

ON THE BOUNDEDNESS OF DUNKL-TYPE FRACTIONAL
INTEGRAL OPERATOR IN THE GENERALIZED
DUNKL-TYPE MORREY SPACES

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Abstract: First, we prove that the Dunkl-type maximal operator M_α is bounded on the generalized Dunkl-type Morrey spaces $\mathcal{M}_{p,\omega,\alpha}$ for $1 < p < \infty$ and from the spaces $\mathcal{M}_{1,\omega,\alpha}$ to the weak spaces $W\mathcal{M}_{1,\omega,\alpha}$.

We prove that the Dunkl-type fractional order integral operator $I^{\beta,\alpha}$, $0 < \beta < 2\alpha + 2$ is bounded from the generalized Dunkl-type Morrey spaces $\mathcal{M}_{p,\omega,\alpha}$ to $\mathcal{M}_{q,\omega^{p/q},\alpha}$, where $\beta/(2\alpha + 2) = 1/p - 1/q$, $1 < p < (2\alpha + 2)/\beta$.

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1. Introduction

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. On the real line, the Dunkl operators are differential-difference operators associated with the reflection group Z_2 on R . In the works [1, 17, 24, 35] the maximal operator associated with the Dunkl operator on R were studied. In this work, we study the fractional maximal function (Dunkl-type fractional maximal function) and the fractional integral (Dunkl-type fractional integral) associated with the Dunkl operator on R . We obtain the necessary and sufficient conditions for the boundedness of the

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Dunkl-type fractional maximal operator, and the Dunkl-type fractional integral operator from the spaces $\mathcal{M}_{p,\omega,\alpha}(R)$ to the spaces $\mathcal{M}_{q,\omega,\alpha}(R)$, $1 < p < q < \infty$, and from the spaces $\mathcal{M}_{1,\omega,\alpha}(R)$ to the weak spaces $W\mathcal{M}_{q,\omega,\alpha}(R)$, $1 < q < \infty$.

For $x \in R^n$ and $r > 0$, let $B(x, r)$ denote the open ball centered at x of radius r .

Let $f \in L_1^{loc}(R^n)$. The maximal operator M and the Riesz potential I^β are defined by

$$Mf(x) = \sup_{t>0} |B(x, t)|^{-1} \int_{B(x, t)} |f(y)| dy,$$

$$I^\beta f(x) = \int_{R^n} \frac{f(y) dy}{|x - y|^{n-\beta}}, \quad 0 < \beta < n,$$

where $|B(x, t)|$ is the Lebesgue measure of the ball $B(x, t)$.

The operators M and I^β play an important role in real and harmonic analysis (see, for example [36] and [32]).

In the theory of partial differential equations, the Morrey spaces $\mathcal{M}_{p,\lambda}(R^n)$ play an important role. They were introduced by C. Morrey in 1938 [28] and defined as follows:

$$\text{For } 0 \leq \lambda \leq n, 1 \leq p < \infty, f \in \mathcal{M}_{p,\lambda}(R^n) \text{ if } f \in L_p^{loc}(R^n) \text{ and}$$

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(R^n)} = \sup_{x \in R^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x, r))} < \infty.$$

If $\lambda = 0$, then $\mathcal{M}_{p,\lambda}(R^n) = L_p(R^n)$, if $\lambda = n$, then $\mathcal{M}_{p,\lambda}(R^n) = L_\infty(R^n)$, if $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda}(R^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on R^n .

These spaces appeared to be quite useful in the study of the local behaviour of the solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

Also by $W\mathcal{M}_{p,\lambda}(R^n)$ we denote the weak Morrey space of all functions $f \in WL_p^{loc}(R^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(R^n)} = \sup_{x \in R^n, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x, r))} < \infty,$$

where $WL_p(R^n)$ denotes the weak L_p -space.

F. Chiarenza and M. Frasca [8] studied the boundedness of the maximal operator M in the Morrey spaces $\mathcal{M}_{p,\lambda}$. Their results can be summarized as follows:

Theorem A. *Let $0 < \alpha < n$ and $0 \leq \lambda < n$, $1 \leq p < \infty$.*

1) *If $1 < p < \infty$, then M is bounded from $\mathcal{M}_{p,\lambda}$ to $\mathcal{M}_{p,\lambda}$.*

2) If $p = 1$, then M is bounded from $\mathcal{M}_{1,\lambda}$ to $W\mathcal{M}_{1,\lambda}$.

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I^β is bounded from $L_p(R^n)$ to $L_q(R^n)$ if and only if $\beta = \frac{n}{p} - \frac{n}{q}$, and for $p = 1 < q < \infty$, I^β is bounded from $L_1(R^n)$ to $WL_q(R^n)$ if and only if $\beta = n - \frac{n}{q}$. D.R. Adams [4] studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement (see, also [6]).

Theorem B. Let $0 < \beta < n$ and $0 \leq \lambda < n$, $1 \leq p < \frac{n-\lambda}{\beta}$.

1) If $1 < p < \frac{n-\lambda}{\beta}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n-\lambda}$ is necessary and sufficient for the boundedness I^β from $\mathcal{M}_{p,\lambda}(R^n)$ to $\mathcal{M}_{q,\lambda}(R^n)$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\beta}{n-\lambda}$ is necessary and sufficient for the boundedness I^β from $\mathcal{M}_{1,\lambda}(R^n)$ to $W\mathcal{M}_{q,\lambda}(R^n)$.

If $\beta = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of Theorem B reduces to the above mentioned result by Hardy-Littlewood-Sobolev.

