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THE RIESZ CAPACITY IN VARIABLE EXPONENT LEBESGUE SPACES

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Abstract: In this paper, we study a capacity theory based on a definition of a Riesz potential in the Euclidean space. Also, we define the Riesz $(\alpha, p(.))$-capacity and discuss the properties of the capacity in the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^n)$.

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1. Introduction

The interest in the variable exponent Lebesgue spaces has increased since the 1990s because of their use in a variety of applications such as mathematical modeling of electrorheological fluids, calculus of variations, differential equations with non-standard growth etc.

The study of variable exponent function spaces in higher dimensions was presented in 1991 an article by Kováčik and Rákosník [10]. They investigated some basic properties of the variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^n)$ and the Sobolev space $W^{k,p(.)}(\mathbb{R}^n)$ such as reflexivity and Hölder inequalities.

Since the variable exponent Lebesgue spaces is not invariant with respect to translations, convolution operators do not behave well in the spaces. For example, the Young theorem is not valid in general in the space $L^{p(.)}(\mathbb{R}^n)$, see

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[14]. Also, the translation operator is not continuous and the convolution with $g \in L^1(\mathbb{R}^n)$ is not continuous as well in general, i.e. the inequality $\|f * g\|_{p(.)} \leq C \|f\|_{p(.)} \|g\|_1$ is not satisfy in general.

The Riesz kernel I_α for $0 < \alpha < n$, is defined as

$$I_\alpha(x) = \frac{1}{\gamma(\alpha, n)} |x|^{\alpha - n},$$

where $\gamma(\alpha, n) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n}{2} - \frac{\alpha}{2}\right)}$. The Riesz potential of a measurable function f on $\mathbb{R}^n$ is defined by the convolution

$$I_\alpha * f(x) = \frac{1}{\gamma(\alpha, n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad \text{see [16].}$$

In this paper, we will omit the value of $\frac{1}{\gamma(\alpha,n)}$ since it is not crucial for our purpose. Throughout his paper, we call the convolution $I_\alpha * f$ simply $I_\alpha f$.

In $\mathbb{R}^n$, the capacity theory for the Riesz and Bessel potential can be found in [2], [3], [11] and [12]. The notion of capacity presents a standard way to describe exceptional sets in many different function spaces. A part of the theory for Riesz and Bessel capacity follows from general results in [8] and [15]. These capacities are very convenient in the function theory for the Sobolev spaces and in the theory of partial differential equations. The applications of these set functions need either a comprehensive knowledge for the exceptional sets or estimate on the rate at which the capacity of a sequence of bounded sets tends to zero. Moreover, they can determine existence of the Sobolev functions and solution for elliptic partial differential equations, as an application see [1].

Nuutinen and Silvestre defined a Riesz capacity in metric spaces in [13]. They studied the fundamental properties of this capacity such as monotonicity, convergence results, relationship between the Hausdorff measure. Our purpose is to investigate the Riesz capacity in n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$ to the variable exponent case.

2. Notation and Preliminaries

We denote by $\mathbb{R}^n$ the Euclidean space of dimension $n \geq 2$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote the open ball with center x and radius r by $B(x, r)$. The measure μ is doubling if there is a fixed constant $c_d \geq 1$, called the doubling constant of μ such that

$$\mu(B(x_0, 2r)) \leq c_d \mu(B(x_0, r))$$

for every ball $B(x_0, r)$ in $\mathbb{R}^n$. The space $C_0^\infty(\mathbb{R}^n)$ is the space of all infinitely differentiable functions with compact support. We denote the family of all measurable functions $p(.) : \mathbb{R}^n \to [1, \infty)$ (called the variable exponent on $\mathbb{R}^n$) by the symbol $\mathcal{P}(\mathbb{R}^n)$. In this paper, the function $p(.)$ always denotes a variable exponent. For $p(.) \in \mathcal{P}(\mathbb{R}^n)$, put

$$p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad p^+ = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$

For every measurable functions f on $\mathbb{R}^n$ we define the function

$$\varrho_{p(.)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.$$

The function $\varrho_{p(.)}$ is convex modular. $L^{p(.)}(\mathbb{R}^n)$ is denoted as the set of all (equivalence classes) measurable functions f on $\mathbb{R}^n$ such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

If $p^+ < \infty$, then $f \in L^{p(.)}(\mathbb{R}^n)$ iff $\varrho_{p(.)}(f) < \infty$. The set $L^{p(.)}(\mathbb{R}^n)$ is a Banach space with the norm $\|.\|_{p(.)}$. If $p(.) = p$ is a constant function, then the Luxemburg coincides with the usual L^p-norm. We denote by $L_+^{p(.)}(\mathbb{R}^n)$ the subset of $L^{p(.)}(\mathbb{R}^n)$ of non-negative functions. The space $L^1_{loc}(\mathbb{R}^n)$ is to be space of all measurable functions f on $\mathbb{R}^n$ such that $f.\chi_K \in L^1(\mathbb{R}^n)$ for any compact subset $K \subset \mathbb{R}^n$.

