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ON F-FACE MAGIC MEAN LABELING OF SOME DUPLICATED GRAPHS

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Abstract: In this paper we introduce the F-face magic mean labeling of planar graphs, an assignment of labels to the edges which induces an assignment of labels to the faces of a graph such that the mean weight of each face is constant. We discuss about the F-face magic mean labelings of some duplicated graphs.

AMS Subject Classification: 05C78
Key Words: labeling, F-face magic mean labeling, F-face magic mean graph

1. Introduction

By a graph, we mean a finite, connected, undirected planar graph without loops or multiple edges. By a planar graph, we mean that it can be drawn in a plane such that no two edges intersect.

Duplication of an edge $e = uv$ by a vertex v' in a graph G is a new graph G' where $V(G') = V(G) \cup \{v'\}$ and $E(G') = E(G) \cup \{uv', v'v\}$. Vertex duplication of a path P_n, denoted by $\widehat{P}_n$ is formed by duplicating all the vertices of $P_n, n \geq 2$. Vertex duplication of a cycle C_n, denoted by $\widehat{C}_n$, is formed by duplicating all the vertices of $C_n, n \geq 3$, where $n \equiv 0 (mod\ 2)$.

The middle graph $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent in $M(G)$ if and only if either they are adjacent vertices

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or edges of G or one is a vertex of G and the other is an edge incident with it.

The Butterfly graph $B_n, n \geq 2$ is obtained from a cycle C_4 having edges v_1v_2, v_2v_3, v_3v_4 and v_4v_1 by duplicating the vertex v_1 by v_1^1, v_1^1 by v_1^2 and v_1^2 by $v_1^3, \ldots, v_1^{n-2}$ by v_1^{n-1}, the edges v_2v_3 by a vertex v_2^1, v_2^1 by $v_2^2, \ldots, v_2^{n-1}$ by v_2^n and the edge v_3v_4 by a vertex v_3^1, v_3^1 by $v_3^2, \ldots, v_3^{n-1}$ by v_3^n.

A function g is called a mean labeling of graph G if $g : V(G) \to \{0, 1, 2, \ldots, |E(G)|\}$ is injective and the induced edge function $g^* : E(G) \to \{1, 2, \ldots, |E(G)|\}$ defined as follows is bijective

$$g^*(e = uv) = \begin{cases} \frac{g(u)+g(v)}{2} & \text{if } g(u) + g(v) \text{ is even} \\ \frac{g(u)+g(v)+1}{2} & \text{if } g(u) + g(v) \text{ is odd.} \end{cases}$$

The graph which admits mean labeling is called a mean graph [5].

A graph G is magic if the edges of G can be labeled by the numbers $1, 2, 3, \ldots, |E(G)|$ so that the sum of the labels of all the edges incident with any vertex is the same [2].

Motivated by these works, we introduce F-face magic mean graph as follows:

An injection $\phi : E(G) \to \{1, 2, \ldots, e\}$ is called a F-face magic mean labeling of G if the induced face labeling

$$\begin{aligned} \phi^*(f_i) &= \left\lfloor \frac{\text{sum of the labels of the edges in the boundary of } f_i}{deg(f_i)} \right\rfloor \\ &= \left\lfloor \frac{\sum\limits_{e_j \text{ is in } f_i} \phi(e_j)}{deg(f_i)} \right\rfloor \\ &= k, \text{ a constant,} \end{aligned}$$

for each face f_i, including the exterior face of G.

In this paper we prove that F-face magic mean labeling exists for certain families of graphs obtained by duplicating the vertices or edges such as $\widehat{P_n}, \widehat{C_{2n}}$, middle graph of C_n, butterfly graph B_n.