2. Definitions, notation and preliminaries

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on R , given by

$$d\mu_\alpha(x) := (2^{\alpha+1}\Gamma(\alpha+1))^{-1} |x|^{2\alpha+1} dx.$$

For every $1 \leq p \leq \infty$, we denote by $L_{p,\alpha}(R) = L_p(R, d\mu_\alpha)$ the spaces of complex-valued functions f , measurable on R such that

$$\|f\|_{p,\alpha} \equiv \|f\|_{L_{p,\alpha}} = \left(\int_R |f(x)|^p d\mu_\alpha(x) \right)^{1/p} < \infty \quad \text{if } p \in [1, \infty),$$

and

$$\|f\|_{\infty,\alpha} \equiv \|f\|_{L_\infty} = \operatorname{ess\,sup}_{x \in R} |f(x)| \quad \text{if } p = \infty.$$

For $1 \leq p < \infty$ we denote by $WL_{p,\alpha}(R)$, the weak $L_{p,\alpha}(R)$ spaces defined as the set of locally integrable functions f with the finite norm

$$\|f\|_{WL_{p,\alpha}} = \sup_{r>0} r (\mu_\alpha \{x \in R : |f(x)| > r\})^{1/p}.$$

Note that

$$L_{p,\alpha} \subset WL_{p,\alpha} \quad \text{and} \quad \|f\|_{WL_{p,\alpha}} \leq \|f\|_{p,\alpha} \quad \text{for all } f \in L_{p,\alpha}(R).$$

Let $B(x, t) = \{y \in R : |y| \in]\max\{0, |x| - t\}, |x| + t[\}$ and $B_t \equiv B(0, t) =]-t, t[, t > 0$. Then

$$\mu_\alpha B_t = b_\alpha t^{2\alpha+2},$$

where $b_\alpha = [2^{\alpha+1}(\alpha+1)\Gamma(\alpha+1)]^{-1}$.

We denote by $BMO_\alpha(R)$ (Dunkl-type BMO space) the set of locally integrable functions f with finite norm (see [14])

$$\|f\|_{*,\alpha} = \sup_{r>0, x \in R} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f_{B_r}(x)| d\mu_\alpha(y) < \infty,$$

where

$$f_{B_r}(x) = \frac{1}{\mu_\alpha B_r} \int_{B_r} \tau_x f(y) d\mu_\alpha(y).$$

For all $x, y, z \in R$, we put

$$W_\alpha(x, y, z) = (1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})\Delta_\alpha(x, y, z)$$

where

$$\sigma_{x,y,z} = \begin{cases} \frac{x^2+y^2-z^2}{2xy} & \text{if } x, y \in R \setminus \{0\}, \\ 0 & \text{otherwise} \end{cases}$$

and Δ_α is the Bessel kernel given by

$$\Delta_\alpha(x, y, z) = \begin{cases} d_\alpha \frac{((|x|+|y|)^2 - z^2)[z^2 - (|x|-|y|)^2])^{\alpha-1/2}}{|xy|^{2\alpha}}, & \text{if } |z| \in A_{x,y}, \\ 0, & \text{otherwise,} \end{cases}$$

where $d_\alpha = (\Gamma(\alpha+1))^2 / (2^{\alpha-1} \sqrt{\pi} \Gamma(\alpha + \frac{1}{2}))$ and $A_{x,y} = [|x| - |y|, |x| + |y|]$.

Properties 1. (see Rösler [37]) *The signed kernel W_α is even with respect to all variables and satisfies the following properties*

$$W_\alpha(x, y, z) = W_\alpha(y, x, z) = W_\alpha(-x, z, y),$$

$$W_\alpha(x, y, z) = W_\alpha(-z, y, -x) = W_\alpha(-x, -y, -z)$$

and

$$\int_R |W_\alpha(x, y, z)| d\mu_\alpha(z) \leq 4.$$

In the sequel, we consider the signed measure $\nu_{x,y}$, on R , given by

$$\nu_{x,y} = \begin{cases} W_\alpha(x, y, z) d\mu_\alpha(z) & \text{if } x, y \in R \setminus \{0\}, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0. \end{cases}$$

Definition 1. For $x, y \in R$ and f a continuous function on R , we put

$$\tau_x f(y) = \int_R f(z) d\nu_{x,y}(z).$$

The operator τ_x , $x \in R$, is called Dunkl translation operator on R and it can be expressed in the following form (see [37])

$$\begin{aligned} \tau_x f(y) = c_\alpha \int_0^\pi f_e((x, y)_\theta) h_1(x, y, \theta) (\sin \theta)^{2\alpha} d\theta \\ + c_\alpha \int_0^\pi f_o((x, y)_\theta) h_2(x, y, \theta) (\sin \theta)^{2\alpha} d\theta, \end{aligned}$$

where $(x, y)_\theta = \sqrt{x^2 + y^2 - 2|xy| \cos \theta}$, $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f , with

$$c_\alpha \equiv \left(\int_0^\pi (\sin \theta)^{2\alpha} d\theta \right)^{-1} = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)},$$

$$h_1(x, y, \theta) = 1 - \operatorname{sgn}(xy) \cos \theta$$

and

$$h_2(x, y, \theta) = \begin{cases} \frac{(x + y) [1 - \operatorname{sgn}(xy) \cos \theta]}{(x, y)_\theta}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0. \end{cases}$$

By the change of variable $z = (x, y)_\theta$, we have also (see [3])

$$\begin{aligned} \tau_x f(y) = c_\alpha \int_0^\pi \{ f((x, y)_\theta) + f(-(x, y)_\theta) \\ + \frac{x + y}{(x, y)_\theta} [f((x, y)_\theta) - f(-(x, y)_\theta)] \} (1 - \cos \theta) (\sin \theta)^{2\alpha} d\theta. \end{aligned}$$

Now we define the Dunkl-type fractional maximal function by

$$M_{\beta, \alpha} f(x) = \sup_{r > 0} (\mu_\alpha B_r)^{\frac{\beta}{2\alpha+2}-1} \int_{B_r} \tau_x |f|(y) d\mu_\alpha(y), \quad 0 \leq \beta < 2\alpha + 2,$$

and the Dunkl-type fractional integral by

$$I^{\beta, \alpha} f(x) = \int_R \tau_x |y|^{\beta-2\alpha-2} f(y) d\mu_\alpha(y), \quad 0 < \beta < 2\alpha + 2.$$

If $\beta = 0$, then $M_\alpha \equiv M_{0, \alpha}$ is the Hardy-Littlewood maximal operator associated with the Dunkl operator (see [1, 17, 24, 35]).

