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EXACT SOLUTIONS OF BOUNDARY-VALUE PROBLEMS

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Abstract: A survey of an approach for obtaining explicit formulae for solving local and nonlocal boundary value problems (BVPs) for some linear partial differential equations is presented. To this end an extension of the Heaviside-Mikusiński operational calculus is used. A multi-dimensional operational calculus is constructed for each of the considered problems. The main steps of construction of exact (closed) solutions using such operational calculi are outlined. It is based on a combination of the Fourier method and an extension of the Duhamel principle to the space variables.

Program tools for numerical computation and visuali - zation of the solutions using the computer algebra system *M*athematica (http://www.wolfram.com/mathematica) are developed.

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1. Introduction

The classical operational calculus is intended mainly to solving initial value problems both for ordinary differential equations (ODEs) and for partial differential equations (PDEs) with constant coefficients [15].

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Here we develop an extension of the Heaviside-Mikusiński operational calculus [18] for BVPs with local and nonlocal boundary conditions, in order to find their solutions in an explicit form.

The essence of our approach is in common use of a combination of the Fourier method and an extension of the Duhamel principle to the space variable in the frames of a nonclassical two-dimensional operational calculus of Mikusiński type.

The main features of this approach are outlined in [3]–[8], [12] as well as in some other papers of the authors.

2. The Classical Duhamel Principle and its Analogon

Local and nonlocal BVPs for the classical equations of mathematical physics in rectangular domains often are solved by Fourier method or some of its extensions, intended for nonlocal cases.

The Duhamel principle had arisen almost simultaneously with the Fourier method. In 1830, J.-M.-C. Duhamel published a Memoire [14], and he had shown there that the solution of the BVP

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(0,t) = 0,\ u(1,t) = \varphi(t),\ u(x,0) = 0$$

could be obtained for arbitrary $\varphi(t)$, provided that we have the solution $U(x,t)$ of the same problem, but for the special choice $\varphi(t) \equiv 1$. It is given by the formula

$$u(x,t) = \frac{\partial}{\partial t}\int_0^t U(x,t-\tau)\varphi(\tau)d\tau \tag{1}$$

in the strip $0 \le x \le 1,\ 0 \le t$, [18].

Using the Fourier method, we can easily find

$$U(x,t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x. \tag{2}$$

It is desirable to extend the Duhamel principle to BVPs with non homogenous initial conditions.

However, for the BVP

$u_t = u_{xx},\ u(0,t) = u(\pi,t) = 0,\ u(x,0) = f(x)$

in the strip $0 \le x \le \pi,\ 0 \le t$,

there is also something like the Duhamel representation (2):

$$u(x,t) = \int_0^{\pi}\left[\theta(x-y,t) - \theta(x+y,t)\right]f(y)dy, \tag{3}$$

where
$$\theta(x,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-n^2 t}\cos nx \tag{4}$$
is the well-known θ-function (see [20], p. 94). The function $\theta(x,t)$ is a solution of the heat equation, but it is not a solution of the same BVP for special choice of $f(x)$ since the series for $\theta(x,t)$ diverges for $t = 0$.

Nevertheless, representation (3) could be used for evaluation of $u(x,t)$ at inner points of the domain $0 \le x \le \pi,\ 0 \le t$.

Representation (3) could be found in many books and papers and sometimes is connected with the name of M. Gevray [20]. No other Duhamel-type representations of this sort for solutions of linear BVPs for PDEs are known to the authors.

3. Convolutions for Boundary Value Problems

The Duhamel principle can be extended for the space variables of a large class of boundary value problems for linear partial differential equations in finite space domains, in which the Fourier method can be applied. To this end we use an approach, suggested by one of the authors [3], for extension of the operational calculus of Heaviside-Mikusiński for functions of two variables. This approach can be applied both to local and to non-local boundary value problems. Here we consider BVPs for three classical equations of mathematical physics in finite domains:

- the heat equation $u_t = u_{xx} + f(x,t)$
- the wave equation $u_{tt} = u_{xx} + f(x,t)$
- the equation of vibrations of a supported beam

  $u_{tt} = -u_{xxxx} + f(x,t)$.

