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SOLVING THE NONLINEAR PENDULUM EQUATION WITH NONHOMOGENEOUS INITIAL CONDITIONS

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Abstract: In this work we solve the simple pendulum nonlinear second order differential equation with nonhomogeneous initial conditions, obtaining a closed-form solution in terms of the Jacobi elliptic functions, and of the incomplete elliptic integral of the first kind. Such a modeling problem can be used to introduce concepts like elliptic integrals and functions to advanced undergraduate students.

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1. Introduction

The equation of motion for an ideal simple pendulum is given by the second order nonlinear ordinary differential equation [10, 5]:

$$\frac{d^2\theta}{dt} + \omega_0^2 \sin\theta = 0, \tag{1}$$

where $w_0 = \sqrt{\frac{g}{l}}$ is its natural angular frequency of oscillation, g is the acceleration of gravity, and l is the lenght of its weightless and extensionless rod.

An usual way to present the simple pendulum model in a first course of general physics is to consider only small angular displacements [6], that is,

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$\theta << 1$. In this case, $\sin\theta \approx \theta$, and then equation (1) can be linearized:

$$\frac{d^2\theta}{dt} + \omega_0^2\theta = 0, \tag{2}$$

resulting in the simple harmonic motion:

$$\theta(t) = \theta_{\max}\sin(\omega_0 t + \varphi), \tag{3}$$

where $\theta_{\max}$ is the maximum angular displacement, and φ is a phase angle, both depending on the initial angular displacement and velocity of the pendulum.

This first approximation leads to the conclusion that, for small angles, the pendulum's period of oscillation is independent of its amplitude, being given by:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}. \tag{4}$$

In order to present a more realistic expression for this period in the case of a larger amplitude of oscillation, it was proposed the use of a more precise approximation [7, 8]:

$$T = 2\pi\sqrt{\frac{l}{g\cos\left(\frac{\theta_{\max}}{2}\right)}}, \quad \theta_{\max} \in [0, \pi/2], \tag{5}$$

since $\theta\cos\left(\frac{\theta_{\max}}{2}\right)$ gives a better approximation for $\sin\theta$ on the interval $[0, \pi/2]$. Besides of that, another approximations were also proposed in [1, 2].

Usually the exact formula for the large-angle period of a simple pendulum is derived from conservation of energy considerations and is presented in Classical Mechanics textbooks, being given by [11, 10]:

$$T = 4\sqrt{\frac{l}{g}}K\left(\sin^2\frac{\theta_{\max}}{2}\right), \tag{6}$$

where $K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-kz^2)}}$ is the complete elliptic integral of the first kind [9]. Beléndez et al. [3], in a very didactic note, have also derived (6) solving directly the nonlinear equation of motion (1), considering the initial angular position $\theta(0) = \theta_{\max}$, and that the pendulum starts at rest, i.e., $\frac{d\theta}{dt}(0) = 0$.

In this work we solve the simple pendulum equation of motion (1), considering nonhomogeneous initial conditions $\theta(0) = \theta_0$ and $\frac{d\theta}{dt}(0) = \phi_0$. In this way, we extend the solution presented in [3] to also consider non-zero initial angular velocities, ϕ_0, obtaining a closed-form expression for $\theta(t)$ given in terms of the

Jacobi elliptic function $\mathrm{sn}(u,k)$, and of the incomplete elliptic integral of the first kind $F(\varphi,k)$. Besides of that, we derive an expression for the period of oscillation $T(\theta_0,\phi_0)$, involving the complete elliptic integral of the first kind $K(k)$.

2. Solving the Initial Value Problem

The dynamics of an ideal simple pendulum is given by the following initial value problem:

$$\begin{cases} \dfrac{d^2\theta}{dt} + \omega_0^2 \sin\theta = 0 \\ \theta(0) = \theta_0 \\ \dfrac{d\theta}{dt}(0) = \phi_0 \end{cases}, \tag{7}$$

where $\theta_0 \in [0,\pi]$ is the initial angular displacement, and ϕ_0 is the initial angular velocity.