Nuutinen and Silvestre defined a metric type of the Riesz potential of order γ such that $0 < \gamma < 1$ and

$$I_\gamma f(x) = \int_X \frac{f(y)}{\mu(B(x, d(x, y)))^{1-\gamma}} d\mu(y),$$

where the metric space (X, d) equipped with a doubling measure μ. The Riesz $(\gamma, p) -$ capacity of a set $E \subset X$ denoted by

$$C_{\gamma,p}(E) = \inf_{f\in R(E)} \|f\|_p^p,$$

where

$$R(E) = \left\{f \in L_+^p(X) : I_\gamma f \ge 1 \text{ on } E\right\}, \quad \text{see [13].}$$

Let $A_1 \subset A_2 \subset ... \subset \mathbb{R}^n$ and $A = \bigcup_{i=1}^{\infty} A_i$. If the convergence satisfies $C(A_i) \longrightarrow C(A)$, the capacity C is called Fatou capacity. We also say that a property holds $C - q.e.$ on $\mathbb{R}^n$ if it holds for all $x \in \mathbb{R}^n$ except those in a set E with $C(E) = 0$, see [4], [11].

3. The Riesz $(\alpha, p(.))$-Capacity

Throughout his paper, we will assume that $1 < p^- \leq p(.) \leq p^+ < n < \infty$ and $0 < \alpha < n$. The usual Riesz potential $I_\alpha f : \mathbb{R}^n \longrightarrow [0, \infty]$ is defined by the integral

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for f measurable function with $f \geq 0$. To define a capacity we denote

$$S_\alpha(A) = \left\{ f \in L_+^{p(.)}(\mathbb{R}^n) : I_\alpha f \geq 1 \text{ on } A \right\}.$$

for the set $A \subset \mathbb{R}^n$. The Riesz $(\alpha, p(.))$- capacity of A is defined by

$$R_{\alpha,p(.)}(A) = \inf_{f \in S_\alpha(A)} \rho_{p(.)}(f).$$

The following theorem is well known by [12].

Theorem 1. *If f is a non-negative function, then $I_\alpha f$ is a lower semi-continuous function.*

Theorem 2. *The Riesz $(\alpha, p(.))$-capacity is an outer measure.*

Proof. By the meaning of the infimum, we get

$$R_{\alpha,p(.)}(\emptyset) = \inf_{f \in S_\alpha(\emptyset)} \rho_{p(.)}(f) = 0.$$

Let $A_1 \subset A_2$ and $f(x) \in S_\alpha(A_1)$. Then the inclusion

$$\left\{ f \in L_+^{p(.)}(\mathbb{R}^n) : I_\alpha f \geq 1 \text{ on } A_2 \right\} \subset \left\{ f \in L_+^{p(.)}(\mathbb{R}^n) : I_\alpha f \geq 1 \text{ on } A_1 \right\}$$

holds. Thus the Riesz $(\alpha, p(.))$- capacity satisfies monotonicity. Now, we will prove the countable subadditivity condition. Assume that $\{A_s\}_{s=1}^{\infty}$ be a sequence and $A = \bigcup_{s=1}^{\infty} A_s$. Also, we can assume that $\sum_{s=1}^{\infty} R_{\alpha,p(.)}(A_s) < \infty$ i.e.