For solving such problems we need some extensions of the direct approach of Mikusiński, using new convolutions.

To this end, we consider two types of convolutions, intended to operational calculi for functions of one variable. We will combine them in a convolution for functions of two variables, in order to build operational calculi for functions of two variables.

3.1. Convolutions for the Differentiation Operator

The basic BVP for the differentiation operator d/dt in the space $C[0,\infty)$ of the continuous functions $f(t)$, $0 \le t < \infty$ is determined by an arbitrary linear functional χ on $C\,[0,\infty)$. It looks as follows:

$$y' = f(t),\ \ \chi(y) = 0. \tag{5}$$

In order the solution y to exist it is necessary to assume $\chi\{1\} \neq 0$. For the simplicity sake, we take $\chi\{1\} = 1$. Then the solution $y = lf(t)$ could be named a generalized integration operator. It has the form

$$l\,f(t) = \int_0^t f(\tau)\,d\tau - \chi_\tau\,\{\int_0^t f(\tau)\,d\tau\}. \tag{6}$$

It is shown in [3] that the operation

$$(f * g)(t) = \chi_\tau\{\int_\tau^t f(t-\sigma+\tau)\,g(\sigma)\,d\sigma\}, \tag{7}$$

is a bilinear, commutative and associative operation such that

$$lf = \{1\} * f.$$

However, this convolution is used in [10] for solving nonlocal Cauchy boundary-value problems in the non-resonance case. The resonance case is considered in [11].

3.2. Convolutions for the Square of the Differentiation Operator d^2/dx^2

Let us consider the space $C\,[0,a]$ of the continuous functions on $[0,a]$. The simplest nonlocal BVP for d^2/dx^2 in $C\,[0,a]$ is given by

$$y'' = f(x),\ \ y(0) = 0,\ \ \Phi\{y\} = 0,$$

where Φ is a linear functional on $C^1\,[0,a]$. In order it to have a solution, it is necessary to assume $\Phi\{x\} \neq 0$. For simplicity sake, we assume that $\Phi\{x\} = 1$.

Its solution $y = Lf(x)$ has the explicit form

$$Lf(x) = \int_0^x (x-\xi)\ f(\xi)\,d\xi - x\,\Phi_\xi\{\int_0^\xi (\xi-\eta)\ f(\eta)\,d\eta\}. \tag{8}$$

In [3] it is proven that the operation

$$(f * g)(x) = -\frac{1}{2}\,\Phi_\xi\left\{\int_0^\xi h(x,\eta)\,d\eta\right\}, \tag{9}$$

with

$$h(x,\eta) =$$

$$\int_x^\eta f(\eta+x-\zeta)\,g(\zeta)\,d\zeta - \int_{-x}^\eta f(|\eta-x-\zeta|)\,g(|\zeta|)\,\mathrm{sgn}(\eta-x-\zeta)\,\zeta\,d\zeta$$

is a bilinear, commutative and associative operation such that

$$Lf(x) = \{x\} * f.$$

Theorem 1. *The operation*

$$\{u(x,t)\} * \{v(x,t)\} = -\frac{1}{2}\tilde{\Phi}_\xi \chi_\tau \{h(x,t;\xi,\tau)\} \tag{10}$$

with

$$h(x,t;\xi,\tau) = \int_\xi^x \int_\tau^t u(x+\xi-\eta, t+\tau-\sigma)\, v(\eta,\sigma)\, d\sigma d\eta$$
$$- \int_{-\xi}^x \int_\tau^t u(|x-\xi-\eta|, t+\tau-\sigma)\, v(|\eta|,\sigma)\, \mathrm{sgn}\,[(x-\xi-\eta)\eta]\; d\sigma\, d\eta$$

and with the functional $\tilde{\Phi}_\xi = \Phi \circ \int_0^\xi$ is a convolution of the operators L and l in $C(\Delta)$ (where $\Delta = (0,a] \times [0,\infty)$), for which $L\,l\,u = \{x\} * u$. The operators $lu = \{1\} \overset{(t)}{*} u(x,t)$ and $Lu = \{x\} \overset{(x)}{*} u(x,t)$ are multipliers of the corresponding convolution algebra [17].