Theorem 2. ([1, 24, 35])

1) If $f \in L_{1,\alpha}(R)$, then for every $s > 0$

$$\mu_\alpha \{x \in R : M_\alpha f(x) > s\} \leq \frac{C_1}{s} \int_R |f(x)| d\mu_\alpha(x),$$

where $C_1 > 0$ is independent of f .

2) If $f \in L_{p,\alpha}(R)$, $1 < p \leq \infty$, then $M_\alpha f \in L_{p,\alpha}(R)$ and

$$\|M_\alpha f\|_{p,\alpha} \leq C_2 \|f\|_{p,\alpha},$$

where $C_2 > 0$ is independent of f .

Corollary 3. If $f \in L_{1,\alpha}^{loc}(R)$, then

$$\lim_{r \rightarrow 0} \frac{1}{\mu_\alpha B_r} \int_{B_r} |\tau_x f(y) - f(x)| d\mu_\alpha(y) = 0$$

for a.e. $x \in R$.

Corollary 4. If $f \in L_{1,\alpha}^{loc}(R)$, then

$$\lim_{r \rightarrow 0} \frac{1}{\mu_\alpha B_r} \int_{B_r} \tau_x f(y) d\mu_\alpha(y) = f(x)$$

for a.e. $x \in R$.

The following theorem is our main result in which we obtain the necessary and sufficient conditions for the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ to be bounded from the spaces $L_{p,\alpha}(R)$ to $L_{q,\alpha}(R)$, $1 < p < q < \infty$ and from the spaces $L_{1,\alpha}(R)$ to the weak spaces $WL_{q,\alpha}(R)$, $1 < q < \infty$.

Theorem 5. ([18]) Let $0 < \beta < 2\alpha + 2$ and $1 \leq p \leq \frac{2\alpha+2}{\beta}$.

1) If $1 < p < \frac{2\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_{\beta,\alpha}$ from $L_{p,\alpha}(R)$ to $L_{q,\alpha}(R)$.

2) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2}$ is necessary and sufficient for the boundedness of $M_{\beta,\alpha}$ from $L_{1,\alpha}(R)$ to $WL_{q,\alpha}(R)$.

3) If $p = \frac{2\alpha+2}{\beta}$, then $M_{\beta,\alpha}$ is bounded from $L_{p,\alpha}(R)$ to $L_\infty(R)$.

For $1 \leq p, \theta \leq \infty$ and $0 < s < 1$, the Besov space for the Dunkl operators on R (Besov-Dunkl space) $B_{p\theta,\alpha}^s(R)$ consists of all functions f in $L_{p,\alpha}(R)$ so that

$$\|f\|_{B_{p\theta,\alpha}^s} = \|f\|_{p,\alpha} + \left(\int_R \frac{\|\tau_x f(\cdot) - f(\cdot)\|_{p,\alpha}^\theta}{|x|^{2\alpha+2+s\theta}} d\mu_\alpha(x) \right)^{1/\theta} < \infty. \quad (1)$$

Besov spaces in the setting of the Dunkl operators have been studied by C. Abdelkefi and M. Sifi [2, 3], R. Bouguila, M.N. Lazhari and M. Assal [5], L. Kamoun [21], Y.Y. Mammadov [25] and V.S. Guliyev, Y.Y. Mammadov [18]. In the following theorem we prove the boundedness of the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ in the Dunkl-type Besov spaces.

Theorem 6. ([18]) *For $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, $1 \leq \theta \leq \infty$ and $0 < s < 1$ the Dunkl-type fractional maximal operator $M_{\beta,\alpha}$ is bounded from $B_{p\theta,\alpha}^s(R)$ to $B_{q\theta,\alpha}^s(R)$. More precisely, there is a constant $C > 0$ such that*

$$\|M_{\beta,\alpha}f\|_{B_{q\theta,\alpha}^s} \leq C\|f\|_{B_{p\theta,\alpha}^s}$$

hold for all $f \in B_{p\theta,\alpha}^s(R)$.

Definition 7. Let $1 \leq p < \infty$, $0 \leq \lambda \leq 2\alpha + 2$. We denote by $\mathcal{M}_{p,\lambda,\alpha}(R)$ the Dunkl-type Morrey space ($\equiv D$ -Morrey space) as the set of locally integrable functions $f(x)$, $x \in R$, with the finite norm

$$\|f\|_{\mathcal{M}_{p,\lambda,\alpha}} = \sup_{t>0, x \in R} \left(t^{-\lambda} \int_{B_t} [\tau_x |f(y)|]^p d\mu_\alpha(y) \right)^{1/p}.$$

Theorem 8. ([19])

1. *If $f \in \mathcal{M}_{1,\lambda,\alpha}(R)$, $0 \leq \lambda < 2\alpha + 2$, then $M_\alpha f \in W\mathcal{M}_{1,\lambda,\alpha}(R)$ and*

$$\|M_\alpha f\|_{W\mathcal{M}_{1,\lambda,\alpha}} \leq C_{1,\lambda,\alpha} \|f\|_{\mathcal{M}_{1,\lambda,\alpha}},$$

where $C_{1,\lambda,\alpha}$ depends only on λ, α and n .