$R_{\alpha,p(.)}(A_s) < \infty$ for every $s \in \mathbb{N}$. Otherwise the proof is clear. Let $\varepsilon > 0$. Again using the meaning of the infimum we can choose $f_s \in S_\alpha(A_s)$ such that

$$\rho_{p(.)}(f) < R_{\alpha,p(.)}(A_s) + \frac{\varepsilon}{2^s}.$$

Denote $f := \sup_{s\in\mathbb{N}} f_s$. It is clear that $(f(x))^{p(.)} \leq \sum_{s=1}^{\infty} (f_s(x))^{p(.)}$ for all $x \in \mathbb{R}^n$. Since $f_s \in L_+^{p(.)}(\mathbb{R}^n)$ are positive measurable functions for every $s \in \mathbb{N}$,

$$\rho_{p(.)}(f) \leq \sum_{s=1}^{\infty} R_{\alpha,p(.)}(A_s) + \varepsilon.$$

Because of the fact that $f_s(x) \leq f(x)$ for all $x \in \mathbb{R}^n$ and $s \in \mathbb{N}$, we write that $I_\alpha f_s(x) \leq I_\alpha f(x)$. Let $x \in \bigcup_{s=1}^{\infty} A_s$. Thus there exists $s \in \mathbb{N}$ such that $x \in A_s$ and then $1 \leq I_\alpha f_s(x) \leq I_\alpha f(x)$. If we consider the definition of R_α, it is clear that $f \in S_\alpha\left(\bigcup_{s=1}^{\infty} A_s\right)$. This yields

$$R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s\right) \leq \sum_{s=1}^{\infty} R_{\alpha,p(.)}(A_s) + \varepsilon.$$

This completes the proof as $\varepsilon \longrightarrow 0$. □

Theorem 3. *If A is a subset of $\mathbb{R}^n$, then*

$$R_{\alpha,p(.)}(A) = \inf_{\substack{A\subset U \\ U \text{ open}}} R_{\alpha,p(.)}(U).$$

Proof. Let $A \subset U$ such that U is open. Using the monotonicity property of the Riesz $(\alpha, p(.))$-capacity,

$$R_{\alpha,p(.)}(A) \leq \inf_{\substack{A\subset U \\ U \text{ open}}} R_{\alpha,p(.)}(U). \tag{1}$$

Now, we can suppose that $R_{\alpha,p(.)}(A) < \infty$. Otherwise the proof is clear. Let $0 < \varepsilon < 1$ and $f \in S_\alpha(A)$. Then we can take a $f \in L_+^{p(.)}(\mathbb{R}^n)$ such that $I_\alpha f \geq 1$ on A and

$$\rho_{p(.)}(f) < R_{\alpha,p(.)}(A) + \varepsilon.$$

Define $f_\varepsilon := \frac{1}{1-\varepsilon} f$ and $T = \{x \in \mathbb{R}^n : I_\alpha f_\varepsilon(x) > 1\}$. Since the Riesz potential I_α is a lower semi-continuous function, T is open. It is easy to show that $f_\varepsilon(x) > f(x)$ for every $x \in \mathbb{R}^n$. If $x \in A$, we reach $I_\alpha f_\varepsilon(x) > I_\alpha f(x) \geq 1$ and then $A \subset T$. Since $f_\varepsilon \in S_\alpha(T)$ and $\frac{1}{1-\varepsilon} > 1$, we have

$$R_{\alpha,p(.)}(T) \leq \left(\frac{1}{1-\varepsilon}\right)^{p^+} \left[R_{\alpha,p(.)}(A) + \varepsilon\right].$$

This yields $R_{\alpha,p(.)}(T) \leq R_{\alpha,p(.)}(A)$ as $\varepsilon \longrightarrow 0$. It can be shown that T is an open set and $A \subset T$. By the definition of the infimum

$$\inf_{\substack{A\subset U \\ U \text{ open}}} R_{\alpha,p(.)}(U) \leq R_{\alpha,p(.)}(T) < \frac{1}{1-\varepsilon} R_{\alpha,p(.)}(A) + \frac{\varepsilon}{1-\varepsilon}.$$

The proof is completed by (1) and previous inequality as $\varepsilon \longrightarrow 0$. □

The previous theorem shows us that the Riesz $(\alpha, p(.))$-capacity is an outer capacity. Then the capacity of a set $A \subset \mathbb{R}^n$ can be provided by approximating with open sets from the outside.

