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ON THE WEAK SOLUTIONS OF THE URYSOHN-STIELTJES FUNCTIONAL INTEGRAL EQUATIONS

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Abstract: The analysis of Urysohn-Stieltjes integral operators has been studied in [1]. Here we study the existence of weakly solution of functional integral equations of Urysohn-Stieltjes type and Hammerstien-Stieltjes type in the reflexive Banach space E. Also, we prove the existence of the weak maximal and weak minimal solutions.

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1. Introduction and Preliminaries

Consider the nonlinear Urysohn-Stieltjes integral equation

$$x(t) = a(t) + \int_0^1 f(t, s, x(s))\, d_s g(t, s),\ t \in I = [0, 1], \tag{1}$$

where $g : I \times I \to R$ and the symbol d_s indicates the integration with respect to s. Equations of type (1) and some of their generalizations were considered in paper (see [3]), for the properties of the Urysohn-Stieltjes integral (see Banaś [1]).

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In this paper, we study the existence of weak solutions $x \in C[I,E]$ of the Urysohn-Stieltjes functional integral equation

$$x(t) = a(t) + \int_0^1 f(t,s,x(\psi(s)))\ d_s g(t,s), \quad t \in I. \tag{2}$$

As an application, we study the existence of weak solutions $x \in C[I,E]$ of the Hammerstien-Stieltjes functional integral equation

$$x(t) = a(t) + \int_0^1 k(t,s) f_1(s,x(\psi(s)))\ d_s g(t,s), \quad t \in I. \tag{3}$$

Also, the existence of the weak maximal and weak minimal solutions will be proved.

Let E be a reflexive Banach space with norm $\| \, . \, \|$ and dual E^*. Denote by $C[I,E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

Now, we shall present some auxiliary results that will be need in this work. Let E be a Banach space (need not be reflexive) and let $x : [a,b] \to E$, then:

(1-) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in [a,b]$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see [14] and [10]). It is evident that in reflexive Banach spaces, if x is weakly continuous function on $[a,b]$, then x is weakly Riemann integrable (see [14]). Since the space of all weakly Riemann-Stieltjes integrable functions is not complete, we will restrict our attention to the existence of weak solutions of equation (2) in the space $C[I,E]$.

Definition 1. Let $f : I \times E \to E$. Then $f(t,u)$ is said to be weakly-weakly continuous at (t_0,u_0) if given $\epsilon > 0$, $\phi \in E^*$ there exists $\delta > 0$ and a weakly open set U containing u_0 such that

$$| \ \phi(f(t,u) - f(t_0,u_0)) \ | < \epsilon$$

whenever

$$| \ t - t_0 \ | < \delta \ \text{ and } \ u \in U.$$

Now, we have the following fixed point theorem, due to O'Regan, in the reflexive Banach space (see [18]) and some propositions which will be used in the sequel (see [12]).

Theorem 2. *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of $C[I,E]$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.*

Proposition 1. *In reflexive Banach space, the subset is weakly relatively compact if and only if it is bounded in the norm topology.*

Proposition 2. *Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\varphi \in E^*$ with $\| \varphi \| = 1$ and $\| y \| = \varphi(y)$.*

2. Main results

In this section, we present our main result by proving the existence of weak solutions for equation (2) in the reflexive Banach space. Let us first state the following assumptions:

(i) $a : I \to I$ is continuous function.

(ii) $\psi : I \to I$ is continuous function such that $\psi(t) \leq t$.

(iii) $f : I \times I \times D \subset E \to E$ satisfies the following conditions:

  (1) $f(., s, x(s))$is continuous function, $\forall s \in I, x \in D \subset E$.

  (2) $f(t, ., .)$ is weakly-weakly continuous function, $\forall t \in I$.

  (3) $\| f(t,s,x) \| \leq m(t,s) + b\|x\|$, $m : I \times I \to I$ is continuous function, b is positive constant for $t, s \in I$, $x \in D$. Moreover, we put $M = \max\{m(t,s) : t, s \in I\}$.