2. *If $f \in \mathcal{M}_{p,\lambda,\alpha}(R)$, $1 < p < \infty$, $0 \leq \lambda < 2\alpha + 2$, then $M_\alpha f \in \mathcal{M}_{p,\lambda,\alpha}(R)$ and*

$$\|M_\alpha f\|_{\mathcal{M}_{p,\lambda,\alpha}} \leq C_{p,\lambda,\alpha} \|f\|_{\mathcal{M}_{p,\lambda,\alpha}},$$

where $C_{p,\lambda,\alpha}$ depends only on p, λ and α .

Theorem 9. ([19]) *Let $0 < \beta < 2\alpha + 2$, $0 \leq \lambda < 2\alpha + 2 - \beta$ and $1 \leq p < \frac{2\alpha+2-\lambda}{\beta}$.*

1) *If $1 < p < \frac{2\alpha+2-\lambda}{\beta}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness $M_{\beta,\alpha}$ from $\mathcal{M}_{p,\lambda,\alpha}(R)$ to $\mathcal{M}_{q,\lambda,\alpha}(R)$.*

2) *If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2-\lambda}$ is necessary and sufficient for the boundedness $M_{\beta,\alpha}$ from $\mathcal{M}_{1,\lambda,\alpha}(R)$ to $W\mathcal{M}_{q,\lambda,\alpha}(R)$.*

For a real parameter $\alpha \geq -1/2$, we consider the Dunkl operator, associated with the reflection group Z_2 on R :

$$\Lambda_\alpha(f)(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{x} \left(\frac{f(x) - f(-x)}{2} \right). \quad (2)$$

Note that $\Lambda_{-1/2} = d/dx$.

For $\alpha \geq -1/2$ and $\lambda \in C$, the initial value problem:

$$\Lambda_\alpha(f)(x) = \lambda f(x), \quad f(0) = 1, \quad x \in R,$$

has a unique solution $E_\alpha(\lambda x)$ called Dunkl kernel [9, 33, 38] and given by

$$E_\alpha(\lambda x) = j_\alpha(i\lambda x) + \frac{\lambda x}{2(\alpha+1)} j_{\alpha+1}(i\lambda x), \quad x \in R,$$

where j_α is the normalized Bessel function of the first kind and order α [39], defined by

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \alpha(n+\alpha+1)}, \quad z \in C.$$

We can write for $x \in R$ and $\lambda \in C$ (see Rösler [37], p. 295)

$$E_\alpha(-i\lambda x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^1 (1-t^2)^{\alpha-1/2} (1-t) e^{i\lambda x t} dt.$$

Note that $E_{-1/2}(\lambda x) = e^{\lambda x}$.

The Dunkl transform \mathcal{F}_α of a function $f \in L_{1,\alpha}(R)$, is given by

$$\mathcal{F}_\alpha f(\lambda) := \int_R E_\alpha(-i\lambda x) f(x) d\mu_\alpha(x), \quad \lambda \in R.$$

Here the integral makes sense since $|E_\alpha(ix)| \leq 1$ for every $x \in R$ [37], p. 295.

Note that $\mathcal{F}_{-1/2}$ agrees with the classical Fourier transform \mathcal{F} , given by:

$$\mathcal{F}f(\lambda) := (2\pi)^{-1/2} \int_R e^{-i\lambda x} f(x) dx, \quad \lambda \in R.$$

Proposition 1. (see Soltani [34])

- (i) If f is an even positive continuous function, then $\tau_x f$ is positive.
- (ii) For all $x \in R$ the operator τ_x extends to $L_{p,\alpha}(R)$, $p \geq 1$ and we have for $f \in L_{p,\alpha}(R)$,

$$\|\tau_x f\|_{p,\alpha} \leq 4\|f\|_{p,\alpha}. \quad (3)$$

- (iii) For all $x, \lambda \in R$ and $f \in L_{1,\alpha}(R)$, we have

$$\mathcal{F}_\alpha(\tau_x f)(\lambda) = E_\alpha(i\lambda x) \mathcal{F}_\alpha f(\lambda).$$

Let f and g be two continuous functions on R with compact support. We define the generalized convolution $*_{\alpha}$ of f and g by

$$f *_{\alpha} g(x) := \int_R \tau_x f(-y) g(y) d\mu_{\alpha}(y), \quad x \in R.$$

The generalized convolution $*_{\alpha}$ is associative and commutative, [37]. Note that $*_{-1/2}$ agrees with the standard convolution $*$.

Proposition 2. (see Soltani [34])

(i) If f is an even positive function and g a positive function with compact support, then $f *_{\alpha} g$ is positive.

(ii) Assume that $p, q, r \in [1, +\infty[$ satisfying $1/p + 1/q = 1 + 1/r$ (the Young condition). Then the map $(f, g) \mapsto f *_{\alpha} g$, defined on $\mathcal{E}_c \times \mathcal{E}_c$, extends to a continuous map from $L_{p,\alpha}(R) \times L_{q,\alpha}(R)$ to $L_{r,\alpha}(R)$, and we have

$$\|f *_{\alpha} g\|_{r,\alpha} \leq 4\|f\|_{p,\alpha} \|g\|_{q,\alpha}.$$

(iii) For all $f \in L_{1,\alpha}(R)$ and $g \in L_{2,\alpha}(R)$, we have

$$\mathcal{F}_{\alpha}(f *_{\alpha} g) = (\mathcal{F}_{\alpha}f)(\mathcal{F}_{\alpha}g).$$

We need the following lemma.