2. Main Results

Theorem 1. *$\widehat{P_n}$, vertex duplication of a path P_n, admits a F-face magic mean labeling for all $n \geq 2$ with face magic mean constant $\frac{3n-2}{2}$ while $n \equiv 0(mod\ 2)$ and $\frac{3(n-1)}{2}$ while $n \equiv 1(mod\ 2)$.*

Proof. Let $\{v_i : 1 \leq i \leq n\}$ be the vertices of P_n and $u_i(1 \leq i \leq n)$ be the respective duplicating vertex of v_i for $1 \leq i \leq n$. Then,

$$\begin{aligned} E(\widehat{P_n}) &= \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_{i-1} : 2 \leq i \leq n\} \\ &\quad \cup \{u_iv_{i+1} : 2 \leq i \leq n\} \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \text{ and} \\ F(\widehat{P_n}) &= \{f_i = (v_iv_{i+1}, v_{i+1}v_{i+2}, v_{i+2}u_{i+1}, u_{i+1}v_i) : 1 \leq i \leq n-2\} \\ &\quad \cup \left\{ f_0 = \left(\bigcup_{i=1}^{n-1} u_iv_{i+1}, \bigcup_{i=2}^{n} u_iv_{i-1}, v_1v_2, v_{n-1}v_1 \right) \right\}. \end{aligned}$$

In $\widehat{P_n}$, $|E(\widehat{P_n})| = 3(n-1)$ and $|F(\widehat{P_n})| = n-1$.

Define $\phi : E(\widehat{P_n}) \to \{1, 2, \ldots, 3n-3\}$ as follows:

$$\begin{aligned} \phi(u_1v_2) &= 1 \\ \phi(u_nv_{n-1}) &= 3n-3 \\ \phi(u_iv_{i-1}) &= n-i+1, \ 2 \leq i \leq n-1 \\ \phi(u_iv_{i+1}) &= 3n-i-2, \ 2 \leq i \leq n-1 \\ \phi(v_iv_{i+1}) &= n+i-1, \ 1 \leq i \leq n-1. \end{aligned}$$

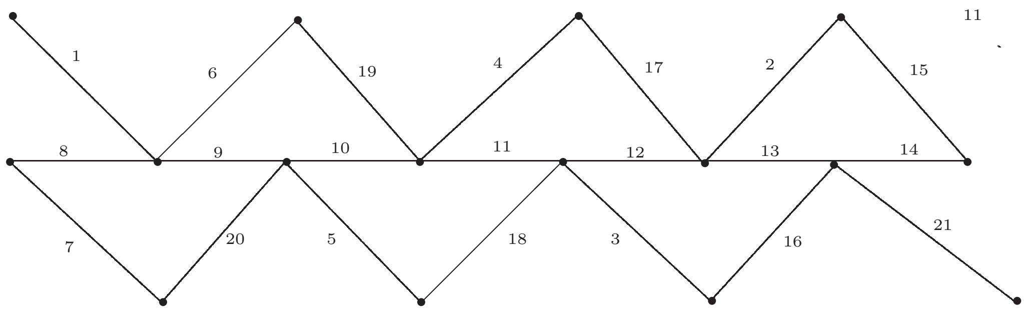
The induced face labeling ϕ^* is obtained as follows:

$$\phi^*(f_i) = \begin{cases} \left\lfloor \frac{3n-2}{2} \right\rfloor & \text{if } n \equiv 0 (mod\ 2) \\ \left\lfloor \frac{3(n-1)}{2} \right\rfloor & \text{if } n \equiv 1 (mod\ 2) \end{cases} = n,$$

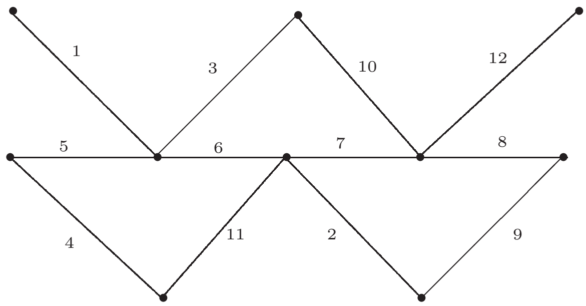
$$\text{and} \quad \phi^*(f_0) = \begin{cases} \left\lfloor \frac{3n-2}{2} \right\rfloor & \text{if } n \equiv 0 (mod\ 2) \\ \left\lfloor \frac{3(n-1)}{2} \right\rfloor & \text{if } n \equiv 1 (mod\ 2) \end{cases} = n.$$