(iv) The functions $t \to g(t,1)$ and $t \to g(t,0)$ are continuous on I, such that

$$\mu = \max\{\sup_t \mid g(t,1) \mid + \sup_t \mid g(t,0) \mid, \quad t \in I\}.$$

(v) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the function $s \to g(t_2, s) - g(t_1, s)$ is nondecreasing on I.

(vi) $g(0, s) = 0$ for any $s \in I$.

Remark 1. Observe that assumptions (v) and (vi) imply that the function $s \to g(t,s)$ is nondecreasing on the interval I, for any fixed $t \in I$ (Remark 1 in [4]). Indeed, putting $t_2 = t,\ t_1 = 0$ in (v) and keeping in mind (vi), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t,s)$ is of bounded variation on I.

Definition 3. By a weak solution to (2) we mean a function $x \in C[I,E]$ which satisfies the integral equation (2). This is equivalent to finding $x \in C[I,E]$ with

$$\varphi(x(t)) = \varphi(a(t) + \int_0^1 f(t,s,x(\psi(s)))\ d_s g(t,s)), \quad t \in I,\ \forall\ \varphi \in E^*.$$

Theorem 4. *Under the assumptions (i)-(vi), the Urysohn-Stieltjes functional integral equation (2) has at least one weak solution $x \in C[I,E]$.*

Proof. Define the operator A by

$$Ax(t) = a(t) + \int_0^1 f(t,s,x(\psi(s)))\ d_s g(t,s), \quad t \in I.$$

For every $x \in C[I,E]$, $f(.,s,x(\psi(s)))$ is continuous on I, and $f(t,.,.)$ is weakly-weakly continuous on I, then $\varphi(f(t,s,x(\psi(s))))$ is continuous for every $\varphi \in E^*$, g is of bounded variation. Hence $f(t,s,x(\psi(s)))$ is weakly Riemann-Stieltjes integrable on I with respect to $s \to g(t,s)$. Thus A makes sense.

Now, define the set Q_r by

$$Q_r = \{x \in C[I,E] :\ \parallel x \parallel \leq r, \quad r = \frac{\|a\| + M\mu}{1 - b\mu}\}.$$

The remainder of the proof will be given in four steps.

First, we will prove that the operator A maps $C[I,E]$ into $C[I,E]$.

Let $\epsilon > 0,\ \ t_1,\ t_2 \in I,\ t_2 > t_1$, and $t_2 - t_1 < \epsilon$, without loss of generality, assume that $Ax(t_2) - Ax(t_1) \neq 0$,

$$\begin{aligned} \parallel Ax(t_2) - Ax(t_1) \parallel \quad &\leq \quad | \varphi(a(t_2) - a(t_1)) | \\ &+ \quad | \int_0^1 \varphi(f(t_2,s,x(\psi(s))))\ d_s g(t_2,s) \\ &- \quad \int_0^1 \varphi(f(t_1,s,x(\psi(s))))\ d_s g(t_1,s) \, | \end{aligned}$$

$$\leq \ \parallel a(t_2) - a(t_1) \parallel + | \int_0^1 \varphi(f(t_2,s,x(\psi(s))))\ d_s g(t_2,s)$$

$$
\begin{aligned}
- & \int_0^1 \varphi(f(t_1,s,x(\psi(s))))\ d_s g(t_2,s) \mid \\
+ & \mid \int_0^1 \varphi(f(t_1,s,x(\psi(s))))\ d_s g(t_2,s) \\
- & \int_0^1 \varphi(f(t_1,s,x(s)))\ d_s g(t_1,s) \mid \\
\leq & \parallel a(t_2) - a(t_1) \parallel
\end{aligned}
$$