Lemma 1. ([18])

Let $0 < \beta < 2\alpha + 2$. Then for $2|x| \leq |y|$ the following inequality is valid

$$\left| \tau_y |x|^{\beta-2\alpha-2} - |y|^{\beta-2\alpha-2} \right| \leq 2^{2\alpha+4-\beta} |y|^{\beta-2\alpha-3} |x|. \quad (4)$$

3. Generalized Dunkl-type Morrey spaces

If in place of the power function r^{λ} in the definition of $\mathcal{M}_{p,\lambda}$ we consider any positive measurable weight function $\phi(x, r)$, then it becomes generalized Morrey space $\mathcal{M}_{p,\omega}$.

Definition 10. Let $\phi(x, r)$ positive measurable weight function on $R^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $\mathcal{M}_{p,\omega}(R^n)$ the generalized Morrey spaces, the spaces of all functions $f \in L_p^{loc}(R^n)$ with finite quasinorm

$$\|f\|_{\mathcal{M}_{p,\omega}(R^n)} = \sup_{x \in R^n, r > 0} \frac{r^{-\frac{n}{p}}}{\omega(x, r)} \|f\|_{L_p(B(x, r))}.$$

T. Mizuhara [27], E. Nakai [30] and V.S. Guliyev [10] (see also [11]) obtained sufficient conditions on weights ω and ω ensuring the boundedness of T from $\mathcal{M}_{p,\omega}$ to $\mathcal{M}_{p,\omega}$. In [30] the following statement was proved, containing the result in [27].

In [10], [11], [27] and [30] there were obtained sufficient conditions on weights ω_1 and ω_2 for the boundedness of the singular operator T from $\mathcal{M}_{p,\omega_1}(R^n)$ to $\mathcal{M}_{p,\omega_2}(R^n)$. In [30] the following conditions was imposed on $w(x, r)$:

$$c^{-1}\omega(x, r) \leq \omega(x, t) \leq c\omega(x, r) \quad (5)$$

whenever $r \leq t \leq 2r$, where $c(\geq 1)$ does not depend on t, r and $x \in R^n$, jointly with the condition:

$$\int_r^\infty \omega(x, t)^p \frac{dt}{t} \leq C \omega(x, r)^p. \quad (6)$$

for the maximal or singular operator and the condition

$$\int_r^\infty t^p \omega(x, t)^p \frac{dt}{t} \leq C r^p \omega(x, r)^p. \quad (7)$$

for potential and fractional maximal operators, where $C(> 0)$ does not depend on r and $x \in R^n$.

In [30] the following statements were proved.

Theorem 11. ([30]) *Let $1 < p < \infty$ and $\omega(x, r)$ satisfy conditions (5)-(6). Then the operators M and T are bounded in $\mathcal{M}_{p,\omega}(R^n)$.*

Theorem 12. ([30]) *Let $1 < p < \infty, 0 < \beta < \frac{n}{p}$, and $\omega(x, t)$ satisfy conditions (5) and (7). Then the operators M^β and I^β are bounded from $\mathcal{M}_{p,\omega}(R^n)$ to $\mathcal{M}_{q,\omega}(R^n)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$.*

The following statement, containing the results in [27], [30] was proved in [10] (see also [11]). Note that Theorems 13 and 14 do not impose the condition (5).

Theorem 13. ([10]) *Let $1 < p < \infty$ and $\omega_1(x, r), \omega_2(x, r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty \omega_1(x, t) \frac{dt}{t} \leq c_1 \omega_2(x, r), \quad (8)$$

with $c_1 > 0$ not depending on $x \in R^n$ and $t > 0$. Then for $p > 1$ the operators M and T are bounded from $\mathcal{M}_{p,\omega_1}(R^n)$ to $\mathcal{M}_{p,\omega_2}(R^n)$ and for $p = 1$ the operators M and T are bounded from $\mathcal{M}_{1,\omega_1}(R^n)$ to $W\mathcal{M}_{1,\omega_2}(R^n)$.

A Spanne type result follows.

Theorem 14. ([10]) *Let $0 < \beta < n$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$ and $\omega_1(x, r)$, $\omega_2(x, r)$ be positive measurable functions satisfying the condition*

$$\int_r^\infty t^\beta \omega_1(x, t) \frac{dt}{t} \leq c_1 \omega_2(x, r). \quad (9)$$

Then for $p > 1$ the operators M^β and I^β are bounded from $\mathcal{M}_{p, \omega_1}(R^n)$ to $\mathcal{M}_{q, \omega_2}(R^n)$ and for $p = 1$ the operators M^β and I^β are bounded from $\mathcal{M}_{1, \omega_1}(R^n)$ to $W\mathcal{M}_{q, \omega_2}(R^n)$.

Let $\omega(x, r)$ be positive measurable weight function on $R \times (0, \infty)$. The norm in the space $\mathcal{M}_{p, \omega, \alpha}(R)$ may be introduced in two forms,

$$\|f\|_{\mathcal{M}_{p, \omega, \alpha}} = \sup_{x \in R, t > 0} \frac{t^{-\frac{2\alpha+2}{p}}}{\omega(x, t)} \|\tau_x |f|\|_{L_{p, \alpha}(B_t)}.$$

If $\omega(x, t) \equiv r^{-\frac{2\alpha+2}{p}}$ then $\mathcal{M}_{p, \omega, \alpha}(R) \equiv L_{p, \alpha}(R)$, if $\omega(x, t) \equiv t^{\frac{\lambda-2\alpha+2}{p}}$, $0 \leq \lambda < 2\alpha + 2$, then $\mathcal{M}_{p, \omega, \alpha}(R) \equiv \mathcal{M}_{p, \lambda, \alpha}(R)$.