Theorem 4. *If A and B are the subsets of $\mathbb{R}^n$, then*

$$R_{\alpha,p(.)}(A \cup B) + R_{\alpha,p(.)}(A \cap B) \leq R_{\alpha,p(.)}(A) + R_{\alpha,p(.)}(B).$$

Proof. Let $\varepsilon > 0$. Choose $f \in S_\alpha(A)$ such that $\rho_{p(.)}(f) \leq R_{\alpha,p(.)}(A) + \varepsilon$, and $g \in S_\alpha(B)$ such that $\rho_{p(.)}(g) \leq R_{\alpha,p(.)}(B) + \varepsilon$. Then we get that $f \in L_+^{p(.)}(\mathbb{R}^n)$ such that $f \geq 1$ on A, and $g \in L_+^{p(.)}(\mathbb{R}^n)$ such that $g \geq 1$ on B. Then it is easy to see that $\max(f,g) \geq 1$ on $A \cup B$ and $\min(f,g) \geq 1$ on $A \cap B$. Thus we get that $\max(f,g) \in S_\alpha(A \cup B)$, $\min(f,g) \in S_\alpha(A \cap B)$. Moreover,

$$\begin{aligned} &\int_{\mathbb{R}^n} |\max(f,g)(x)|^{p(x)}\,dx + \int_{\mathbb{R}^n} |\min(f,g)(x)|^{p(x)}\,dx \\ &= \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,dx + \int_{\mathbb{R}^n} |g(x)|^{p(x)}\,dx. \end{aligned}$$

By the definition of Riesz $(\alpha, p(.))$-capacity, we have

$$R_{\alpha,p(.)}(A \cup B) + R_{\alpha,p(.)}(A \cap B) \leq R_{\alpha,p(.)}(A) + R_{\alpha,p(.)}(B) + 2\varepsilon.$$

The claim follows as $\varepsilon \longrightarrow 0$. □

Theorem 5. *Let K_s is a decreasing sequence of compact subsets of $\mathbb{R}^n$ for $s \in \mathbb{N}$. Then*

$$\lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(K_s) = R_{\alpha,p(.)}\left(\bigcap_{s=1}^{\infty} K_s\right).$$

Proof. It is obvious that the inclusion $\bigcap_{s=1}^{\infty} K_s \subset K_s$ satisfies for all $s \in \mathbb{N}$. This yields $R_{\alpha,p(.)}\left(\bigcap_{s=1}^{\infty} K_s\right) \leq R_{\alpha,p(.)}(K_s)$ for all $s \in \mathbb{N}$ by the monotonicity of the Riesz $(\alpha, p(.))$- capacity. This shows that decreasing family $\left\{R_{\alpha,p(.)}(K_s)\right\}_{s\in\mathbb{N}}$ is bounded below with $R_{\alpha,p(.)}\left(\bigcap_{s=1}^{\infty} K_s\right)$. Hence, we have

$$\lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(K_s) = \inf_{s\in\mathbb{N}} R_{\alpha,p(.)}(K_s) \geq R_{\alpha,p(.)}\left(\bigcap_{s=1}^{\infty} K_s\right) \tag{2}$$

Now, let $\bigcap_{s=1}^{\infty} K_s \subset U$ and U is open. Since $\bigcap_{s=1}^{\infty} K_s$ is compact as well, there is a positive integer k such that $K_s \subset U$ for all $s \geq k$. It follows by the monotonicity of the Riesz $(\alpha, p(.))$-capacity that

$$\lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(K_s) \leq R_{\alpha,p(.)}(U).$$

If we consider the meaning of the infimum and the fact that $R_{\alpha,p(.)}$ is an outer capacity, then we get

$$\lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(K_s) \leq \inf_{\substack{\bigcap_{s=1}^{\infty} K_s\subset U \\ U \text{ is open}}} R_{\alpha,p(.)}(U) = R_{\alpha,p(.)}\left(\bigcap_{s=1}^{\infty} K_s\right). \tag{3}$$

The claim is obvious by (2) and (3). □

Theorem 6. *If* $(A_s)_{s\in\mathbb{N}} \subset \mathbb{R}^n$ *with* $R_{\alpha,p(.)}(A_s) = 0$, *then* $R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s\right) = 0$.

Proof. Let $0 < \varepsilon < 1$ be given. By the hypothesis, since $R_{\alpha,p(.)}(A_s) = 0$, we can choose $f_s \in S_\alpha(A_s)$ such that $\rho_{p(.)}(f_s) \leq \frac{\varepsilon}{2^s}$. Define a function $f = \sup_{s\in\mathbb{N}} f_s$. Because of the fact that $f_s(x) \leq f(x)$ for all $x \in \mathbb{R}^n$ and $s \in \mathbb{N}$, we obtain $I_\alpha f_s(x) \leq I_\alpha f(x)$. Let $x \in \bigcup_{s=1}^{\infty} A_s$. Thus there exists $s \in \mathbb{N}$ such that $x \in A_s$ and then $1 \leq I_\alpha f_s(x) \leq I_\alpha f(x)$. If we consider the definition of S_α, then it is clear that $f \in S_\alpha\left(\bigcup_{s=1}^{\infty} A_s\right)$. Moreover, if we consider the fact $f(x)^{p(.)} \leq \sum_{s=1}^{\infty} f_s(x)^{p(.)}$ for all $x \in \mathbb{R}^n$, then we have

$$R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s\right) \leq \int\limits_{\mathbb{R}^n} \left(\sum_{s=1}^{\infty} |f_s(x)|^{p(x)}\right) dx.$$

