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ON SATURATION ORDER OF FUNCTIONS OF SOME VARIABLES BY SINGULAR INTEGRALS

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Abstract: In the paper we consider approximation of functions $f(x) \in L^p(R_n)$, by α-singular integrals, determine approximation order and saturation class.

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1. Introduction

Let R_n be n dimensional Euclidean space and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $t = (t_1, t_2, ..., t_n)$, $K_\lambda(t) = \prod_{l=1}^{n} K_l, \lambda_l(t_l)$, where $K_{l,\lambda_l}(t_l)$ ($t_l \in R_1$, $\lambda_l > 0$, $1 \leq l \leq n$) are one-dimensional kernels satisfying the following conditions:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} K_{l,\lambda_l}(t_l) dt_l = 1, \quad \|K_{l,\lambda_l}(t_l)\|_{L(R_1)} \leq M_l < \infty,$$

$$\lim_{\lambda_l \to \infty} \int_{|t_l| \leq \delta} |K_{l,\lambda_l}(t_l)| \, dt_l = 0 \tag{1}$$

for every $1 \leq l \leq n$.

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Consider for every $1 \le l \le n$, the α-singular integral of the general form

$$Q^{(\alpha)}_{l,\,\lambda_l}(f,\,x) = \frac{1}{(\sqrt{2\pi})^n} \int\limits_{R_n} \left\{ \sum_{s_1,\,\ldots,\,s_n=1}^{\infty} \left[\prod_{l=1}^{n} (-1)^{s_l-1} \begin{pmatrix} \alpha \\ s_l \end{pmatrix} \right] \right. \\ \left. \times\, f(x_1 - s_1 t_1,\,\ldots,\,x_n - s_n t_n) \right\} \left[\prod_{l=1}^{n} K_{l,\,\lambda_l}(t_l) \right] dt_1 \ldots dt_n, \tag{2}$$

where $\alpha > 0$ is any real number, and one-dimensional kernels satisfy conditions (1).

Note that if $f(x) \in L^p(R_n)$ $(1 \le p < \infty)$ and the kernel $K_{l,\,\lambda_l}(t_l)$ satisfies conditions (1), then singular integral (2) exists almost everywhere on R_n and the following relations are valid:

a) $\left\| Q^{(\alpha)}_{l,\,\lambda_l}(f,\,x) \right\|_{L^p(R_n)} \le C_2 \left\| f(x) \right\|_{L^p(R_n)} \cdot \prod\limits_{l=1}^{n} \left\| K_{l,\,\lambda_l}(t_l) \right\|_{L(R_1)}$;

b) $\lim\limits_{\substack{\lambda_1 \to \infty \\ \cdots \\ \lambda_n \to \infty}} \left\| Q^{(\alpha)}_{l,\,\lambda_l}(f,\,x) - f(x) \right\|_{L^p(R_n)} = 0;$

c) $\lim\limits_{\lambda_\mu \to \infty} \left\| Q^{(\alpha)}_{l,\,\lambda_l}(f,\,x) - f(x) \right\|_{L^p(R_n)} = \left\| Q^{(\alpha)}_{l,\,\lambda_l}(f,\,x) - f(x) \right\|_{L^p(R_n)}$
for every $1 \le \mu \le n$ and $\lim\limits_{\lambda_\mu \to \infty}$ means that $\lambda_\mu \to \infty$ for every $1 \le \mu \le n$ $(\mu \ne l)$;

d)
$$\left\| f(x) - Q^{(\alpha)}_{l,\,\lambda_l}(f,x) \right\|_{L^p(R_n)} \le \sum_{\mu=1}^{n-1} \left\{ \sum_{S_1,\,\ldots,\,S_n=1}^{\infty} \left[\prod_{l=\mu+1}^{n} \begin{pmatrix} \alpha \\ s_l \end{pmatrix} \right] \right. \\ \left. \times \left[\prod_{l=\mu+1}^{n} K_{l,\,\lambda_l}(t_l) \right]_{L(R_1)} \right\} + \left\| f(x) - Q^{(\alpha)}_{n,\,\lambda_n}(f,\,x) \right\|_{L^p(R_n)}.$$

In the sequel, we will assume

$$G^{(\alpha)}_{l,\,\lambda_l}(u_l) = \sum_{S_l=1}^{\infty} (-1)^{s_l-1} \begin{pmatrix} \alpha \\ s_l \end{pmatrix} K^{\wedge}_{l,\,\lambda_l}(u_l s_l)$$

at $\lambda_l > 0$ for every $1 \le l \le n$, where $K^{\wedge}_{l,\,\lambda_l}(u_l)$ is Fourier transformation of the functions $K_{l,\,\lambda_l}(t_l)$.