Following [3], we start to solve this second order nonlinear differential equation multiplying it by $\frac{d\theta}{dt}$:

$$\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}\omega_0^2\sin\theta = 0,$$

and noting that it can be rewritten as $\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega_0^2\cos\theta\right] = 0$. Then, integrating this equation on the time interval $[0,t]$, $t \geq 0$, we obtain

$$\left[\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \omega_0^2\cos\theta\right]_0^t = 0,$$

which implies:

$$\frac{1}{2}\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{2}\left(\frac{d\theta}{dt}\Big|_{t=0}\right)^2 = \omega_0^2(\cos\theta(t) - \cos\theta_0),$$

and therefore:

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2(\cos\theta(t) - \cos\theta_0) + \phi_0^2. \tag{8}$$

Now, defining the new variables $y(t) = \sin\left(\frac{\theta(t)}{2}\right)$ and $k = \sin^2\left(\frac{\theta_0}{2}\right)$, we can rewrite equation (8) as:

$$\left(\frac{dy}{dt}\right)^2 = \omega_0^2 k\left(1 - \frac{y^2}{k}\right)(1-y^2) + \frac{\phi_0^2}{4}(1-y^2), \tag{9}$$

and considering the scaling $\tau = \omega_0 t$ and $z = \frac{y}{\sqrt{k}}$ follows that $\frac{dy}{dt} = \omega_0\sqrt{k}\frac{dz}{d\tau}$, which plugged on (9) yields:

$$\begin{aligned}\left(\frac{dz}{d\tau}\right)^2 &= \left(1-z^2\right)\left(1-kz^2\right) + \frac{\phi_0^2}{4\omega_0^2 k}(1-kz^2) \\ &= (1-kz^2)(\gamma_0^2 - z^2),\end{aligned} \tag{10}$$

where $\gamma_0^2 = 1 + \frac{\phi_0^2}{4\omega_0^2 k}$. Taking the square root on both sides of (10), and after some algebra, we have that $d\tau = \pm\frac{dz}{\sqrt{(1-kz^2)(\gamma_0^2-z^2)}}$. Integrating on the interval $[0,\tau]$ we obtain $\tau = \pm\frac{1}{\sqrt{k}}\int_1^z \frac{d\xi}{\sqrt{\left(\frac{1}{k}-\xi^2\right)(\gamma_0^2-\xi^2)}}$, since $z(0) = 1$. Then:

$$\pm\tau = \frac{1}{\sqrt{k}}\left[\int_0^z \frac{d\xi}{\sqrt{\left(\frac{1}{k}-\xi^2\right)\left(\gamma_0^2-\xi^2\right)}} - \int_0^1 \frac{d\xi}{\sqrt{\left(\frac{1}{k}-\xi^2\right)\left(\gamma_0^2-\xi^2\right)}}\right].$$

Now, we can write the first integral in this expression as the inverse Jacobi elliptic function $\mathrm{sn}^{-1}(\cdot,\cdot)$, and the second integral as the incomplete elliptic integral of the first kind $F(\cdot,\cdot)$ (Formula 219.00, pp. 58, [4]):

$$\begin{aligned}&\pm\,\tau = \mathrm{sn}^{-1}\left(\frac{z}{\gamma_0}, \gamma_0^2 k\right) - F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) \\ &\Leftrightarrow \mathrm{sn}^{-1}\left(\frac{z}{\gamma_0}, \gamma_0^2 k\right) = F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) \pm \tau.\end{aligned} \tag{11}$$