Since $f_s \in L_+^{p(.)}(\mathbb{R}^n)$ are positive measurable functions for every $s \in \mathbb{N}$, we get that $R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s\right) < \sum_{s=1}^{\infty} \frac{\varepsilon}{2^s} = \varepsilon$ by Beppo-Levi's theorem. That is the desired. □

Theorem 7. *Let $r > 0$. Then the inequality*

$$R_{\alpha,p(.)}(B(x,r)) \leq C \max\left\{|B(x,r)|^{1-p^-}, |B(x,r)|^{1-p^+}\right\}$$

holds where the constant C depends on α, p^+, p^- and dimension n.

Proof. Define a function $h = \frac{(2r)^{n-\alpha}}{|B(x,r)|}\chi_{B(x,r)}$. It is easy to see that $h \geq 0$ and $\rho_{p(.)}(h) < \infty$, and hence $h \in L_+^{p(.)}(\mathbb{R}^n)$. From the definition of the Riesz $(\alpha, p(.))$-capacity, we obtain

$$I_\alpha h(z) = \frac{(2r)^{n-\alpha}}{|B(x,r)|} \int\limits_{B(x,r)} \frac{dy}{|z-y|^{n-\alpha}}$$

for every $z \in B(x,r)$. If we consider the facts that $z \in B(x,r)$ and $y \in B(x,r)$, then we have $|z-y| \leq |z-x| + |x-y| < 2r$. Thus, we have $h \in S_\alpha(B(x,r))$. Hence, the desired inequality is satisfied for $C = \max\left\{(2r)^{(n-\alpha)p^-}, (2r)^{(n-\alpha)p^+}\right\}$. □

Theorem 8. *Assume that $f \in L_+^{p(.)}(\mathbb{R}^n)$. Then the inequality*

$$R_{\alpha,p(.)}(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}) \leq \max\left\{\lambda^{-p^+}, \lambda^{-p^-}\right\} \rho_{p(.)}(f)$$

is satisfied for $0 < \lambda < \infty$.

Proof. Let $f \in L_+^{p(.)}(\mathbb{R}^n)$ and $0 < \lambda < \infty$. Denote $f_\lambda := \frac{f}{\lambda}$ and $T := \{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}$. Thus we have $I_\alpha f_\lambda(x) > 1$ for all $x \in T$ that is $f_\lambda \in S_\alpha(T)$. This follows the claim. □

Theorem 9. *Let $A \subset \mathbb{R}^n$. Then $R_{\alpha,p(.)}(A) = 0$ if and only if there exists $f \in L_+^{p(.)}(\mathbb{R}^n)$ such that $I_\alpha f(x) = \infty$ for every $x \in A$.*

Proof. First, assume that $R_{\alpha,p(.)}(A) = 0$. By the definition of the Riesz $(\alpha, p(.))$- capacity we can take $f_s \in S_\alpha(A)$ such that $\rho_{p(.)}(f_s) < \frac{1}{2^s}$ for each integer s. Denote $f := \sum_{s=1}^{\infty} f_s$. If we consider the fact $f(x)^{p(.)} \leq \sum_{s=1}^{\infty} f_s(x)^{p(.)}$ for all $x \in \mathbb{R}^n$ and the Beppo-Levi's theorem, then we have

$$\rho_{p(.)}(f) \leq \sum_{s=1}^{\infty} \rho_{p(.)}(f_s) < \sum_{s=1}^{\infty} \frac{1}{2^s} < \infty.$$

Thus $f \in L_+^{p(.)}(\mathbb{R}^n)$. Since $f_s \in L_+^{p(.)}(\mathbb{R}^n)$, it is clear that the function $\frac{f_s(y)}{|x-y|^{n-\alpha}}$ is a positive measurable function for every $s \in \mathbb{N}$. Again by Beppo-Levi's theorem, we have $I_\alpha f(x) = \sum_{s=1}^{\infty} I_\alpha f_s(x)$ .If we consider the fact that $f_s \in S_\alpha(A)$, then $I_\alpha f(x) = \infty$ on A.