Denote by F the set of all infinitely differentiable functions with a compact support. Introduce the class of functions

$$M^l_F(\psi) \equiv \left\{ \psi(x) \in F,\ \ \eta_l(u_l)\psi^{\wedge}(u) \right\} = r^{\wedge}_{\psi}(u)$$

for some $r_\psi(x) \in F$, $\ \eta_l(u_l) \ne 0$, $\ 1 \le l \le n$.

Theorem 1. *Let $f(x) \in L^p(R_n)$ $(1 \le p < \infty)$ and one-dimensional kernels $K_{l,\lambda_l}(t_l)$ $(t_l \in R_1,\ \lambda_l > 0,\ l = \overline{1, n})$ of singular integrals (2) be such that the function*

$$\beta_{\lambda_l}^{(\alpha)}(u_l) = \frac{1 - G_{l,\lambda_l}^{(\alpha)}(u_l)}{\tau_l(\lambda_l)\eta_l(u_l)} \quad (\tau_l(\lambda_l) > 0,\ \lim_{\lambda_l \to 0} \tau_l(\lambda_l) = 0)$$

be Fourier-Stieltjes transformation of some function

$$\mu_\lambda^{(\alpha)}(t) \in NBV(-\infty; +\infty)$$

(i.e. $\frac{1}{\sqrt{2}} \int\limits_{-\infty}^{\infty} d\mu_{\lambda_l}^{(\alpha)}(t) = 1$ *and* $\int\limits_{-\infty}^{\delta} + \int\limits_{\delta}^{+\infty} \left| d\mu_{\lambda_l}^{(\alpha)}(t) \right| \to 0$ *as* $\lambda_l \to \infty,\ \ 1 \le l \le n$*).*

Then:

I. if

$$\left\| f(x) - Q_{l,\lambda_l}^{(\alpha)}(f, x) \right\|_{L^p(R_n)} = O\ \left(\sum_{l=1}^{n} \tau_l(\lambda_l) \right) \tag{3}$$

as $\lambda \to \infty$, *then* $f(x) = 0$ *almost everywhere on* R_n*;*

II. The following relations are equivalent:

A)

$$\left\| f(x) - Q_{l,\lambda_l}^{(\alpha)}(f, x) \right\|_{L^p(R_n)} = O\ \left(\sum_{l=1}^{n} \tau_l(\lambda_l) \right) \tag{4}$$

as $\lambda \to \infty$ *(this means that* $\lambda_l \to \infty$ *for every* $1 \le l \le n$ *separately);*

B) there exist a bounded measure υ *on* R_n *and the function* $l(x) \in L^p(R_n)$ *such that for every* $\psi(x) \in M_F^l(\psi)$ *the following relation is valid*

$$\int\limits_{R_n} r_\psi(x) f(x) dx = \begin{cases} \int\limits_{R_n} \psi(x) d\upsilon(x) & \text{for} \quad p = 1, \\ \int\limits_{R_n} \psi(x) l(x) & \text{for} \quad 1 < p < \infty. \end{cases} \tag{5}$$

Proof. Let us consider the case $1 < p < \infty$.

I. According to c) we have

$$\left\| f(x) - Q_{l,\lambda_l}^{(\alpha)}(f, x) \right\|_{L^p(R_n)} = O\ \ (\tau_l(\lambda_l))\ \ (1 \le l \le n), \tag{6}$$

as $\lambda_l \to \infty$. Then for any function $\psi(x) \in F$ we have

$$\lim_{\lambda_l \to \infty} \int\limits_{R_n} \frac{f(x) - Q_{l,\lambda_l}^{(\alpha)}(f, x)}{\tau_l(\lambda_l)} \psi(x) dx = O\ \ \ (1 \le l \le n). \tag{7}$$

As the singular integral (2) is a convolution type integral, we find

$$\int\limits_{R_n} \frac{f(x) - Q^{(\alpha)}_{l,\lambda_l}(f,\, x)}{\tau_l(\lambda_l)}\psi(x)dx = \int\limits_{R_n} \frac{\varphi(x) - Q^{(\alpha)}_{l,\lambda_l}(\psi,\, x)}{\tau_l(\lambda_l)} f(x)dx \tag{8}$$

for every $\psi(x) \in F$.

