lambda_k |\Delta_v^m x_k|}{\rho} \right) \leq 1 \right\}.$$

Since M_k are increasing and $a = 1$, we get $A_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) \subset B_{\mathcal{M}}^{\lambda}(\Delta_v^m, u)$.

Hence

$$\|x\|_{\Delta}^{\lambda} = \inf B_{\mathcal{M}}^{\lambda}(\Delta_v^m, u) \leq \inf A_{\lambda}^{\mathcal{M}}(\Delta_v^m, u) = \|x\|_{\lambda}^{\Delta}, \tag{1}$$

i.e., $\|T(x)\|_{\Delta}^{\lambda} \leq \|x\|_{\lambda}^{\Delta}$. Thus T is continuous with $\|T\| \leq 1 = a^2$.

If $a \neq 1$, define $\beta_k = \frac{\lambda_k}{a}$, $k \in \mathbb{N}$. Then $\beta_k \leq 1$ and from (1), it follows that

$$\|x\|_{\Delta}^{\beta} \leq \|x\|_{\beta}^{\Delta} \text{ for } x \in \ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u). \tag{2}$$

Hence from (2)

$$\|T(x)\|_{\Delta}^{\lambda} = \|x\|_{\Delta}^{\lambda} \leq a^2 \|x\|_{\lambda}^{\Delta},$$

i.e., T is continuous with $\|T\| \leq a^2$. This completes the proof. □

Theorem 18. *If $\{\lambda_k\}$ is unbounded with $\sup_{k \geq 1} \lambda_k^{-1} = d \geq 1$, $\lambda_k > 0$ for all k, then $\ell_{\mathcal{M}}^{\lambda}(\Delta_v^m, u)$ is properly contained in $\ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u)$ and the inclusion map $U : \ell_{\mathcal{M}}^{\lambda}(\Delta_v^m, u) \to \ell_{\lambda}^{\mathcal{M}}(\Delta_v^m, u)$ is continuous with $\|U\| \leq d^2$.*

Proof. The proof of this theorem is similar to that of Theorem 17 and so is omitted. □

References

[1] Ç. A. Bektaş and G. Atıci, On some new modular sequence spaces, *Bol. Soc. Paran. Mat.*, **31** (2013), 55-65.

[2] H. Kızmaz, On certain sequence spaces, *Canad. Math. Bull.*, **24** (1981), 169-176.

[3] H. Nakano, Modulared sequence spaces, *Proc. Jap. Acad.*, **27** Reprinted in: *Semi-ordered Linear Spaces*, Tokyo (1955).

[4] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1965), 379-390.

[5] J.Y.T. Woo, On modular sequence spaces, *Studia Math.*, **48** (1973), 271-289.

[6] M. Et and A. Esi, On Köthe-Toeplitz duals of generalized difference sequence spaces, *Bull. Malaysian Math Sc. Soc.*, **23** (2000), 25-32.

[7] M. Et and R. Çolak, On generalized difference sequence spaces. *Soochow J. Math.*, **21** (1995), 377-386.

[8] M. Gupta and S. Pradhan, On certain type of modular sequence spaces, *Turk. J. Math.*, **32** (2008), 293-303.

[9] P.K. Kamthan and M. Gupta, *Sequence Spaces and Series*, Marcel Dekker, New York (1981).