This theorem gives us an operation $(u * v)(x,t)$ in $C(\Delta)$, which is a convolution of each of both operators l and L.

3.3. Construction of an Operational Calculus for the Operators L and l in $C((0,a] \times [0,\infty))$

Consider the ring $\mathfrak{M}$ of the multipliers of the convolution algebra $[C(\Delta), *]$, where $\Delta = [0,a] \times [0,\infty)$.

Denote by $\mathcal{M}$ the ring of the fractions $\dfrac{M}{N}$, where $M \in \mathfrak{M}$, $N \in \mathfrak{M}$, N being non-divisor of 0 in $\mathfrak{M}$. Such fractions are called multipliers fractions.

In $\mathcal{M}$ there can be embedded both the ring $(\mathcal{C}, *)$ and the numerical field ($\mathbb{R}$ or $\mathbb{C}$) and also, the convolution algebras $(\mathcal{C}[0,a], \overset{(x)}{*})$ and $(\mathcal{C}[0,\infty), \overset{(t)}{*})$.

Of course, $\mathfrak{M}$ also is a part of $\mathcal{M}$, since $M = \dfrac{M}{I}$, where I is the identity operator. Hereafter, we will denote I simply by 1.

Let $f = \{f(x)\}$ be a function of the variable x only and $\varphi = \{\varphi(t)\}$ is a function of the variable t only, but considered as elements of C.

The operators

$$[f]_t : u \mapsto f \overset{(x)}{*} u$$

and

$$[\varphi]_x : u \mapsto \varphi \overset{(t)}{*} v$$

are said to be numerical operators with respect to t and x respectively. In these notations we have $L = [x]_t$ and $l = [1]_x$. They belongs to $\mathcal{M}$. We denote $s = \dfrac{1}{l}$ and $S = \dfrac{1}{L}$.

3.4. Basic Formulae of the Operational Calculus for *l* and *L*

These are

$$\frac{\partial u}{\partial t} = su - [\chi_\tau\{u(x,\tau)\}]_t \tag{11}$$

and

$$\frac{\partial^2 u}{\partial x^2} = Su - [\Phi_\xi\{u(\xi,t)\}]_x\,,$$

where the indices t and x mean that the corresponding functions of t and x are considered as "partial" numerical operators.

These formulae express the relation between the partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ and the products su and Su $\left(s = \frac{1}{l},\ S = \frac{1}{L}\right)$.

4. Duhamel-Type Representations of Solutions of BVP

In order to illustrate the application of the operational calculus, briefly described above, let us consider the following class of BVP:

$$\begin{aligned} &u_t = u_{xx} + F(x,t),\ 0 < x < a,\ t > 0 \\ &u(0,t) = 0,\ \Phi_\xi\{u(\xi,t)\} = 0 \\ &\chi_\tau\{u(x,\tau)\} = f(x), \end{aligned}$$

where Φ and χ are linear functionals respectively in $C^1[0,a]$ and $C[0,\infty]$.

Using the main formulae (11), we reduce the problem to the single equation:

$$(s - S)u = [f(x)]_t + \{F(x,t)\}\,.$$

Assuming that $s - S$ is not a divisor of 0 (this assumption is equivalent to the requirement for uniqueness of the solution), we can write the following form of the solution in $\mathcal{M}$:

$$u = \frac{1}{s-S}\,[f(x)]_t + \frac{1}{s-S}\{F(x,t)\}. \tag{12}$$

Consider the partial solution $\Omega(x,t)$ of the equation for $F(x,t) \equiv 0$ and $f(x) \equiv x$. This solution is an algebraic object and it has the form:

$$\Omega = \frac{1}{S(s-S)}\,, \tag{13}$$

since $[f(x)]_t = [x]_t = \frac{1}{S}$.