Furthermore, from $\psi(x) \in M^l_F(\psi)$ and theorem on convolution of Fourier transformation, we have

$$\left[\frac{f(x) - Q^{(\alpha)}_{l,\lambda_l}(f,\, x)}{\tau_l(\lambda_l)}\right]^{\wedge}(u) = \frac{1 - Q^{(\alpha)}_{l,\lambda_l}(u_l)}{\tau_l(\lambda_l)}\psi^{\wedge}(u)$$

$$= \left[\mu^{(\alpha)}_{\lambda_l}(t_l)\right]^{\vee}(u_l) r^{\wedge}_{\psi}(u) = \left[\frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} r_{\psi}(x - t_l) d\mu^{(\alpha)}_{\lambda_l}(t_l)\right]^{\wedge}(u).$$

Hence by the uniqueness of the Fourier transformation, we find:

$$\frac{\psi(x) - Q^{(\alpha)}_{l,\lambda_l}(f,\, x)}{\tau_l(\lambda_l)} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} r_{\psi}(x - t_l) d\mu^{(\alpha)}_{\lambda_l}(t_l).$$

From the last equality it follows that

$$\left\| \frac{\psi(x) - Q^{(\alpha)}_{l,\lambda_l}(f,\, x)}{\tau_l(\lambda_l)} - r_{\psi}(x) \right\|_{L^p(R_n)}$$

$$\leq \frac{1}{\sqrt{2\pi}} \left\| r_{\psi}(x - t_l) - r_{\psi}(x) \right\|_{L^p(R_n)} d\mu^{(\alpha)}_{\lambda_l}(t_l) \to 0$$

as $\lambda_l \to \infty \quad (1 \leq l \leq n)$.

Hence we have

$$\lim_{\lambda_l \to \infty} \int\limits_{R_n} \frac{\psi(x) - Q^{(\alpha)}_{l,\lambda_l}(\psi,\, x)}{\tau_l(\lambda_l)} f(x)d(x) = \int\limits_{R_n} r_{\psi}(x) f(x) dx \tag{9}$$

for $f(x) \in L^p(R_n)$.

Therefore by (7) and (8),

$$\int\limits_{R_n} r_{\psi}(x) f(x) dx = 0$$

for $r_\psi(x) \in F$. Hence we conclude, that $f(x) = 0$ almost everywhere on R_n.

II. A)$\Rightarrow$B). Taking into account c), from (4) we get

$$\left\| f(x) - Q^{(\alpha)}_{l,\lambda_l}(f, x) \right\|_{L^p(R_n)} = O \quad (\tau_l(\lambda_l)) \quad (\lambda_l \to \infty).$$

Then by the theorem on weak compactness (see [4], p.16) there exist the function $l(x) \in L^p(R_n)$ and sequence of numbers l_i $\left(\lim\limits_{l_i \to \infty} \lambda_{l_i} = \infty\right)$ such that

$$\lim_{l_i \to \infty} \int\limits_{R_n} \frac{f(x) - Q^{(\alpha)}_{l,\lambda_l}(f, x)}{\tau_l(\lambda_l)} \psi(x) d(x) = \int\limits_{R_n} \psi(x) l(x) dx \tag{10}$$

for any function $\psi(x) \in F$.

As the singular integral (2) is a convolution type integral, then taking account (9), we find

$$\lim_{\lambda_l \to \infty} \int\limits_{R_n} \frac{f(x) - Q^{(\alpha)}_{l,\lambda_l}(f, x)}{\tau_l(\lambda_l)} \psi(x) d(x)$$

$$= \lim_{\lambda_l \to \infty} \int\limits_{R_n} \frac{\psi(x) - Q^{(\alpha)}_{l,\lambda_l}(\psi, x)}{\tau_l(\lambda_l)} f(x) dx = \int\limits_{R_n} r_\psi(x) f(x) dx. \tag{11}$$

Comparing (10) and (11), we have

$$\int\limits_{R_n} r_\psi(x) f(x) dx = \int\limits_{R_n} \psi(x) l(x) dx,$$

i.e. B) is valid.