$$
\begin{aligned}
+ & \int_0^1 \mid \varphi(f(t_2,s,x(\psi(s))) - f(t_1,s,x(\psi(s)))) \mid \ d_s(\bigvee_{z=0}^{s} g(t_2,z)) \\
+ & \int_0^1 \mid \varphi(f(t_1,s,x(\psi(s)))) \mid \ d_s(\bigvee_{z=0}^{s} [g(t_2,z) - g(t_1,z))]) \\
\leq & \parallel a(t_2) - a(t_1) \parallel + \parallel f(t_2,s,x) - f(t_1,s,x) \parallel \int_0^1 \ d_s g(t_2,s) \\
+ & \int_0^1 \parallel f(t_1,s,x) \parallel \ d_s[g(t_2,s) - g(t_1,s)] \\
\leq & \parallel a(t_2) - a(t_1) \parallel + \parallel f(t_2,s,x) - f(t_1,s,x) \parallel [g(t_2,1) - g(t_2,0)] \\
+ & \int_0^1 m(t,s)\ d_s[g(t_2,s) - g(t_1,s))] + \int_0^1 b\|x\|\ d_s[g(t_2,s) - g(t_1,s)] \\
\leq & \parallel a(t_2) - a(t_1) \parallel + \parallel f(t_2,s,x) - f(t_1,s,x) \parallel [g(t_2,1) - g(t_2,0)] \\
+ & M \int_0^1 \ d_s[g(t_2,s) - g(t_1,s)] + br \int_0^1 \ d_s[g(t_2,s) - g(t_1,s)] \\
\leq & \parallel a(t_2) - a(t_1) \parallel + \parallel f(t_2,s,x) - f(t_1,s,x) \parallel [g(t_2,1) - g(t_2,0)] \\
+ & (M + br)[(g(t_2,1) - g(t_1,1)) - (g(t_2,0) - g(t_1,0))] \\
\leq & \parallel a(t_2) - a(t_1) \parallel + \parallel f(t_2,s,x) - f(t_1,s,x) \parallel [g(t_2,1) - g(t_2,0)] \\
+ & (M + br)[\mid g(t_2,1) - g(t_1,1) \mid + \mid g(t_2,0) - g(t_1,0) \mid].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\parallel Ax(t_2) \ - \ & Ax(t_1) \parallel \leq \parallel a(t_2) - a(t_1) \parallel \\
+ & \parallel f(t_2,s,x) - f(t_1,s,x) \parallel [g(t_2,1) - g(t_2,0)] \\
+ & (M + br)[\mid g(t_2,1) - g(t_1,1) \mid + \mid g(t_2,0) - g(t_1,0) \mid],
\end{aligned}
$$

then from the continuity of the function g assumption (iv) we deduce that A maps $C[I,E]$ into $C[I,E]$.

Secondly, we will prove that the operator A maps Q_r into Q_r.

Take $x \in Q_r$, without loss of generality assume $Ax \neq 0$, $t \in I$. By Proposition 2, we have

$$\begin{aligned}
\| Ax(t) \| &= \varphi(Ax(t)) \\
&\leq |\, \varphi(a(t)) \,| + |\, \varphi(\int_0^1 f(t,s,x(\psi(s)))\ d_s g(t,s)) \,| \\
&\leq \| a \| + \int_0^1 |\, \varphi(f(t,s,x(\psi(s)))) \,|\ \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| a \| + \int_0^1 \| f(t,s,x) \|\ \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| a \| + \int_0^1 m(t,s)\ d_s g(t,s) + \int_0^1 b\|x\|\ d_s g(t,s) \\
&\leq \| a \| + M \int_0^1 \ d_s g(t,s) + br \int_0^1 \ d_s g(t,s) \\
&\leq \| a \| + M[\ g(t,1) - g(t,0)\ ] + br[\ g(t,1) - g(t,0)\ ] \\
&\leq \| a \| + (M + br)[\ \sup_{t\in I} |\, g(t,1) \,| + \sup_{t\in I} |\, g(t,0) \,|\ \ ] \\
&\leq \| a \| + (M + br)\mu.
\end{aligned}$$

Then,

$$\| Ax(t) \| \ \leq \ \| a \| + (M + br)\mu = r.$$

Hence $Ax \in Q_r$ which prove that $A : Q_r \to Q_r$ and AQ_r is bounded in $C[I,E]$.

Thirdly, we will prove that $AQ_r(t)$ is relatively weakly compact in E.