Therefore, ϕ is a F-face magic mean labeling of $\widehat{P_n}$. Thus $\widehat{P_n}$ is a F-face magic mean graph with face magic mean constant $\left\lfloor \frac{3n-2}{2} \right\rfloor$, where $n \geq 2$. □

The F-face magic mean labeling of $\widehat{P_8}$ and $\widehat{P_5}$ are shown in Figure 1.



F-face magic mean labeling of $\widehat{P_8}$ with face magic mean constant 11



F-face magic mean labeling of $\widehat{P_5}$ with face magic mean constant 6

Figure 1

Theorem 2. *$\widehat{C_n} (n \geq 3)$ admits a F-face magic mean labeling with face magic mean constant $\frac{3n}{2}$ if and only if n is even.*

Proof. While n is odd, $\widehat{C_n}$ is a non-planar graph. Consider n is even. Let $\{v_i : 1 \leq i \leq n\}$ be the vertices of C_n and u_i be the respective duplicating vertex of v_i, for $1 \leq i \leq n$. Then,

$$\begin{aligned} E(\widehat{C_n}) &= \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_{i-1} : 2 \leq i \leq n\} \\ &\cup \{u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_n v_1, v_n u_1, v_n v_1\} \text{ and} \\ F(\widehat{C_n}) &= \{f_i = (v_{i-1} v_i, v_i v_{i+1}, u_i v_{i+1}, u_i v_{i-1}) : 2 \leq i \leq n-1\} \\ &\cup \{f_1 = (v_1 v_2, v_n v_1, u_1 v_n, u_1 v_2), f_n = (v_{n-1} v_n, v_n v_1, v_1 u_n, v_{n-1} u_n)\} \\ &\cup \left\{ f_I = \left(\bigcup_{\substack{i=2 \\ i \equiv 0 (mod\ 2)}}^{n} v_{i-1} u_i, \bigcup_{\substack{i=2 \\ i \equiv 0 (mod\ 2)}} u_i v_{i-1}, u_n v_1 \right) \right., \end{aligned}$$

$$f_0 = \left(\bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{n-1} u_i v_{i+1}, \bigcup_{\substack{i=3 \\ i\equiv 1(mod\ 2)}}^{n-1} u_i v_{i-1}, v_n u_1 \right) \Bigg\}.$$

In $\widehat{C_n}$, $|E(\widehat{C_n})| = 3n$ and $|F(\widehat{C_n})| = n+2$.

Define $\phi : E(\widehat{C_n}) \to \{1, 2, \ldots, 3n\}$ as follows:

$$\phi(u_i v_{i+1}) = \begin{cases} i, & 1 \le i \le n-1 \text{ and } i \text{ is odd} \\ n-1+i, & 1 \le i \le n-1 \text{ and } i \text{ is even}, \end{cases}$$
$$\phi(v_i v_{i+1}) = \begin{cases} 3n-i, & 1 \le i \le n-1 \text{ and } i \text{ is odd} \\ i, & 1 \le i \le n-1 \text{ and } i \text{ is even}, \end{cases}$$
$$\phi(u_i v_{i-1}) = \begin{cases} 3n+3-i, & 2 \le i \le n \text{ and } i \text{ is odd} \\ 2n+2-1, & 2 \le i \le n \text{ and } i \text{ is even}, \end{cases}$$

$$\phi(u_n v_1) = 2n-1,$$
$$\phi(v_n u_1) = 2n+2 \text{ and}$$
$$\phi(v_n v_1) = n.$$

The induced face labeling ϕ^* is obtained as follows:

$$\phi^*(f_i) = \frac{3n}{2}, 1 \le i \le n,$$
$$\phi^*(f_I) = \frac{3n}{2} \text{ and}$$
$$\phi^*(f_0) = \frac{3n}{2}.$$

Therefore, $\widehat{C_n}$ is a F-face magic mean graph with face magic mean constant $\frac{3n}{2}$. □

The F-face magic mean labeling of $\widehat{C_8}$ having face magic mean constant 12 is shown in Figure 2.