Now, suppose that there exists $f \in L_+^{p(.)}(\mathbb{R}^n)$ such that $I_\alpha f(x) = \infty$ for every $x \in A$. Then we have

$$\begin{aligned} R_{\alpha,p(.)}(A) &\leq R_{\alpha,p(.)}(\{x \in \mathbb{R}^n : I_\alpha f(x) > \lambda\}) \\ &\leq \max\left\{\lambda^{-p^-}, \lambda^{-p^+}\right\} \rho_{p(.)}(f) \end{aligned}$$

for every $\lambda > 0$ by Theorem 8. The claim follows by letting $\lambda \longrightarrow \infty$. □

Corollary 10. *If $f, g, h \in L^{p(.)}(\mathbb{R}^n)$, then*

$$I_\alpha(f+g) = I_\alpha(f) + I_\alpha(g), \qquad R_{\alpha,p(.)} - q.e.$$

and

$$I_\alpha(\lambda h) = \lambda I_\alpha(h), \qquad R_{\alpha,p(.)} - q.e.$$

for every λ finite constant.

Now we will investigate several convergence results of Riesz $(\alpha, p(.))$-capacity.

Definition 11. A sequence $\{f_s\}$ converges to f in sense to capacity if

$$\lim_{s \longrightarrow \infty} R_{\alpha,p(.)}(\{x \in \mathbb{R}^n : |f_s(x) - f(x)| \geq \varepsilon\}) = 0$$

for all $\varepsilon > 0$. Also, this convergence is denoted by $f_s \longrightarrow f$ in $R_{\alpha,p(.)}$.

Theorem 12. *Suppose that $\{f_s\} \subset L_+^{p(.)}(\mathbb{R}^n)$ and $f \in L_+^{p(.)}(\mathbb{R}^n)$. Each of following statements is a consequence of the previous one:*

(i) $f_s \longrightarrow f$ *in* $L^{p(.)}(\mathbb{R}^n)$

(ii) $I_\alpha f_s \longrightarrow I_\alpha f$ *in* $R_{\alpha,p(.)}$

(iii) There exists a subsequence $\{f_{s_t}\} \subset \{f_s\}$ *such that* $I_\alpha f_{s_t} \longrightarrow I_\alpha f$ *pointwise* $R_{\alpha,p(.)}-q.e.$

Proof. Let $\varepsilon > 0$. Using the Theorem 9, the potentials $I_\alpha f_s$ and $I_\alpha f$ are finite $R_{\alpha,p(.)}-$q.e. If we consider Corollary 10 and Theorem 8, respectively, then we have

$$R_{\alpha,p(.)}\left(\left\{x \in \mathbb{R}^n : |I_\alpha f_s(x) - I_\alpha f(x)| \geq \varepsilon\right\}\right) \leq \varepsilon^{-p^+} \rho_{p(.)}(f_s - f).$$

Also, it is known that convergence in norm implies convergence in modular. This shows that *(i)* implies *(ii)*.

Now assume that $I_\alpha f_s \longrightarrow I_\alpha f$ *in* $R_{\alpha,p(.)}$. Let $\varepsilon := \frac{1}{2^t} > 0$ for every $t \in \mathbb{N}$. Then there exists a subsequence $\{f_{s_t}\} \subset \{f_s\}$ such that

$$R_{\alpha,p(.)}\left(\left\{x \in \mathbb{R}^n : |I_\alpha f_{s_t}(x) - I_\alpha f(x)| \geq \frac{1}{2^t}\right\}\right) < \frac{1}{2^t}$$

Denote $A_t := \left\{x \in \mathbb{R}^n : |I_\alpha f_{s_t}(x) - I_\alpha f(x)| > \frac{1}{2^t}\right\}$ and $A := \bigcap_{k=1}^{\infty} \bigcup_{t=k}^{\infty} A_t$. It is easy to show that $A \subset \bigcup_{t=k}^{\infty} A_t$ for all k. It follows by the monotonicity property of the Riesz $(\alpha, p(.))$-capacity that

$$R_{\alpha,p(.)}(A) \leq R_{\alpha,p(.)}\left(\bigcup_{t=k}^{\infty} A_j\right) \leq \sum_{t=k}^{\infty} R_{\alpha,p(.)}(A_t) \leq \sum_{t=k}^{\infty} \frac{1}{2^t}$$