Note that Q_r is nonempty, uniformly bounded and strongly equicontinuous subset of $C[I,E]$, by the uniform boundedness of AQ_r, according to Proposition 1, AQ_r is relatively weakly compact.

Finally, we will prove that the operator A is weakly sequentially continuous.

Let $\{x_n(t)\}$ be sequence in Q_r weakly convergent to $x(t)$ in E, since Q_r is closed we have $x \in Q_r$. Fix $t,s \in I$, since f satisfies (1)-(2), then we have $f(t,s,x_n(\psi(s)))$ converges weakly to $f(t,s,x(\psi(s)))$. Furthermore, $(\forall\ \varphi \in E^*)$ $\varphi(f(t,s,x_n(\psi(s))))$ convergence strongly to $\varphi(f(t,s,x(\psi(s))))$.

Applying Lebesgue dominated convergence theorem,

$$\varphi(\int_0^1 f(t,s,x_n(\psi(s)))\ d_s g(t,s)) = \int_0^1 \varphi(f(t,s,x_n(\psi(s))))\ d_s g(t,s)$$

$$\to \int_0^1 \varphi(f(t,s,x(\psi(s))))\ d_s g(t,s),\ \forall \varphi \in E^*,\ t \in I,$$

i.e. $\varphi(Ax_n(t)) \to \varphi(Ax(t)),\ \forall\, t \in I,\ Ax_n(t)$ converging weakly to $Ax(t)$ in E.

Thus, A is weakly sequentially continuous on Q_r.

Since all conditions of Theorem 2 are satisfied, then the operator A has at least one fixed point $x \in Q_r$ and the Urysohn-Stieltjes functional integral equation (2) has at least one weak solution. □

3. Hammerstien-Stieltjes function integral equation

Consider the following assumption:

$(iii)^*$ Let $f_1 : I \times E \to E$ and $k : I \times I \to R_+$ assume that f_1, k satisfy the following assumptions:

- $(1)^*$ $f_1(s, x(\psi(s)))$ is weakly-weakly continuous function.
- $(2)^*$ There exists a continuous function $m_1(t)$ and constant $b > 0$ such that

$$\| f_1(t,x) \| \leq m_1(t) + b\|x\|,$$

  for $t, s \in I,\ x \in E$. Moreover, we put $M_1 = \max\{m_1(t) :\ t \in I\},\ M_1 > 0$.
- $(3)^*$ $k(t,s)$ is continuous function such that
  $K = \sup_t \mid k(t,s) \mid$ and K is positive constant.

Definition 5. By a weak solution to (3) we mean a function $x \in C[I,E]$ which satisfies the integral equation (3). This is equivalent to finding $x \in C[I,E]$ with

$$\varphi(x(t)) \ = \ \varphi(a(t) + \int_0^1 k(t,s) f_1(s, x(\psi(s)))\ d_s g(t,s)),\ t \in I\ \forall\ \varphi \in E^*.$$

New for the existence of a weak solution of (3), we have the following theorem.

Theorem 6. *Let the assumptions (i),(ii),(iv)-(vi) and $(iii)^*$ be satisfied. Then the Hammerstien-Stieltjes functional integral equation (3) has at least one weak solution $x \in C[I,E]$.*

Proof. Let

$$f(t,s,x(\psi(s))) = k(t,s)f_1(s,x(\psi(s))).$$

Then from the assumption $(iii)^*$, we find that the assumptions of Theorem 2 are satisfied and result follows. □

4. The weak maximal and weak minimal solutions

Now we give the following definition.

Definition 7. Let $q(t)$ be a weak solution of (2) Then $q(t)$ is said to be a weak maximal solution of (2) if every weak solution x(t) of (2) satisfies the inequality

$$\varphi(x(t)) < \varphi(q(t)), \quad \forall \varphi \in E^*.$$

A weak minimal solution $s(t)$ can be defined by similar way by reversing the above inequality, i.e.

$$\varphi(x(t)) > \varphi(s(t)), \quad \forall \varphi \in E^*.$$

In this section, we assume that f satisfies the following assumption:

(4) for any $x, y \in E$ satisfying $\varphi(x(t)) < \varphi(y(t)), \quad \forall \varphi \in E^*$ implies that

$$\varphi(f(t,s,x(\psi(s)))) < \varphi(f(t,s,y(\psi(s)))).$$

Lemma 1. *Let $f(t,s,x)$ satisfy assumptions of Theorem 2 and let $x(t), y(t) \in C[I,E]$ on I satisfy*

$$\varphi(x(t)) \leq \varphi(a(t)) + \int_0^1 \varphi(f(t,s,x(\psi(s))))d_s g(t,s)$$

$$\varphi(y(t)) \geq \varphi(a(t)) + \int_0^1 \varphi(f(t,s,x(\psi(s))))d_s g(t,s)$$

for all $\varphi \in E^*$, where one of them is strict.

If $(f(t,s,x))$ satisfies assumption (4), then

$$\varphi(x(t)) < \varphi(y(t)).$$

Proof. Let the conclusion (4) be false, then there exists t_1 such that

$$\varphi(x(t_1)) = \varphi(y(t_1)), \quad t_1 > 0$$

and

$$\varphi(x(t)) < \varphi(y(t)), \quad 0 < t < t_1.$$

Since $(f(t,s,x))$ satisfies assumption (4), we get

$$\begin{aligned} \varphi(x(t_1)) &\leq \varphi(g(t_1)) + \int_0^1 \varphi(f(t_1, s, x(\psi(s))))d_s g(t,s) \\ &< \varphi(g(t_1)) + \int_0^1 \varphi(f(t_1, s, y(\psi(s))))\ d_s g(t,s) \\ &< \varphi(y(t_1)). \end{aligned}$$

Which contradicts the fact that $\varphi(x(t_1)) = \varphi(y(t_1))$, then

$$\varphi(x(t)) < \varphi(y(t)).$$

□

Theorem 8. *Let the assumptions of Theorem 2 be satisfied. If $f(t,x)$ satisfies assumption (4), then there exist a weak maximal and weak minimal solutions of (2).*

Proof. First, we shall prove the existence of the weak maximal solution of (2). Let $\epsilon > 0$ be given. Now consider the integral equation

$$x_\epsilon(t) = a(t) + \int_0^1 f_\epsilon(t, s, x_\epsilon(\psi(s)))\ d_s g(t,s), \tag{4}$$

where

$$f_\epsilon(t, s, x_\epsilon(\psi(s))) = f(t, s, x_\epsilon(\psi(s))) + \epsilon.$$

Clearly the function $f_\epsilon(t,s,x_\epsilon)$ satisfies the conditions (1)-(3) of Theorem 2, and

$$\|f_\epsilon(t,s,x_\epsilon)\| \leq m(t,s) + b\|x\| + \epsilon = m(t,s) + b_1\|x\|.$$

Therefore equation (4) has a weak solution $x_\epsilon \in C[I,E]$ according to Theorem 2. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then,

$$x_{\epsilon_1}(t) = a(t) + \int_0^1 f_{\epsilon_1}(t, s, x_{\epsilon_1}(\psi(s)))\ d_s g(t,s)$$

$$x_{\epsilon_1}(t) = a(t) + \int_0^1 (f(t, s, x_{\epsilon_1}(\psi(s))) + \epsilon_1)\ d_s g(t,s)$$

implies that

$$\varphi(x_{\epsilon_1}(t)) > \varphi(a(t)) + \int_0^1 \varphi(f(t,s,x_{\epsilon_1}(\psi(s)))) + \epsilon_2 \; d_s g(t,s), \tag{5}$$

$$\varphi(x_{\epsilon_2}(t)) = \varphi(a(t)) + \int_0^1 \varphi(f(t,s,x_{\epsilon_2}(\psi(s)))) + \epsilon_2 \; d_s g(t,s). \tag{6}$$

Using Lemma 1, then (5) and (6) imply

$$\varphi(x_{\epsilon_2}(t)) < \varphi(x_{\epsilon_1}(t)), \quad t \in [0,1].$$

As shown before in the proof of Theorem 2, the family of functions $x(t)$ defined by (4) is uniformly bounded and of strongly equicontinuous functions. Hence by the Arzela-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0,1]$ and denote this limit by $q(t)$. From the weakly continuity of the function f_{ϵ_n} and applying the Lebesgue Dominated Convergence Theorem, we get

$$q(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = a(t) + \int_0^1 (f(t,s,q(\psi(s)))) \; d_s g(t,s),$$

which proves that $q(t)$ as a solution of (2).