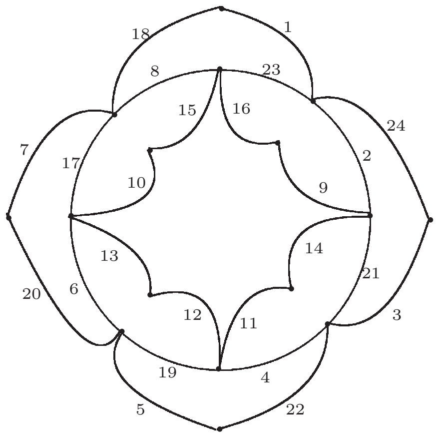


Figure 2: F-face magic mean labeling of $\widehat{C_8}$

Theorem 3. *The middle graph $M(C_n)$ of a cycle C_n is a F-face magic mean graph, for all $n \geq 3$.*

Proof. Let the vertex set, edge set and face set of $G = M(C_n)$ be

$$\begin{aligned} V(G) &= \{v_i : 1 \leq i \leq n\} \cup \{u_i(= e_i) : 1 \leq i \leq n\}, \\ E(G) &= \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{u_iv_i : 1 \leq i \leq n\} \\ &\quad \cup \{u_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_nv_1\} \quad \text{and} \end{aligned}$$

$$\begin{aligned} F(G) = &\{f_i = (u_iv_i, u_iv_{i+1}, v_iv_{i+1}) : 1 \leq i \leq n-1\} \\ &\cup \left\{ f_n = (u_nv_n, u_nv_1, v_nv_1), f_I = \left(\bigcup_{i=1}^{n-1} v_iv_{i+1}, v_nv_1 \right) \right. \\ &\left. f_0 = \left(\bigcup_{i=1}^{n} u_iv_i, \bigcup_{i=1}^{n-1} u_iv_{i+1}, u_nv_1 \right) \right\}, \quad \text{respectively.} \end{aligned}$$

In $G, |E(G)| = 3n$ and $|F(G)| = n + 2$.

Case (i) $n \equiv 0(mod\ 2)$.

Define $\phi : E(G) \to \{1, 2, \ldots, 3n\}$ as follows:

$$\phi(u_iv_i) = i, \quad 1 \leq i \leq n,$$

$$\phi(v_iv_{i+1}) = \begin{cases} \frac{3n}{2} + i, & 1 \leq i \leq \frac{n}{2} \\ \frac{n}{2} + i, & \frac{n}{2} + 1 \leq i \leq n-1, \end{cases}$$

$$\phi(v_nv_1) = \frac{3n}{2},$$
$$\phi(u_iv_{i+1}) = \begin{cases} 3n-2i+2, & 1 \le i \le \frac{n}{2} \\ 4n-2i+1, & \frac{n}{2}+1 \le i \le n-1 \end{cases} \quad \text{and}$$
$$\phi(u_nv_1) = 2n+1.$$

The induced face labeling on f_i is obtained as follows:

$$\phi^*(f_i) = \begin{cases} \left\lfloor \frac{9n+4}{6} \right\rfloor, & 1 \le i \le \frac{n}{2} \\ \left\lfloor \frac{9n+2}{6} \right\rfloor, & \frac{n}{2}+1 \le i \le n, \end{cases}$$
$$\phi^*(f_I) = \left\lfloor \frac{3n+1}{2} \right\rfloor \quad \text{and}$$
$$\phi^*(f_0) = \left\lfloor \frac{3n+1}{2} \right\rfloor.$$

Thus ϕ is a face magic mean labeling with face magic mean constant $\frac{3n}{2}$.