4. The Dunkl-type fractional integral operator in the spaces $\mathcal{M}_{p, \omega, \alpha}(R)$

Theorem 15. *Let $1 \leq p < \infty$ and the $\omega(x, r)$ positive measurable weight function on $R \times (0, \infty)$ satisfying the condition*

$$\int_r^\infty \omega(x, t) \frac{dt}{t} \leq C \omega(x, r). \quad (10)$$

Then for $p > 1$ the maximal operator M is bounded from $\mathcal{M}_{p, \omega, \alpha}(R)$ to $\mathcal{M}_{p, \omega, \alpha}(R)$ and for $p = 1$ the maximal operator M is bounded from $\mathcal{M}_{1, \omega, \alpha}(R)$ to $W\mathcal{M}_{1, \omega, \alpha}(R)$.

Proof. The maximal function $M_\alpha f(x)$ may be interpreted as a maximal function defined on a space of homogeneous type. By this we mean a topological space X equipped with a continuous pseudometric ρ and a positive measure μ satisfying

$$\nu E(x, 2r) \leq C_0 \nu E(x, r) \quad (11)$$

with a constant C_0 being independent of x and $r > 0$. Here $E(x, r) = \{y \in X : \rho(x, y) < r\}$, $\rho(x, y) = |x - y|$. Let (X, ρ, μ) be a space of homogeneous type,

where $X = R$, $\rho(x, y) = |x - y|$, $d\nu(x) = d\mu_\alpha(x)$. It is clear that this measure satisfies the doubling condition (11).

Define

$$M_\nu f(x) = \sup_{r>0} (\nu B(x, r))^{-1} \int_{B(x, r)} |f(y)| d\nu(y).$$

It is well known that the maximal operator M_ν is bounded from $L_1(X, \nu)$ to $WL_1(X, \nu)$ and is bounded on $L_p(X, \nu)$ for $1 < p < \infty$ (see [7]).

The following inequality was proved in [24]

$$M_\alpha f(x) \leq C M_\nu f(x), \quad (12)$$

where $C > 0$ is independent of f .

Using inequality (12) we have

$$\begin{aligned} & \left(\int_{B_r} [\tau_x (M_\alpha f(y))]^p d\mu_\alpha(y) \right)^{1/p} \\ &= \left(\int_R [\tau_x (M_\alpha f(y))]^p \chi_{B_r}(y) d\mu_\alpha(y) \right)^{1/p} \\ &\leq C \left(\int_Y (M_\nu f(y))^p \chi_{B(x, r)}(y) d\nu(y) \right)^{1/p}. \end{aligned}$$

In [20] there was proved that the analogue of the Fefferman-Stein theorem for the maximal operator defined on a space of homogeneous type is valid, if condition (11) is satisfied. Therefore,

$$\int_Y (M_\nu \varphi(y))^p \psi(y) d\nu(y) \leq C_p \int_Y |\varphi(y)|^p M_\nu \psi(y) d\nu(y). \quad (13)$$

Then taking $\varphi(y) = f(y)$ and $\psi(y) \equiv \chi_{B(x, r)}(y)$ we obtain from inequality (13) that

$$\begin{aligned} & \left(\int_{B_r} [\tau_x (M_\alpha f(y))]^p d\mu_\alpha(y) \right)^{1/p} \\ &\leq C \left(\int_Y (M_\nu f(y))^p \chi_{B(x, r)}(y) d\nu(y) \right)^{1/p} \\ &\leq C_p \left(\int_Y |f(y)|^p M_\nu \chi_{B(x, r)}(y) d\nu(y) \right)^{1/p} \\ &= C_p \left(\int_R [\tau_x |f(y)|]^p M \chi_{B_r}(y) d\mu_\alpha(y) \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
& \leq C_p \left(\int_{B_r} [\tau_x |f(y)|]^p d\mu_\alpha(y) \right. \\
& \quad \left. + C_p \sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} [\tau_x |f(y)|]^p M \chi_{B_r}(y) d\mu_\alpha(y) \right)^{1/p} \\
& \leq C_p \left(\int_{p_r} [\tau_x |f(y)|]^p d\mu_\alpha(y) \right)^{1/p} + \\
& \quad C_p \left(\sum_{j=1}^{\infty} \int_{B_{2^{j+1}r} \setminus B_{2^j r}} [\tau_x |f(y)|]^p \frac{r^{2\alpha+2}}{(|y|+r)^{2\alpha+2}} d\mu_\alpha(y) \right)^{1/p} \\
& \leq C_p \|f\|_{\mathcal{M}_{p,\omega,\alpha}} r^{\frac{2\alpha+2}{p}} \left(\omega(x, r) + \sum_{j=1}^{\infty} \frac{1}{(2^j+1)^{2\alpha+2}} (2^{j+1}r)^{\frac{2\alpha+2}{p}} \omega(x, 2^{j+1}r) \right) \\
& \leq C_3 \|f\|_{\mathcal{M}_{p,\omega,\alpha}} r^{\frac{2\alpha+2}{p}} \left(\omega(x, r) + C \int_r^\infty \omega(x, t) \frac{dt}{t} \right) \\
& \leq C_4 r^{\frac{2\alpha+2}{p}} \omega(x, r) \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.
\end{aligned}$$

Thus, Theorem 15 is proved. \square

For the Dunkl-type fractional integral operator the following Hardy-Littlewood-Sobolev type theorem in the generalized Dunkl-type Morrey spaces is valid.

Theorem 16. *Let $0 < \beta < 2\alpha + 2$, and $1 \leq p < \frac{2\alpha+2}{\beta}$, ω satisfy the conditions (10) and*

$$\int_t^\infty \omega(x, r) r^{\beta-1} dr \leq C \omega(x, r) r^\beta, \quad (14)$$

1) If $1 < p < \frac{2\alpha+2}{\beta}$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, then $I^{\beta,\alpha}$ is bounded from $\mathcal{M}_{p,\omega,\alpha}(R)$ to $\mathcal{M}_{q,\omega^{q/p},\alpha}(R)$.