Theorem 2. *If* $\Omega(x,t)$ *is a function in* $C(\Delta)$*, the problem*

$$u_t = u_{xx},\ \ u(0,t) = 0,\ \ \Phi_\xi\{u(\xi,t)\} = 0,\ \ \chi_\tau\{u(x,\tau)\} = f(x)$$

with $f(0) = 0$, $\Phi\{f\} = 0$ *and* $f \in C^2[0,a]$ *has a classical solution* $u(x,t)$ *of the form*

$$u(x,t) = \frac{\partial^2}{\partial x^2}\left\{\Omega(x,t) \overset{(x)}{*} f(x)\right\}. \tag{14}$$

The proof is given in [2].

Having in mind this theorem, the formulae (11), (14) and also the forms of Ω, Φ_ξ and χ_τ, we can obtain a representation of the solution of a given BVP for the heat equation.

In a similar way we can obtain formulae for the solutions of BVP for the wave equation and for the equation of a supported beam.

Let us note that a direct approach to the construction of operational calculi connected with linear nonlocal boundary value problems for a large class of linear evolution equations with several space variables and one time variable is proposed in [13].

5. BVPs for Equations of Mathematical Physics

We consider local and non-local BVPs for the heat equation, for the wave equation and for the equation of a supported beam. In all problems the partial solutions are denoted by Ω and they are obtained in a form of series, separately for every problem.

5.1. Heat Equation

A. Local BVP

Consider the BVP:

$$u_t = u_{xx},\ \ 0 \le x \le 1,\ t \ge 0,\ u(0,t) = 0,\ u(1,t) = 0,\ \ u(x,0) = f(x).$$

Using Theorem 2 and also (10) and (14), for $\Phi_\xi\{u(\xi,t)\} = u(1,t)$ we obtain the following form of the solution:

$$u(x,t) = \int_0^1 [\Omega(1-x-\xi,t) - \Omega(1+x-\xi,t)]\, f(\xi) d\xi, \tag{15}$$

where

$$\Omega(x,t) = \sum_{n=1}^{\infty} (-1)^n \exp(-n^2\pi^2 t) \cos n\pi x$$

is a solution of the problem for $f(x) = x$.

B. Non-local BVP

The so-called "Samarskii–Ionkin problem" (see [16]) has the form:

$$u_t = u_{xx},\ u(0,t) = 0,\ \int_0^1 u(x,\tau)d\tau = 0,\ u(x,0) = f(x).$$

This is a BVP with $\Phi_\xi\{u(\xi,t)\} = \int_0^1 f(\xi)d\xi$.

After simplification of (14), the following representation is obtained (see [3]):

$$u(x,t) = -2\int_0^x \Omega(x-\xi,t)f(\xi)d\xi - \int_x^1 \Omega(1+x-\xi,t)f(\xi)d\xi$$
$$+\int_{-x}^1 \Omega(1-x-\xi,t)f(|\xi|)\mathrm{sgn}\xi d\xi, \tag{16}$$

where
$$\Omega(x,t) = \sum_{n=1}^{\infty}\{-2x\cos 2n\pi x + 8\pi nt\sin 2n\pi x\}\, e^{-4n^2\pi^2 t}.$$

5.2. String Equation

A. Local BVP

Consider the BVP:

$$\begin{aligned} &u_{tt} = u_{xx} + F(x,t), \quad 0 < x < a,\ 0 < t < \infty,\\ &u(0,t) = 0,\ u(a,t) = 0\\ &u(x,0) = f(x),\ u_t(x,0) = g(x). \end{aligned}$$

The following representation is obtained for $f(x) \equiv 0$:

$$u(x,t) =$$

$$-\frac{1}{2}\int_x^1 \Omega(1+x-\xi)g'(\xi)d\xi + \frac{1}{2}\int_{-x}^1 \Omega(1-x-\xi)g'(|\xi|)d\xi, \tag{17}$$

where
$$\Omega(x,t) = \frac{2}{\pi^2}\sum_{n=1}^{\infty}((-1)^{n-1}/n^2)\sin n\pi x\sin n\pi t$$

is a solution of the problem, but for the special choice $g(x) \equiv x$.