Applying the Jacobi elliptic function $\mathrm{sn}(\cdot,\cdot)$ on both sides of this equation:

$$\frac{z}{\gamma_0} = \mathrm{sn}\left(F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) \pm \tau, \gamma_0^2 k\right)$$

and coming back to the original variables:

$$\sin\left(\frac{\theta}{2}\right) = \gamma_0\sqrt{k}\,\mathrm{sn}\left(F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) \pm \tau, \gamma_0^2 k\right),$$

we finally obtain the solution for the angular displacement as a function of time:

$$\theta(t) = 2\arcsin\left(\gamma_0\sqrt{k}\,\mathrm{sn}\left(F\left(\arcsin\left(\frac{1}{\gamma_0}\right), \gamma_0^2 k\right) \pm \omega_0 t, \gamma_0^2 k\right)\right), \tag{12}$$

which can be written as:

$$\theta(t) = 2\arcsin\left(\eta_0\,\mathrm{sn}\left(F\left(\arcsin\left(\frac{\sqrt{k}}{\eta_0}\right), \eta_0^2\right) \pm \omega_0 t, \eta_0^2\right)\right), \tag{13}$$

where:

$$\eta_0 = \sqrt{k + \frac{\phi_0^2}{4\omega_0^2}}, \; k = \sin^2\left(\frac{\theta_0}{2}\right), \text{ and } \omega_0 = \sqrt{\frac{g}{l}}. \tag{14}$$

Note that expression (13) represents two possible solutions: the positive branch gives the angular position for positive initial angular velocities ($\phi_0 > 0$), while the negative branch gives the solution if the initial angular velocity is non-positive ($\phi_0 \le 0$):

$$\theta(t) = \begin{cases} 2\arcsin\left(\eta_0 \operatorname{sn}\left(\alpha(t), \eta_0^2\right)\right), & \text{if } \phi_0 > 0 \\ 2\arcsin\left(\eta_0 \operatorname{sn}\left(\beta(t), \eta_0^2\right)\right), & \text{if } \phi_0 \le 0 \end{cases}, \tag{15}$$

where:

$$\begin{aligned} \alpha(t) &= F\left(\arcsin\left(\frac{\sqrt{k}}{\eta_0}\right), \eta_0^2\right) + \omega_0 t, \\ \beta(t) &= F\left(\arcsin\left(\frac{\sqrt{k}}{\eta_0}\right), \eta_0^2\right) - \omega_0 t. \end{aligned} \tag{16}$$

The time derivative of (15) gives the pendulum's angular velocity as a function of time:

$$\frac{d\theta(t)}{dt} = \phi(t) = \begin{cases} \dfrac{2\omega_0\eta_0 \operatorname{cn}\left(\alpha(t), \eta_0^2\right) \operatorname{dn}\left(\alpha(t), \eta_0^2\right)}{\sqrt{1 - \eta_0^2 \operatorname{sn}^2\left(\alpha(t), \eta_0^2\right)}}, & \text{if } \phi_0 > 0, \\ -\dfrac{2\omega_0\eta_0 \operatorname{cn}\left(\beta(t), \eta_0^2\right) \operatorname{dn}\left(\beta(t), \eta_0^2\right)}{\sqrt{1 - \eta_0^2 \operatorname{sn}^2\left(\beta(t), \eta_0^2\right)}}, & \text{if } \phi_0 \le 0. \end{cases} \tag{17}$$

The maxima/minima of $\theta(t)$ can be identified setting (17) to zero. Since the zeros of $\operatorname{dn}(\cdot, \eta_0^2)$ are all complex numbers, $\frac{d\theta(t)}{dt} = 0$ if and only if $\operatorname{cn}(\cdot, \eta_0^2) = 0$, what occurs when $\alpha(t) = \beta(t) = (2n+1)K(\eta_0^2)$, $n \in \mathbb{Z}$. In this way, the period of oscillation of the pendulum will be given by:

$$T = (\theta_0, \phi_0) = \frac{4}{\omega_0} K(\eta_0^2) = \frac{4}{\omega_0} K\left(\sin^2\left(\frac{\theta_0}{2}\right) + \frac{\phi_0^2}{4\omega_0^2}\right). \tag{18}$$