for all k. Hence, the upper limit set A has zero capacity. If $x \in \mathbb{R}^n - A$, then there is a $k = k(x)$ such that $x \in \mathbb{R}^n - A_t$ for $t \geq k$. Thus

$$|I_\alpha f_{s_t}(x) - I_\alpha f(x)| \leq \frac{1}{2^t}$$

for $t \geq k$. Taking the limit from the last inequality letting $t \longrightarrow \infty$, we have $I_\alpha f_{s_t}(x) \longrightarrow I_\alpha f(x)$ for $x \in \mathbb{R}^n - A_t$. Hence the proof is completed. □

Definition 13. Let $(X, \|.\|)$ be a normed space and X' is the dual of the X. A sequence (x_n) converges weakly to x_0 if $f(x_n)$ converges strongly to $f(x_0)$ for all $f \in X'$ and $n \in \mathbb{N}$. This convergence denoted by $x_n \longrightarrow x$ weakly or $x_n \rightharpoonup x$. Also, it is known that every subsequence of a the weakly convergence sequence converges weakly to same vector, see [7], [16].

Definition 14. A normed space X has the Banach-Saks property if $\frac{1}{t}\sum_{s=1}^{t} g_s \longrightarrow g$ whenever $g_s \longrightarrow g$ weakly, see [6]. Moreover, every uniformly convex space has the Banach–Saks property by [9].

Theorem 15. *If $\{f_s\} \subset L^{p(.)}(\mathbb{R}^n)$ and $f \in L^{p(.)}(\mathbb{R}^n)$, then*

(i) If $f_s \longrightarrow f$ weakly in $L^{p(.)}(\mathbb{R}^n)$, then

$$\liminf_{s\longrightarrow\infty} I_\alpha f_s \leq I_\alpha f \leq \limsup_{s\longrightarrow\infty} I_\alpha f_s, \qquad R_{\alpha,p(.)} - q.e.$$

(ii) If $f_s \longrightarrow f$ weakly in $L_+^{p(.)}(\mathbb{R}^n)$, then

$$I_\alpha f \leq \liminf_{s\longrightarrow\infty} I_\alpha f_s$$

and

$$I_\alpha f = \liminf_{s\longrightarrow\infty} I_\alpha f_s, \qquad R_{\alpha,p(.)} - q.e.$$

Proof. *(i)* Since the space $L^{p(.)}(\mathbb{R}^n)$ is a uniformly convex space [10], one has Banach-Saks property and there exists a subsequence $\{f_s^*\}$ can find such that a sequence $\{h_m\}$, where

$$h_m := \frac{1}{m}\sum_{s=1}^{m} f_s^*,$$

converges strongly to f in $L^{p(.)}(\mathbb{R}^n)$. If we consider the Theorem 12, then we can find a subsequence $\{h_m^*\}$ such that

$$I_\alpha f = \lim_{m\longrightarrow\infty} I_\alpha h_m^*, \qquad R_{\alpha,p(.)} - q.e.$$

This yields

$$I_\alpha f = \lim_{m\longrightarrow\infty} I_\alpha h_m^* \geq \liminf_{s\longrightarrow\infty} I_\alpha f_s, \qquad R_{\alpha,p(.)} - q.e. \tag{4}$$

If we replace f_s and f by $-f_s$ and $-f$, respectively in (4), then we have

$$\liminf_{s\longrightarrow\infty} I_\alpha(-f_s) \leq I_\alpha(-f), \qquad R_{\alpha,p(.)} - q.e.$$

or

$$I_\alpha f \leq \limsup_{s\longrightarrow\infty} I_\alpha f_s, \qquad R_{\alpha,p(.)} - q.e. \tag{5}$$

The claim is valid by (4) and (5).

(ii) Since the Riesz potential I_α is lower semi-continuous, we have

$$I_\alpha f \leq \lim_{s\longrightarrow\infty} \inf I_\alpha f_s \tag{6}$$

where $f_s \longrightarrow f$ weakly in $L_+^{p(.)}(\mathbb{R}^n)$. Moreover, the inequality

$$\lim_{s\longrightarrow\infty} \inf I_\alpha f_s \leq I_\alpha f, \qquad R_{\alpha,p(.)} - q.e. \tag{7}$$

holds by the (i). If we consider the inequalities (6) and (7), then the proof is completed.