B. Non-local BVPs

Consider the following BVP for the string equation:

$$u_{tt} = u_{xx}, \quad 0 < x < 1,\, 0 < t < \infty$$
$$u(0,t) = 0,\ \int_0^1 u(\xi,t)d\xi = 0$$
$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

S. Beilin (see [1]) considers problems of this type and states conditions for the existence and uniqueness of the solution. We derived a formula, convenient for numerical computation of the solution.

Case 1. $f(x) \equiv 0$, $g(x) \neq 0$.

We use the solution $\Omega(x,t)$ for $g(x) \equiv x^3/6 - x/12$ and $f(x) \equiv 0$; we have:

$$\Omega(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{x \cos(2\,n\,\pi\,x)\,\sin(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} \right.$$
$$\left. + \left(\frac{t\,\cos(2\,n\,\pi\,t)}{4\,n^3\,\pi^3} - \frac{3\,\sin(2\,n\,\pi\,t)}{8\,n^4\,\pi^4} \right) \sin(2\,n\,\pi\,x) \right\}.$$

The solution $u(x,t)$ has the form (see [19])

$$u(x,t) = \frac{\partial^2}{\partial x^2} \left\{ \Omega(x,t) \overset{(x)}{*} g(x) \right\}$$

and after its simplification the following representation is derived:

$$u(x,t) = -2 \int_0^x \Omega_x(x-\xi,t) g'(\xi) d\xi \tag{18}$$
$$- \int_x^1 \Omega_x(1+x-\xi,t) g'(\xi) d\xi + \int_{-x}^1 \Omega_x(1-x-\xi,t) g'(|\xi|) d\xi.$$

Case 2. $f(x) \neq 0$ and $g(x) \equiv 0$.

The representation of the solution now has the form (see [19])

$$u(x,t) = \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} \left(\Omega(x,t) \overset{(x)}{*} f(x) \right).$$

For the purposes of simplification of this representation we introduce

$$\tilde{\Omega}(x,t) = \int_0^t \Omega_x(x,\tau)\, d\tau,$$

where $\Omega(x,t)$ is a solution of the problem under consideration for the special choice $f(x) \equiv x^3/6 - x/12$ and $g(x) \equiv 0$,

$$\tilde{\Omega}(x,t) \;=\; \sum_{n=1}^{\infty} \left\{ - \frac{x\,\cos(2\,n\,\pi\,x)\,\sin(n\,\pi\,t)^2}{n^2\,\pi^2} \right.$$
$$\left. - \frac{t\,\cos(n\,\pi\,t)\,\sin(n\,\pi\,t)\,\sin(2\,n\,\pi\,x)}{n^2\,\pi^2} + \frac{\sin(n\,\pi\,t)^2\,\sin(2\,n\,\pi\,x)}{n^3\,\pi^3} \right\}.$$

The following representation of $u(x,t)$ is derived:

$$u(x,t) = \tag{19}$$

$$-2\int_0^x \tilde{\Omega}_x(x-\xi,t)\, f''(\xi)\, d\xi - \int_x^1 \tilde{\Omega}_x(1+x-\xi,t)\, f''(\xi)\, d\xi$$

$$+\int_{-x}^1 \operatorname{sgn} x\, \tilde{\Omega}_x(1-x-\xi,t)\, f''(\xi)\, d\xi \;-\; 2\,\tilde{\Omega}(1,t)\, f''(x) + f(x).$$

5.3. Equation of a Free Supported Beam

A. Local BVPs

Consider the following problem for the equation of a free supported beam (see [15]):

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4}, \quad 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0,\ u_{xx}(0,t) = 0,\ u(1,t) = 0,\ u_{xx}(1,t) = 0$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