Case (ii) $n \equiv 1(mod\ 2)$.

Define $\phi : E(G) \to \{1, 2, \ldots, 3n\}$ as follows:

$$\phi(v_iv_{i+1}) = \begin{cases} \frac{3(n+1)}{2} + i - 2, & 1 \le i \le \frac{n+1}{2} \\ \frac{n-1}{2} + i, & \frac{n+3}{2} \le i \le n-1, \end{cases}$$
$$\phi(v_nv_1) = \frac{3n-1}{2},$$
$$\phi(u_iv_i) = i, \text{ for } 1 \le i \le n,$$

$$\phi(u_iv_{i+1}) = \begin{cases} 3n-2i+2, & 1 \le i \le \frac{n+1}{2} \\ 4n-2i+2, & \frac{n+3}{2} \le i \le n-1 \end{cases} \quad \text{and}$$
$$\phi(u_nv_1) = 2n+2.$$

The induced face labeling ϕ^* is obtained as follows:

$$\phi^*(f_i) = \frac{3n+1}{2}, \quad 1 \le i \le n,$$
$$\phi^*(f_I) = \frac{3n+1}{2} \quad \text{and} \quad \phi^*(f_0) = \frac{3n+1}{2}.$$

Thus ϕ is a F-face magic mean labeling with face magic mean constant $\frac{3n+1}{2}$.

Hence $M(C_n)$ is a F-face magic mean graph for all $n \ge 3$. □

Figure 3 illustrates the F-face magic mean labeling of $M(C_8)$ and $M(C_7)$.

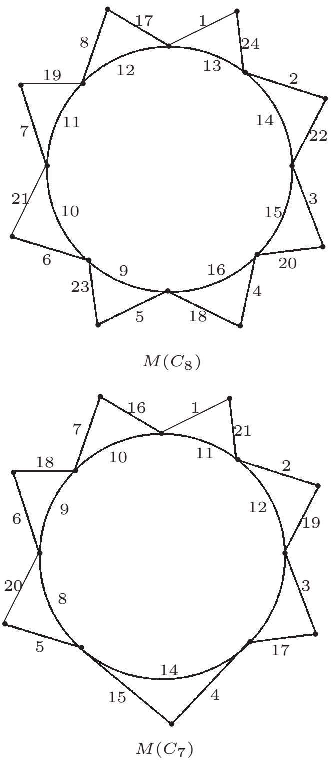


Figure 3

Theorem 4. *Butterfly graph $B_n, n \geq 2$ is a F-face magic mean graph having face magic mean constant $3n+1$.*

Proof. Let $n \geq 2$.

Let $V(B_n) = \{v_i : 1 \leq i \leq 4\} \cup \{v_1^j : 1 \leq j \leq n-1\} \cup \{v_2^j : 1 \leq j \leq n\} \cup \{v_3^j : 1 \leq j \leq n\}$. Then,

$$\begin{aligned} E(B_n) = \{&v_1v_2, v_2v_3, v_3v_4, v_4v_1\} \cup \{v_2v_1^j : 1 \leq j \leq n-1\} \\ &\cup \{v_2v_2^j : 1 \leq j \leq n\} \cup \{v_3v_2^j : 1 \leq j \leq n\} \cup \{v_3v_3^j : 1 \leq j \leq n\} \\ &\cup \{v_4v_3^j : 1 \leq j \leq n\} \cup \{v_4v_1^j : 1 \leq j \leq n-1\} \text{ and} \\ F(B_n) = \{&f_1 = (v_1v_2, v_2v_3, v_3v_4, v_4v_1), f_2 = (v_2v_2', v_2'v_3, v_3v_2), \\ &f_3 = (v_3v_3', v_3'v_4, v_4v_3), f_1' = (v_1v_2, v_2v_1', v_1'v_4, v_4v_1)\} \\ &\cup \{f_1^j = (v_1^jv_2, v_2v_1^{j-1}, v_1^{j-1}v_4, v_4v_1^j) : 2 \leq j \leq n-1\} \end{aligned}$$