□

Theorem 16. *The Riesz $(\alpha, p(.))$-capacity is a Fatou capacity.*

Proof. Let A_s is a increasing sequence of subsets of $\mathbb{R}^n$ for $s \in \mathbb{N}$. It is clear that $A_s \subset \bigcup_{s=1}^{\infty} A_s$ for all $s \in \mathbb{N}$. Note first that monotonicity of the Riesz $(\alpha, p(.))$-capacity yields that

$$\lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(A_s) \leq R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s\right).$$

To prove the opposite inequality, we can suppose that $\lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(A_s) < \infty$. Otherwise the proof is obvious. Let $f_s \in S_\alpha(A_s)$ be given for $s \in \mathbb{N}$. Thus by the meaning of the infimum $f_s \geq 0$ is the function such that

$$\rho_{p(.)}(f_s) \leq R_{\alpha,p(.)}(A_s) + \frac{1}{s}.$$

Thus $\{f_s\}_{s\in\mathbb{N}}$ is bounded in $L^{p(.)}(\mathbb{R}^n)$ and there exists a subsequence $\{f_{s_t}\}_{t\in\mathbb{N}}$ that converges weakly to a function $f \in L_+^{p(.)}(\mathbb{R}^n)$, see [16]. Since $f_s \in S_\alpha(A_s)$, we get that $I_\alpha f_s \geq 1$ on A_s. Using the theorem 15 (ii), we obtain

$$I_\alpha f \geq 1 \text{ on } \bigcup_{s=1}^{\infty} A_s, \qquad R_{\alpha,p(.)} - q.e.$$

Again if we consider the Theorem 15 (ii), then there exists a set $B \subset \bigcup_{s=1}^{\infty} A_s$ with $R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s - B\right) = 0$ such that $I_\alpha f \geq 1$ on B, that is $f \in S_\alpha(B)$. Since the Riesz $(\alpha, p(.))$-capacity is an outer measure, we get $R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s\right) =$

$R_{\alpha,p(.)}(B)$. It is known that $\rho_{p(.)}$ modular function is lower semi-continuous, see [6]. Hence we have

$$R_{\alpha,p(.)}\left(\bigcup_{s=1}^{\infty} A_s\right) \leq \lim \inf_{t\longrightarrow\infty} \rho_{p(.)}(f_{s_t}) \leq \lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(A_s).$$

□

Corollary 17. *Let $\{A_s\}_{s\in\mathbb{N}}$ is a subset of $\mathbb{R}^n$. Then*

$$R_{\alpha,p(.)}\left(\lim \inf_{s\longrightarrow\infty} A_s\right) \leq \lim \inf_{s\longrightarrow\infty} R_{\alpha,p(.)}(A_s).$$

Proof. Denote $B := \lim\inf_{s\longrightarrow\infty} A_s = \bigcup_{t=1}^{\infty} \bigcap_{k\geq t}^{\infty} A_k$ and $B_s := \bigcup_{t=1}^{s} \bigcap_{k\geq t}^{\infty} A_k$. It is easy to see that $B_s \subset B_{s+1} \subset ...$ Moreover, we have

$$\bigcup_{s=1}^{\infty} B_s = \bigcup_{s=1}^{\infty}\left(\bigcup_{t=1}^{s}\bigcap_{k\geq t}^{\infty} A_k\right) = \bigcup_{t=1}^{\infty}\bigcap_{k\geq t}^{\infty} A_k = B.$$

Since the Riesz $(\alpha, p(.))$- capacity is a Fatou capacity,

$$R_{\alpha,p(.)}(B) = \lim_{s\longrightarrow\infty} R_{\alpha,p(.)}(B_s) \leq \lim \inf_{s\longrightarrow\infty} R_{\alpha,p(.)}(A_s).$$

That is the desired. □

A set function which satisfies the hypothesis of Theorem 2, Theorem 5 and Theorem 16 is called Choquet capacity, see [5]. Hence we get the following result.

Corollary 18. *The set function $A \longrightarrow R_{\alpha,p(.)}(A)$, $A \subset \mathbb{R}^n$, is a Choquet capacity. In particular, all Borel sets $A \subset \mathbb{R}^n$ are capacitable, that is,*

$$R_{\alpha,p(.)}(A) = \inf_{\substack{A\subset U \\ U \text{ open}}} R_{\alpha,p(.)}(U) = \sup_{\substack{K\subset A \\ K \text{ compact}}} R_{\alpha,p(.)}(K).$$

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