For the case $f(x) \equiv 0$ we obtain:

$$u(x,t) = -\frac{1}{2}\int_x^1 \Omega_x(1+x-\xi,t)\, g(\xi) d\xi \tag{20}$$

$$+\frac{1}{2}\int_{-x}^1 \Omega_x(1-x-\xi,t) g(|\xi|) \operatorname{sgn} \xi\, d\xi,$$

where $\Omega_x(x,t) = \dfrac{2}{\pi^2}\displaystyle\sum_{n=1}^{\infty}((-1)^{n-1}/n^2) \sin(n\pi)^2 t \cos n\pi x.$

B. Non-local BVPs

Consider the problem

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4}, \quad 0 < x < 1,\ 0 < t < \infty,$$

$$u(0,t) = 0,\ u_{xx}(0,t) = 0$$

$$\int_0^1 u(\xi,t) d\xi = 0,\ u_x(1,t) - u_x(0,t) = 0$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

Case 1. $f(x) \equiv 0,\ g(x) \neq 0.$

The solution $u(x,t)$ in this case has the form (see [19])

$$u(x,t) = \frac{\partial^2}{\partial x^2}\left\{\Omega(x,t) \overset{(x)}{*} f(x)\right\}.$$

After some simplifications an explicit representation is obtained:

$$u(x,t) = -2\int_0^x \Omega_x(x-\xi,t)g'(\xi)d\xi \tag{21}$$

$$-\int_x^1 \Omega_x(1+x-\xi,t)g'(\xi)d\xi + \int_{-x}^1 \Omega_x(1-x-\xi,t)g'(|\xi|)d\xi,$$

where

$$\Omega_x(x,t) = \sum_{n=1}^{\infty}\left\{\frac{\cos(2\,n\,\pi\,x)\,\left(8\,n^2\,\pi^2\,t\,\cos(4\,n^2\,\pi^2\,t) - 3\,\sin(4\,n^2\,\pi^2\,t)\right)}{8\,n^4\,\pi^4} - \frac{x\,\sin(4\,n^2\,\pi^2\,t)\,\sin(2\,n\,\pi\,x)}{4\,n^3\,\pi^3})\right\}.$$

Case 2. $f(x) \neq 0$, $g(x) \equiv 0$.

The following representation is obtained after simplification applied in a way, similar to those in *Case 2* for the string equation (for more details see [19]):

$$u(x,t) = \tag{22}$$

$$-2\int_0^x \tilde{\Omega}_{xx}(x-\xi,t)f^{iv}(\xi)d\xi + \int_x^1 \tilde{\Omega}_{xx}(1+x-\xi,t)f^{iv}(\xi)d\xi$$
$$-\int_{-x}^1 \tilde{\Omega}_{xx}(1-x-\xi,t)f^{iv}(|\xi|)\mathrm{sign}(\xi)d\xi + 2f^{iv}(\xi)(\tilde{\Omega}_x(0,t)$$
$$-\tilde{\Omega}_x(1,t)) + f(x),$$