As a special case, if the initial angular velocity is zero, $\phi_0 = 0$, then $\theta_0 = \theta_{\max}$, and $\eta_0 = \sqrt{k}$. Therefore, (15)-(16) reduces to:

$$\theta(t) = 2\arcsin\left(\sqrt{k} \operatorname{sn}\left(F\left(\frac{\pi}{2}, k\right) - \omega_0 t, k\right)\right).$$

Since from Formula 110.06 (pp. 9, [4]) we have $F\left(\frac{\pi}{2},k\right)=K(k)$, where $K(\cdot)$ is the complete elliptic integral of the first kind, we recover the solution presented in [3]:

$$\theta(t)=2\arcsin\left(\sqrt{k}\,\mathrm{sn}\left(K(k)-\omega_0 t,k\right)\right). \tag{19}$$

In the same way, if $\phi_0=0$, the period given by (18) reduces to (6).

3. Concluding Remarks

In this paper we have derived a closed-form solution for the angular displacement of an ideal simple pendulum in terms of the Jacobi elliptic function $\mathrm{sn}(u,k)$, and of the incomplete elliptic integral of the first kind $F(\varphi,k)$, given arbitrary initial angular displacement (θ_0) and velocity (ϕ_0):

$$\theta(t)=\begin{cases}2\arcsin\left(\eta_0\ \mathrm{sn}\left(\alpha(t),\eta_0^2\right)\right), & \text{if } \phi_0>0,\\ 2\arcsin\left(\eta_0\ \mathrm{sn}\left(\beta(t),\eta_0^2\right)\right), & \text{if } \phi_0\le 0,\end{cases}$$

where:

$$\alpha(t)=F\left(\arcsin\left(\frac{\sqrt{k}}{\eta_0}\right),\eta_0^2\right)+\omega_0 t,$$

$$\beta(t)=F\left(\arcsin\left(\frac{\sqrt{k}}{\eta_0}\right),\eta_0^2\right)-\omega_0 t,$$

and

$$\eta_0=\sqrt{k+\frac{\phi_0^2}{4\omega_0^2}},\ k=\sin^2\left(\frac{\theta_0}{2}\right),\ \text{and } \omega_0=\sqrt{\frac{g}{l}}.$$

We also have obtained that the pendulum's angular velocity as a function of time is given by:

$$\phi(t)=\begin{cases}\dfrac{2\omega_0\eta_0\mathrm{cn}\left(\alpha(t),\eta_0^2\right)\mathrm{dn}\left(\alpha(t),\eta_0^2\right)}{\sqrt{1-\eta_0^2\ \mathrm{sn}^2\left(\alpha(t),\eta_0^2\right)}}, & \text{if } \phi_0>0,\\ -\dfrac{2\omega_0\eta_0\mathrm{cn}\left(\beta(t),\eta_0^2\right)\mathrm{dn}\left(\beta(t),\eta_0^2\right)}{\sqrt{1-\eta_0^2\ \mathrm{sn}^2\left(\beta(t),\eta_0^2\right)}}, & \text{if } \phi_0\le 0,\end{cases}$$

where $\mathrm{cn}(u,k)$ and $\mathrm{dn}(u,k)$ are Jacobi elliptic functions.

In addition, we have shown that the pendulum will oscillate with a period equals to:

$$T = (\theta_0, \phi_0) = \frac{4}{\omega_0} K \left(\sin^2 \left(\frac{\theta_0}{2} \right) + \frac{\phi_0^2}{4\omega_0^2} \right).$$

which depends on both initial angular position and velocity, and where $K(k)$ is the complete elliptic integral of the first kind.

Finally, such an exercise may be used to initiate advanced undergraduate students to concepts such as nonlinear ordinary differential equations, and elliptic integrals and functions.

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