2) If $p = 1$, $1 - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, then $I^{\beta,\alpha}$ is bounded from $\mathcal{M}_{1,\omega,\alpha}(R)$ to $W\mathcal{M}_{q,\omega^q,\alpha}(R)$.

Proof. 1) Let $f \in \mathcal{M}_{p,\omega,\alpha}(R)$.

Then,

$$I^{\beta,\alpha}f(x) = \left(\int_{B_t} + \int_{R \setminus B_t} \right) \tau_x f(y) |y|^{-2\alpha-2} d\mu_\alpha(y) \\ = F_1(x) + F_2(x). \quad (15)$$

Let $I^{\beta,\nu}f$ be the fractional integral operator on the space of homogeneous type (X, d, ν) :

$$I^{\beta,\nu}f(x) = \int_Y f(y) d(x, y)^{\alpha-1} d\nu(y)$$

Also, in the work [22], [23] it was proved:

Proposition 3. *Let $0 < \beta < 1$, $1 \leq p < \frac{1}{\beta}$, $\frac{1}{p} - \frac{1}{q} = \beta$. Then the following two conditions are equivalent:*

1) *There is a constant $C > 0$ such that for any $f \in L_{p,\varphi}(Y)$ the inequality*

$$\|I^{\beta,\nu}(f\varphi^\beta)\|_{L_{q,\varphi}} \leq C\|f\|_{L_{p,\varphi}}$$

holds.

2) $\varphi \in A_{1+\frac{q}{p}}(Y)$, $\frac{1}{p} + \frac{1}{p'} = 1$.

By Proposition 3 and $\varphi(y) = (M\chi_{B(x,r)}(y))^\theta \in A_p(Y)$, $0 < \theta < 1$, we have

$$\begin{aligned} & \left(\int_{B_t} \tau_x |F_1(y)|^q d\mu_\alpha(y) \right)^{1/q} \\ & \leq \left(\int_R \tau_x |F_1(y)|^q (M_\alpha \chi_{B_t}(y))^\theta d\mu_\alpha(y) \right)^{1/q} \\ & \leq \left(\int_Y \left| I^{\beta,\nu}(f\varphi^\beta)(y) \right|^q \varphi(y) d\nu(y) \right)^{1/q} \\ & \leq C_2 \left(\int_Y |f(y)|^p (M_\nu \chi_{B_t}(y))^\theta d\nu(y) \right)^{1/p} \\ & = C_2 \left(\int_R \tau_x |f(y)|^p (M_\alpha \chi_{B_t}(y))^\theta d\mu_\alpha(y) \right)^{1/p} \\ & \leq C_2 \left(\int_{B_t} \tau_x |f(y)|^p d\mu_\alpha(y) \right)^{1/p} \\ & + C_2 \left(\sum_{j=1}^{\infty} \int_{B_{2^{j+1}t} \setminus E_{2^j t}} \tau_x |f(y)|^p (M_\alpha \chi_{B_t}(y))^\theta d\mu_\alpha(y) \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
& \leq C_2 \left(\int_{B_t} \tau_x |f(y)|^p d\mu_\alpha(y) \right)^{1/p} \\
& + C_2 \left(\sum_{j=1}^{\infty} \int_{B_{2^{j+1}t} \setminus B_{2^j t}} \tau_x |f(y)|^p \frac{t^{(2\alpha+2)\theta}}{(|y|+t)^{(2\alpha+2)\theta}} d\mu_\alpha(y) \right)^{1/p} \\
& \leq C_3 \|f\|_{\mathcal{M}_{p,\omega,\alpha}} \left(r^{\frac{2\alpha+2}{p}} \omega(x,t) + \sum_{j=1}^{\infty} \frac{1}{(2^j+1)^{(2\alpha+2)\theta}} (2^{j+1}t)^{\frac{2\alpha+2}{p}} \omega(x, 2^{j+1}t) \right)^{1/p} \\
& \leq C_3 \|f\|_{\mathcal{M}_{p,\omega,\alpha}} t^{\frac{2\alpha+2}{p}} \left(\omega(x,t) + C \int_t^\infty \omega(x,r) \frac{dr}{r} \right) \\
& \leq C_4 t^{\frac{2\alpha+2}{p}} \omega(x,t) \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.
\end{aligned}$$

Hence,

$$\|F_1\|_{\mathcal{M}_{q,\omega,\alpha}} = \sup_{x \in R, t > 0} t^{-\frac{2\alpha+2}{q}} \omega^{-1}(x,t) \left(\int_{B_t} \tau_x |F_1(y)|^q d\mu_\alpha(y) \right)^{\frac{1}{q}} \leq C_4 \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.$$