where

$$\tilde{\Omega}_{xx}(x,t) = \left\{-\frac{x\,\cos(2\,n\,\pi\,x)\sin(2\,n^2\,\pi^2\,t)^2}{4\,n^4\,\pi^4}\right.$$
$$-\frac{\left(4\,n^2\,\pi^2\,t\,\cos(2\,n^2\,\pi^2\,t) - 3\,\sin(2\,n^2\,\pi^2\,t)\right)\,\sin(2\,n^2\,\pi^2\,t)\,\sin(2\,\pi\,x)}{4\,n^7\,\pi^5}$$
$$\left.-\frac{\sin(2\,n^2\,\pi^2\,t)^2\,\sin(2\,n\,\pi\,x)}{4\,n^5\,\pi^5}\right\}.$$

```
{{0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.},
 {0., 0.122599, 0.157754, 0.0669366, -0.115165, -0.293893, -0.360363, -0.248572,
  0.0238813, 0.352929, 0.587785}, {0., 0.157754, 0.189536, 0.0425892,
  -0.226956, -0.475528, -0.542465, -0.336482, 0.104357, 0.611667, 0.951057},
 {0., 0.0669366, 0.0425892, -0.104357, -0.317774, -0.475528,
  -0.451647, -0.189536, 0.251303, 0.702484, 0.951057},
 {0., -0.115165, -0.226956, -0.317774, -0.352929, -0.293893,
  -0.122599, 0.136138, 0.408592, 0.590693, 0.587785},
 {0., -0.293893, -0.475528, -0.475528, -0.293893, 3.33067×10^-16,
  0.293893, 0.475528, 0.475528, 0.293893, 2.77556×10^-15},
 {0., -0.360363, -0.542465, -0.451647, -0.122599, 0.293893, 0.598127, 0.633283,
  0.360829, -0.115165, -0.587785}, {0., -0.248572, -0.336482, -0.189536,
  0.136138, 0.475528, 0.633283, 0.483428, 0.0425892, -0.520849, -0.951057},
 {0., 0.0238813, 0.104357, 0.251303, 0.408592, 0.475528, 0.360829, 0.0425892,
  -0.39825, -0.793302, -0.951057}, {0., 0.352929, 0.611667, 0.702484,
  0.590693, 0.293893, -0.115165, -0.520849, -0.793302, -0.828458, -0.587785},
 {0., 0.587785, 0.951057, 0.951057, 0.587785, 2.22045×10^-16,
  -0.587785, -0.951057, -0.951057, -0.587785, -5.28466×10^-14}}
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Figure 1: Table of numerical values of the solution

6. Program Packages for the Considered BVPs

The representations (15)–(22) of the solutions of BVPs considered above, are convenient for numerical computation of an arbitrary number of values of the solutions. A visualization of each solution can be made as well.

For practical application of these capabilities 3 program packages for the computer algebra system *Mathematica* were developed (for the 3 types of the considered equations). In each of these packages functions for solving local and nonlocal BVPs are implemented.

After loading the chosen package, a call to the respective function has to be performed, for example:

DSolveOCStringNonl1$[g, u, x, t, \{x, 0, 1, 0.1\}, \{t, 0, 1, 0.1\}, 5]$

for solving a nonlocal BVP for the string equation, Case 1.

As a result, a table of numerical values of the solution in the given intervals is returned, followed by visualization of the boundary function and the solution (see **Figures 1, 2 and 3**, respectively).

In the included illustrative example a nonlocal BVP for the string equation, Case 1, is solved for $g(x) = 2\pi x \cos 2\pi x + \dfrac{3}{2} \sin 2\pi x$ and for degree of truncation of the series representing the partial solution $\Omega(x, t)$, equal to 5 (the last parameter of the call stands for that).

For the cases when the exact solution is known, a comparison of its values

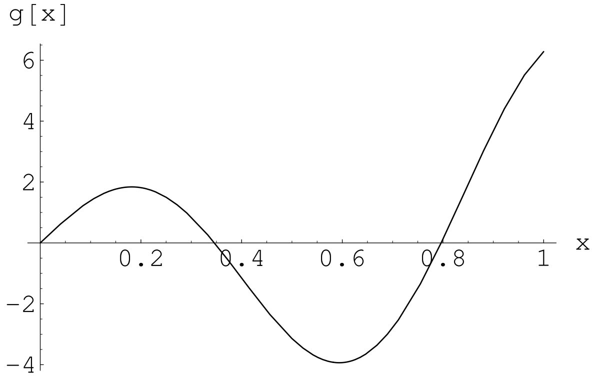

Figure 2: Function $g(x)$

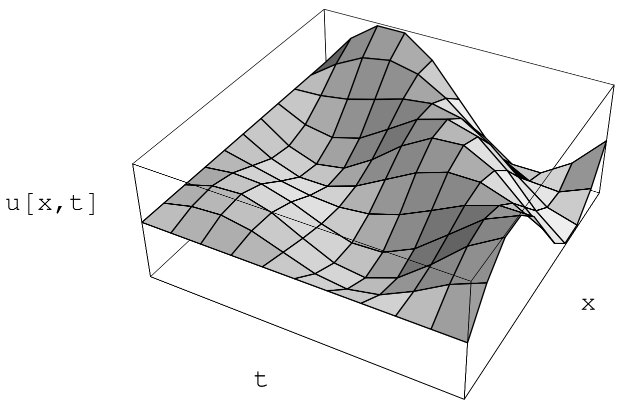

Figure 3: Solution $u(x, t)$

in a set of points with the values of "our" solution in the same points could be easily made. A number of experiments were made and errors of computation of order $10^{-9} - 10^{-13}$ were found.