$$\cup \{f_2^j = (v_2^{j+1}v_3, v_3v_2^j, v_2^jv_2, v_2v_2^{j+1}) : 1 \le j \le n-1\}$$
$$\cup \{f_3^j = (v_3^{j+1}v_4, v_4v_3^j, v_3^jv_3, v_3v_3^{j+1}) : 1 \le j \le n-1\}$$
$$\cup \{f_0 = (v_2v_2^n, v_2^nv_3, v_3v_3^n, v_3^nv_4, v_4v_1^{n-1}, v_1^{n-1}v_2)\}.$$

In butterfly graph $B_n, |E(B_n)| = 6n+2$ and $F(B_n)| = 3n+1$.

Define $\phi : E(B_n) \to \{1, 2, \ldots, 6n+2\}$ as follows:

$$\begin{aligned}
\phi(v_1v_2) &= 4n+2,\\
\phi(v_2v_3) &= 3n+2,\\
\phi(v_3v_4) &= 3n+1,\\
\phi(v_4v_1) &= 2n+1,\\
\phi(v_2v_2^j) &= j, 1 \le j \le n,\\
\phi(v_3v_2^j) &= 6n+3-j, 1 \le j \le n,\\
\phi(v_3v_3^j) &= n+j, 1 \le j \le n,\\
\phi(v_4v_3^j) &= 5n+3-j, 1 \le j \le n,\\
\phi(v_2v_1^j) &= 4n+2-j, 1 \le j \le n-1 \text{ and}\\
\phi(v_4v_1^j) &= 2n+j, 1 \le j \le n-1.
\end{aligned}$$

The induced face labeling ϕ^* is obtained as follows:

$$\begin{aligned}
\phi^*(f_1) &= \left\lfloor \frac{12n+6}{4} \right\rfloor, \phi^*(f_2) = \left\lfloor \frac{9n+4}{3} \right\rfloor,\\
\phi^*(f_1') &= \left\lfloor \frac{12n+5}{4} \right\rfloor, \phi^*(f_1^j) = 3n+1, 2 \le j \le n-1.\\
\phi^*(f_2^j) &= \left\lfloor \frac{12n+6}{4} \right\rfloor, 2 \le j \le n-1,\\
\phi^*(f_3^j) &= \left\lfloor \frac{12n+6}{4} \right\rfloor, 2 \le j \le n-1 \text{ and} \phi^*(f_0) = \left\lfloor \frac{18n+8}{6} \right\rfloor.
\end{aligned}$$

Thus ϕ is a F-face magic mean labeling of B_n.

Hence B_n is a F-face magic mean graph having face magic mean constant $3n+1$. □

Figure 4 illustrates the F-face magic mean labeling of butterfly graph B_4.

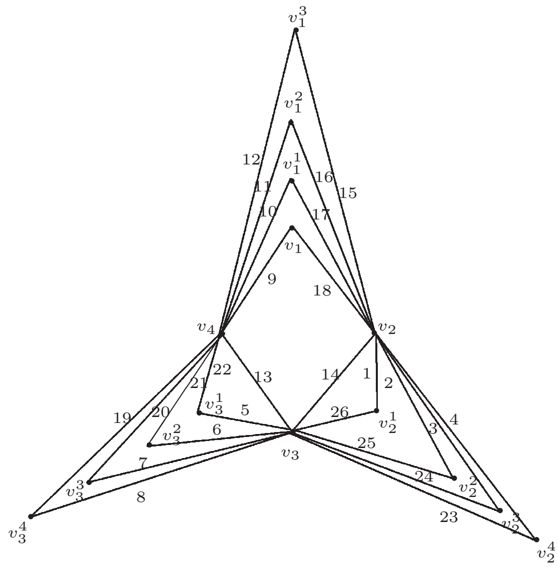


Figure 4: F-face magic mean labeling of B_4.

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