Now prove B)$\Rightarrow$A). As

$$Q^{(\alpha)}_{l,\lambda_l}(f) = \int\limits_{R_n} \frac{\psi(x) - Q^{(\alpha)}_{l,\lambda_l}(\psi, x)}{\tau_l(\lambda_l)} f(x) dx$$

$$= \int\limits_{R_n} \frac{f(x) - Q^{(\alpha)}_{l,\lambda_l}(f, x)}{\tau_l(\lambda_l)} \psi(x) dx = Q^{(\alpha)}_{l,\lambda_l}(\psi),$$

then as in the proof of relation b)$\Rightarrow$c) of the theorem in [7], we have

$$\frac{f(x) - Q^{(\alpha)}_{l,\lambda_l}(f, x)}{\tau_l(\lambda_l)} = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} l(x + t_l) d\mu^{(\alpha)}_{\lambda_l}(t_l) \quad (1 \le l \le n),$$

or

$$\left\| \frac{f(x) - Q^{(\alpha)}_{l,\,\lambda_l}(f,\,x)}{\tau_l(\lambda_l)} - r_\psi(x) \right\|_{L^p(R_n)}$$

$$\le \frac{1}{\sqrt{2\pi}}\, \|l(x)\|_{L^p(R_n)} \int\limits_{-\infty}^{\infty} \left| d\mu^{(\alpha)}_{\lambda_l}(t_l) \right| \le M.$$

That is, independently of λ_l $(1 \le l \le n)$,

$$\left\| f(x) - Q^{(\alpha)}_{l,\,\lambda_l}(f,\,x) \right\|_{L^p(R_n)} = O\quad (\tau_l(\lambda_l)) \quad (\lambda_l \to \infty).$$

Taking into account d) from the last equality we find

$$\left\| f(x) - Q^{(\alpha)}_{\lambda}(f,\,x) \right\|_{L^p(R_n)} = O\quad \left(\sum_{l=1}^{n} \tau_l(\lambda_l) \right)$$

as $\lambda \to \infty$, i.e. A) is valid.

The theorem has been proved for $1 < p < \infty$. For $p = 1$, it is proved in the same way.

Apply this theorem to Fejer's specific linear operator,

$$\sigma_\lambda(f,\,x) = \frac{1}{\prod\limits_{l=1}^{n} (2\pi\lambda_l)} \int\limits_{R_n} f(x-t) \prod_{l=1}^{n} \left(\frac{\sin \frac{1}{2}\lambda_l t_l}{\frac{1}{2} t_l} \right)^2 dt \tag{12}$$

in the case $1 \le p \le 2$. In this case $\alpha = 1$ and

$$K_\lambda = \frac{1}{\prod\limits_{l=1}^{n} (2\pi\lambda_l)} \prod_{l=1}^{n} \left(\frac{\sin \frac{1}{2}\lambda_l t_l}{\frac{1}{2} t_l} \right)^2 = \prod_{l=1}^{n} K_{l,\,\lambda_l}(t_l).$$

Since

$$\left[K_{l,\,\lambda_l}(t_l)\right]^\wedge (u_l) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left(1 - \frac{|u_l|}{\lambda_l}\right) & \text{for} \quad |u_l| < \lambda_l\,, \\ 0 & \text{for} \quad |u_l| \ge \lambda_l, \end{cases}$$

then

$$G^{(\alpha)}_{l,\,\lambda_l}(u_l) = \begin{cases} 1 - \frac{|u_l|}{\lambda_l} & \text{for} \quad |u_l| < \lambda_l\,, \\ 0 & \text{for} \quad |u_l| \ge \lambda_l. \end{cases}$$