Finally, we shall show that $q(t)$ is the weak maximal solution of (2). To do this, let $x(t)$ be any weak solution of (2). Then,

$$\begin{aligned}\varphi(x_\epsilon(t)) &= \varphi(a(t)) + \int_0^1 (\varphi(f(t,s,x_\epsilon(\psi(s)))) + \epsilon) \; d_s g(t,s) \\ &> \varphi(a(t)) + \int_0^1 \varphi(f(t,s,x_\epsilon(\psi(s)))) \; d_s g(t,s),\end{aligned}$$

and

$$\varphi(x(t)) = \varphi(a(t)) + \int_0^1 (\varphi(f(t,s,x(\psi(s)))) \; d_s g(t,s),$$

and applying Lemma 1, we get

$$\varphi(x_\epsilon(t)) > \varphi(x(t)).$$

From the uniqueness of the maximal solution (see [10]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in [0,1]$ as $\epsilon \to 0$.

By similar way as above, we can prove that $s(t)$ is the weak minimal solution of (2).

The weak maximal and minimal solutions of (3) can be defined in the same fashion as done above. □

Now, the function f_1 is assumed to satisfy the following assumption:

(4*) for any $x, y \in E$ satisfying $\varphi(x(t)) < \varphi(y(t)),\ \ \forall\varphi \in E^*$ implies that

$$\varphi(f_1(s, x(\psi(s)))) < \varphi(f_1(s, y(\psi(s)))).$$

Now the following lemma can be proved.

Lemma 2. *Let $f_1(t, x), k(t, s)$ satisfy the assumptions of Theorem 3 and let $x(t), y(t) \in C[I, E]$ on I satisfy*

$$\varphi(x(t)) \leq \varphi(a(t)) + \int_0^1 k(t, s)\varphi(f_1(s, x(\psi(s))))d_s g(t, s),$$

$$\varphi(y(t)) \geq \varphi(a(t)) + \int_0^1 k(t, s)\varphi(f_1(s, x(\psi(s))))d_s g(t, s),$$

where one of these is strict. If $f_1(t, x)$ satisfies assumption (4). Then,*

$$\varphi(x(t)) < \varphi(y(t)).$$

Theorem 9. *Let the assumptions of Theorem 3 be satisfied. If $f_1(t, x)$ satisfies assumption (4*), then there exist a weak maximal and weak minimal solutions of (3).*

Proof. First, we shall prove the existence of the weak maximal solution of (3). Let $\epsilon > 0$ be given. Now consider the integral equation

$$x_\epsilon(t) = a(t) + \int_0^1 k(t, s) f_{1\epsilon}(s, x_\epsilon(\psi(s)))\ d_s g(t, s), \tag{7}$$

where

$$f_{1\epsilon}(s, x_\epsilon(s)) = f_1(s, x_\epsilon(\psi(s))) + \epsilon.$$

Clearly the function $f_{1\epsilon}(s, x_\epsilon)$ satisfies the conditions $(1)^*, (2)^*$ of Theorem 3, and

$$\|f_{1\epsilon}(s, x_\epsilon)\| \leq m_1(t) + b\|x\| + \epsilon = m_1(t) + b_1\|x\|.$$