Now we estimate $|F_2(x)|$. By the Hölder inequality we have

$$\begin{aligned}
|F_2(x)| & \leq \int_{R \setminus B_t} |y|^{\beta-2\alpha-2} \tau_x |f(y)| d\mu_\alpha(y) \\
& = \sum_{j=1}^{\infty} \int_{B_{2^{j+1}t} \setminus B_{2^j t}} |y|^{\beta-2\alpha-2} \tau_x |f(y)| d\mu_\alpha(y) \\
& \leq \sum_{j=1}^{\infty} \left(\int_{B_{2^{j+1}t} \setminus E_{2^j t}} |y|^{(\beta-2\alpha-2)p'} d\mu_\alpha(y) \right)^{\frac{1}{p'}} \left(\int_{B_{2^{j+1}t} \setminus B_{2^j t}} \tau_x |f(y)|^p d\mu_\alpha(y) \right)^{\frac{1}{p}} \\
& \leq C \|f\|_{\mathcal{M}_{p,\omega,\alpha}} \sum_{j=1}^{\infty} (2^j t)^\beta \omega(x, 2^j t) \leq C \|f\|_{\mathcal{M}_{p,\omega,\alpha}} \int_t^\infty \omega(x,r) r^{\beta-1} dr \\
& \leq C t^\beta \omega(x,t) \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|F_2\|_{\mathcal{M}_{q,\omega,\alpha}} & = \sup_{x \in R, t > 0} t^{-\frac{2\alpha+2}{q}} \omega^{-1}(x,t) \left(\int_{B_t} \tau_x |F_2(y)|^q d\mu_\alpha(y) \right)^{\frac{1}{q}} \\
& \leq C \sup_{x \in R, t > 0} \omega^{-1}(x,t) t^\beta \|f\|_{\mathcal{M}_{p,\omega,\alpha}} \omega(x,t) \|\chi_{B_t}\|_{L_{q,\alpha}} \leq C \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.
\end{aligned}$$

Therefore $I^{\beta,\alpha}f \in \mathcal{M}_{q,\omega^{q/p},\alpha}(R)$ and

$$\left\| I^{\beta,\alpha}f \right\|_{\mathcal{M}_{q,\omega^{q/p},\alpha}} \leq C \|f\|_{\mathcal{M}_{p,\omega,\alpha}}.$$

2) Let $f \in \mathcal{M}_{1,\omega,\alpha}(R)$. By the (15), we get

$$\begin{aligned} |F_1| &\leq \int_{E_t} \tau_x |f(y)| |y|^{\beta-2\alpha-2} d\mu_\alpha(y) \\ &\leq \sum_{k=-\infty}^{-1} \left(2^k t\right)^{\beta-2\alpha-2} \int_{B_{2^{k+1}t} \setminus B_{2^k t}} \tau_x |f(y)| d\mu_\alpha(y). \end{aligned}$$

Hence

$$|F_1(x)| \leq C t^\beta M_\alpha f(x). \quad (16)$$

Then

$$\begin{aligned} &\mu_\alpha \left(\left\{ y \in B_t : \tau_x |I^{\beta,\alpha}f(y)| > 2s \right\} \right) \\ &\leq \mu_\alpha(\{y \in B_t : \tau_x |F_1(y)| > s\}) + \mu_\alpha(\{y \in B_t : \tau_x |F_2(y)| > s\}). \end{aligned}$$

Taking into account inequality (16) and Theorem 8, we have

$$\begin{aligned} &\mu_\alpha(\{y \in B_t : \tau_x |F_1(y)| > s\}) \\ &\leq \mu_\alpha \left(\left\{ y \in B_t : \tau_x (M_\alpha f(y)) > \frac{s}{C t^\beta} \right\} \right) \leq \frac{C t^\beta}{s} \cdot \omega(x, t) \|f\|_{\mathcal{M}_{1,\omega,\alpha}}, \end{aligned}$$

and thus if $C t^{\frac{-2\alpha-2}{q}} \omega(x, t) \|f\|_{\mathcal{M}_{1,\omega,\alpha}} = s$, then $|F_2(x)| \leq \beta$ and consequently, $\mu_\alpha(\{y \in B_t : \tau_x |F_2(y)| > s\}) = 0$.

Finally,

$$\begin{aligned} \mu_\alpha(\{y \in B_t : \tau_x |I^{\beta,\alpha}f(y)| > 2s\}) &\leq \frac{C}{s} \omega(x, t) t^\alpha \|f\|_{\mathcal{M}_{1,\omega,\alpha}} \\ &= C \omega^q(x, t) \left(\frac{\|f\|_{\mathcal{M}_{1,\omega,\alpha}}}{s} \right)^q. \end{aligned}$$

Thus Theorem 16 is proved. \square

Theorem 17. For $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+2}$, $1 \leq \theta \leq \infty$ and $0 < s < 1$ the Dunkl-type fractional integral operator $I^{\beta,\alpha}$ is bounded from $B_{p\theta,\omega,\alpha}^s(R)$ to $B_{q\theta,\omega^{q/p},\alpha}^s(R)$. More precisely, there is a constant $C > 0$ such that

$$\|I^{\beta,\alpha}f\|_{B_{q\theta,\omega^{q/p},\alpha}^s} \leq C \|f\|_{B_{p\theta,\omega,\alpha}^s}$$

holds for all $f \in B_{p\theta,\omega,\alpha}^s(R)$.

Proof. For $x \in R$, let τ_x be the generalized translation by x . By definition of the generalized Dunkl-Besov-Morrey type spaces it suffices to show that

$$\|\tau_x I^{\beta,\alpha} f - I^{\beta,\alpha} f\|_{p,\alpha} \leq C \|\tau_x f - f\|_{p,\alpha}.$$

It is easy to see that τ_x commutes with $I^{\beta,\alpha}$, i.e. $\tau_x I^{\beta,\alpha} f = I^{\beta,\alpha}(\tau_x f)$. Hence we have

$$|\tau_x I^{\beta,\alpha} f - I^{\beta,\alpha} f| = |I^{\beta,\alpha}(\tau_x f) - I^{\beta,\alpha} f| \leq I^{\beta,\alpha}(|\tau_x f - f|).$$

Taking $L_{q,\alpha}(R)$ norm on both ends of the above inequality, by the boundedness of $I^{\beta,\alpha}$ from $\mathcal{M}_{p,\omega,\alpha}(R)$ to $\mathcal{M}_{q,\omega^{q/p},\alpha}(R)$, we obtain the desired result. Theorem 17 is proved. \square

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