As far as we know, *Mathematica* system is not able to solve nonlocal BVPs.

7. Resonance Vibrations of String and Beam under Integral Boundary Conditions

As an application of our approach to finding exact solutions of BVPs we study a real problem.

The Tacoma Narrows bridge collapse on November 7, 1940 still had not obtained unanimous explanation. The physical phenomenon resonance is often pointed out as a possible explanation of the bridge failure. Many authors of studies of this disaster reject such an explanation.

If we consider the bridge as linear elastic system subjected to local boundary value conditions, it is not possible the resonance phenomenon to occur. Nevertheless, considering such a system as subjected to nonlocal boundary value condition of energetic type (integral boundary-value condition), there always occur resonances on all frequencies.

Let us consider again the linear nonlocal boundary value problems for the equations of a vibrating string and for a free supported beam and their Duhamel–type representations of the solutions, derived by the presented approach:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$0 < x < 1, 0 < t < \infty$$

$$u(0,t) = 0, \int_0^1 u(\xi,t)d\xi = 0$$

$$u(x,0) = f(x),$$

$$u_t(x,0) = g(x),$$

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial^4 u}{\partial x^4}$$

$$0 < x < 1, 0 < t < \infty$$

$$\int_0^1 u(\xi,t)d\xi = 0,$$

$$u_x(1,t) - u_x(0,t) = 0$$

$$u(0,t) = 0, u_{xx}(0,t) = 0$$

$$u(x,0) = f(x),\ u_t(x,0) = g(x).$$

For simplicity sake the cases when $f(x) \equiv 0$, $g(x) \neq 0$ for the string equation and $f(x) \neq 0$, $g(x) \equiv 0$ for the beam equation are considered. The representations of their solutions are as follows:

$$u(x,t) = -2\int_0^x \Omega_x(x-\xi,t)g'(\xi)d\xi - \int_x^1 \Omega_x(1+x-\xi,t)g'(\xi)d\xi + \int_{-x}^1 \Omega_x(1-x-\xi,t)g'(|\xi|)d\xi,$$

for the string equation, and

$$u(x,t) = -2\int_0^x \tilde{\Omega}_{xx}(x-\xi,t)f^{iv}(\xi)d\xi + \int_x^1 \tilde{\Omega}_{xx}(1+x-\xi,t)f^{iv}(\xi)d\xi$$

$$-\int_{-x}^{1} \tilde{\Omega}_{xx}(1-x-\xi,t) f^{iv}(|\xi|)\operatorname{sign}(\xi) d\xi + 2 f^{iv}(\xi)(\tilde{\Omega}_x(0,t)$$
$$-\tilde{\Omega}_x(1,t)) + f(x),$$

for the beam equation (formulae (18) and (22), respectively).

Analyzing all components of these representations, we can conclude that both formulae have the following general form:

$$u(x,t) = u_1(x,t) + t\, u_2(x,t),$$

where $u_1(x,t)$ and $u_2(x,t)$ are periodic with respect to t and hence bounded for fixed x. The resonance effect is due to the aperiodic term $t\, u_2(x,t)$. When its absolute value exceeds some fixed quantity, a demolition occurs.

Similar considerations are presented in more details in [9]. It is shown there that under a quite simple integral boundary condition, resonance inevitably occurs even in the case of absence of external forces. Conservation of the integral of displacement in time has a clear physical meaning: conservation of the bridge potential energy in the course of vibration.

By this phenomenon, we explain resonance vibration of the Tacoma bridge. This explanation is also applicable to the resonance vibration observed on the Volgograd bridge in May 2010, so that it was even closed for traffic for some time [9].

However, we do not discuss the prevention of such resonance vibration. The design issues need special investigation.

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