Therefore equation (7) has a weak solution $x_\epsilon \in C[I, E]$ according to Theorem 3. Let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$. Then,

$$x_{\epsilon_1}(t) = a(t) + \int_0^1 k(t, s) f_{1\epsilon_1}(s, x_{\epsilon_1}(\psi(s)))\ d_s g(t, s),$$

$$x_{\epsilon_1}(t) = a(t) + \int_0^1 k(t, s)(f_1(s, x_{\epsilon_1}(\psi(s))) + \epsilon_1)\ d_s g(t, s),$$

implies that

$$\varphi(x_{\epsilon_1}(t)) > \varphi(a(t)) + \int_0^1 k(t,s)\varphi(f_1(s, x_{\epsilon_1}(\psi(s)))) + \epsilon_2 \; d_s g(t,s), \qquad (8)$$

$$\varphi(x_{\epsilon_2}(t)) = \varphi(a(t)) + \int_0^1 k(t,s)\varphi(f_1(s, x_{\epsilon_2}(\psi(s)))) + \epsilon_2 \; d_s g(t,s). \qquad (9)$$

Using Lemma 2, then (8) and (9) implies

$$\varphi(x_{\epsilon_2}(t)) < \varphi(x_{\epsilon_1}(t)), \quad t \in [0,1].$$

As shown before in the proof of Theorem 3, the family of functions $x_\epsilon(t)$ defined by (7) is uniformly bounded and of strongly equicontinuous functions. Hence by the Arzela-Ascoli Theorem, there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0,1]$ and denote this limit by $q(t)$. From the weakly continuity of the function $f_{1\epsilon_n}$ and applying Lebesgue Dominated Convergence Theorem, we get

$$q(t) = \lim_{n\to\infty} x_{\epsilon_n}(t) = a(t) + \int_0^1 k(t,s) f_1(s, q(\psi(s))) \; d_s g(t,s),$$

which proves that $q(t)$ as a solution of (3).

Finally, we shall show that $q(t)$ is the weak maximal solution of (3). To do this, let $x(t)$ be any weak solution of (3). Then

$$\begin{aligned} \varphi(x_\epsilon(t)) \quad &= \quad \varphi(a(t)) + \int_0^1 k(t,s)(\varphi(f_1(s, x_\epsilon(\psi(s)))) + \epsilon) \; d_s g(t,s) \\ &> \quad \varphi(a(t)) + \int_0^1 k(t,s)\varphi(f_1(s, x_\epsilon(\psi(s)))) \; d_s g(t,s), \end{aligned}$$

and

$$\varphi(x(t)) = \varphi(a(t)) + \int_0^1 k(t,s)\varphi(f_1(s, x(\psi(s)))) \; d_s g(t,s),$$

applying Lemma 2, we get

$$\varphi(x_\epsilon(t)) > \varphi(x(t))$$

from the uniqueness of the maximal solution (see [10]), it is clear that $x_\epsilon(t)$ tends to $q(t)$ uniformly in $t \in [0,1]$ as $\epsilon \to 0$.

By similar way as done above we can prove that $s(t)$ is the weak minimal solution of (3). □

In what follows, we provide some examples illustrating the above obtained results.

Example 1. Consider the function $g : I \times I \to R$ defined by the formula

$$g(t,s) = t^3 + ts, \quad t \in I,$$

It can be easily seen that the function $g(t,s)$ satisfy assumptions (iv)-(vi) given in Theorem 2. In this case, the Urysohn-Stieltjes functional integral equation (2) has the form

$$x(t) = a(t) + \int_0^1 tf(t,s,x(\psi(s)))\, ds, \quad t \in I. \tag{10}$$

Therefore, the equation (10) has at least one weak solution $x \in C[I,E]$, if the functions a, ψ and f satisfy the assumptions (i)-(iii).

Example 2. Consider the function $g : I \times I \to R$ defined by the formula

$$g(t,s) = \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in (0,1], \quad s \in I, \\ 0, & \text{for } t = 0, \quad s \in I. \end{cases}$$

Also, the function $g(t,s)$ satisfy assumptions (iv)-(vi) given in Theorem 2. In this case, the Urysohn-Stieltjes functional integral equation (2) has the form

$$x(t) = a(t) + \int_0^1 \frac{t}{t+s} f(t,s,x(\psi(s)))\, ds, \quad t \in I. \tag{11}$$

Therefore, the equation (11) has at least one weak solution $x \in C[I,E]$, if the functions a, ψ and f satisfy the assumptions (i)-(iii).

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