Therefore, the functions $\tau_l(\lambda_l) = \frac{1}{\lambda_l}$ and $\eta_l(u_l) = |u_l|$ satisfy the relations

$$\frac{1-G^{(\alpha)}_{l,\lambda_l}(u_l)}{\tau_l(\lambda_l)\,|u_l|}=\begin{cases}1 & \text{for} \quad |u_l|<\lambda_l\,,\\ \frac{\lambda_l}{u_l} & \text{for} \quad |u_l|\geq\lambda_l.\end{cases} \tag{13}$$

It is known [6] that

$$\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left(\frac{\sin t_l}{t_l}-C_it_l\right)e^{-t_l\frac{u_l}{\lambda_l}}dt_l=\begin{cases}1 & \text{for} \quad |u_l|<\lambda_l\,,\\ \frac{\lambda_l}{u_l} & \text{for} \quad |u_l|\geq\lambda_l,\end{cases} \tag{14}$$

where $C_it_l=-\int\limits_{t_l}^{\infty}\frac{\cos u}{u}du.$

Introduce the function

$$l_{\lambda_l}(t_l)=\lambda_l\sqrt{\frac{2}{\pi}}\left(\frac{\sin t_l\lambda_l}{t_l}-C_i(t_l\lambda_l)\right).$$

Since

$$\int\limits_{-\infty}^{\infty}|l_{\lambda_l}(t_l)|\,dt_l=\lambda_l\sqrt{\frac{2}{\pi}}\int\limits_{-\infty}^{\infty}\left|\frac{\sin t_l\lambda_l}{t_l}-C_i(t_l\lambda_l)\right|dt_l$$

$$=\sqrt{\frac{2}{\pi}}\int\limits_{-\infty}^{\infty}\left|\frac{\sin t_l}{t_l}-C_it_l\right|dt_l\leq M_1<\infty,$$

then $l_{\lambda_l}(t_l)\in L(R_n)$.

On the other hand, if by

$$\mu_{\lambda_l}(t_l)=\int\limits_{-\infty}^{t_l}l_{\lambda_l}(u_l)du_l\ \ (1\leq l\leq u)$$

we denote the uniformly bounded measure on R_1, then by

$$\int\limits_{-\infty}^{\infty}\frac{\sin t_l}{t_l}dt_l=\pi \ \ \text{and} \ \ \int\limits_{-\infty}^{\infty}C_it_ldt_l=0,$$

we get $\mu_{\lambda_l}(-\infty)=0,\ \ \mu_{\lambda_l}(+\infty)=\sqrt{2\pi}$ for all values of λ_l and

$$[Var\mu_{\lambda_l}(t_l)]^{\infty}_{-\infty}=\int\limits_{-\infty}^{\infty}|l_{\lambda_l}(u_l)|\,du_l\leq M_2<\infty.$$

The comparison of (13) and (14) shows that the function

$$\frac{1 - C^{(\alpha)}_{l,\,\lambda_l}(u_l)}{\tau_l(\lambda_l)\eta_l(u_l)}$$

is Fourier-Stieltes transformation of normalized function with bounded variation.

Consequently, the conditions of the theorem are satisfied for Fejer's singular integral.

Therefore we have the following

Corollary 2. *Let $f(x) \in L^p(R_n)$ $(1 \le p \le 2)$. Then for the relation*

$$\|\sigma_\lambda(f,\,x) - f(x)\|_{L^p} = O\left(\sum_{l=1}^{n} \frac{1}{\lambda_l}\right)$$

to hold as $\lambda_l \to \infty$, it is necessary and sufficient that almost everywhere $f(x) = 0$ on R_n.

Corollary 3. *Let $f(x) \in L^p(R_n)$ $(1 \le p \le 2)$. For the relation*

$$\|\sigma_\lambda(f,\,x) - f(x)\|_{L^p(R_n)} = O\left(\sum_{l=1}^{n} \frac{1}{\lambda_l}\right)$$

to hold as $\lambda_l \to \infty$, it is necessary and sufficient that $f(x) \in n_p(f)$, where

$$n_p(f) = \begin{cases} f(x) \in L(R_n)/f(x) \in B\bigcup(R_n), & \text{for } p = 1, \\ f(x) \in L^p(R_n)/f(x) \in AC_{loc}(R_1), & \text{with respect to } x_l, \\ u\frac{\partial f(x)}{\partial x_l} \in L^p(R_n), \ \ (1 \le p \le 2, \ 1 \le l \le n). & \end{cases}$$

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