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$(2n-1)$-POINT NONLINEAR TERNARY INTERPOLATING SUBDIVISION SCHEMES

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Abstract: Nonlinear interpolating subdivision schemes have been introduced in recent years to reduce Gibbs phenomenon near irregular initial data points. In this article, we presented a class of $(2n-1)$-point nonlinear ternary interpolating subdivision schemes. It is shown that several of the existing nonlinear ternary interpolating subdivision schemes become special cases for our proposed class of schemes. Convergence for one special case of 5-point nonlinear interpolating subdivision schemes is proved.

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1. Introduction

Subdivision schemes are well known for their applications in animation, computer graphics and geometric modeling. Smooth curves and surfaces are generated from the initial control points by updating the existing data points and inserting additional data points in-between. In ternary interpolating subdivision schemes, initial data points are kept intact and two additional points are added around each existing point at every iteration. In an effort to keep the existing initial data points, the limit curve generated by the interpolating subdivision schemes produces oscillations also known as Gibbs phenomenon near

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big jumps in the initial data points. On the other hand, approximating subdivision schemes produce smooth curves but the limiting curve does not necessarily pass through the initial data points. In several applications Gibbs phenomenon is highly undesirable but at the same time it is required that the limiting curve contain all initial data points.

To avoid Gibbs phenomenon while keeping all initial data points in the limiting curve, nonlinear subdivision Schemes like ENO, WENO, PPH and PCHIP ([1], [2], [3], [4], [5], [6], [8], [9], [10]) were introduced during last several years. Recently we have introduced two families of nonlinear ternary interpolating subdivision schemes to reduce undesirable oscillations. In [4], we introduced a family of 3-point nonlinear ternary interpolating subdivision schemes and proved that it is C^1 continuous. In [5], we developed a family of 5-point nonlinear ternary interpolating subdivision schemes and showed that this family is C^2 continuous. In this article, we introduce a general form of $(2n-1)$-point nonlinear ternary interpolating subdivision schemes and show that several of existing subdivision schemes including ([4], [5]) become special cases of our proposed family of schemes.

We have the following arrangement for this article. In Section 2, preliminary concepts and their properties are discussed. In Section 3, a general formula for the $(2n-1)$-point nonlinear ternary interpolating subdivision schemes is presented and some special cases are discussed. Convergence of one special case, 5-point nonlinear ternary interpolating subdivision schemes is proved in Section 4. In Section 5, numerical results are presented.

2. Preliminaries

In this section, we revisit some basic results.

2.1. Averages

For $(x_1, x_2) \in R^2$, we define a nonlinear function called Modified Geometric Mean or MGM as

$$MGM(x_1, x_2) = \begin{cases} sign(x_1)\sqrt{x_1 x_2} & \text{if} \quad x_1 x_2 > 0 \\ 0 & \text{if} \quad x_1 x_2 \leq 0, \end{cases} \tag{2.1}$$

where $sign(x_1) = 1$ if $x_1 \geq 0$ and $sign(x_1) = -1$ if $x_1 < 0$. The nonlinear function MGM defined above has several interesting properties like:

$$MGM(x_1, x_2) = MGM(x_2, x_1), \tag{2.2}$$

$$MGM(-x_1, -x_2) = -MGM(x_1, x_2), \tag{2.3}$$

$$|MGM(x_1, x_2)| \leq |AM(x_1, x_2)| \leq \max(|x_1|, |x_2|), \tag{2.4}$$

$$|MGM(x_1, x_2) - AM(x_1, x_2)| \leq |AM(x_1, x_2)|, \tag{2.5}$$

where $AM(x_1, x_2) = \dfrac{x_1 + x_2}{2}$.

The proofs of properties (2.1) - (2.5) are trivial.

We recall the PPH function $(x, y) \in R^2$ defined by ([1], [2]) as

$$PPH(x, y) = \begin{cases} (1 + sign(xy))\frac{xy}{x+y} & \text{for} \quad xy > 0, \\ 0 & \text{if} \quad xy \leq 0. \end{cases} \tag{2.6}$$

The PPH function defined above also satisfies the properties in (2.1) - (2.5).

2.2. Nonlinear Interpolating Subdivision Schemes

A general form of $(2n - 1)$-points nonlinear univariate ternary interpolating subdivision scheme S_{NL} which maps set of data points $f^k = \{f_i^k\}_{i \in Z}$ into the next refinement level of data points $f^{k+1} = \{f_i^{k+1}\}_{i \in Z}$ is defined as

$$S_{NL}(f^k) = f^{k+1} = S(f^k) + F(d^{(2n-3)} f^k). \tag{2.7}$$

Here S is a linear interpolating subdivision scheme defined as,

$$\begin{aligned} S(f^k)_{3i-1} &= \sum_{j=-(n-1)}^{n-1} a_j f_{i+j}^k, \\ S(f^k)_{3i} &= f_i^k, \\ S(f^k)_{3i+1} &= \sum_{j=-(n-1)}^{n-1} a_{-j} f_{i+j}^k, \end{aligned} \tag{2.8}$$

with a necessary condition of uniform convergence

$$\sum_{j=-(n-1)}^{n-1} a_j = \sum_{j=-(n-1)}^{n-1} a_{-j} = 1, \tag{2.9}$$

and $F(d^{(2n-3)} f^k)$ is given by

$$\begin{aligned} F(d^{(2n-3)} f^k)_{3i-1} &= M_{2n-1}\{AV(d^{(2n-3)} f_{i-(n-1)}^k, d^{(2n-3)} f_{i-(n-2)}^k) \\ &\quad - AM(d^{(2n-3)} f_{i-(n-1)}^k, d^{(2n-3)} f_{i-(n-2)}^k)\}, \\ F(d^{(2n-3)} f^k)_{3i} &= 0, \\ F(d^{(2n-3)} f^k)_{3i+1} &= -M_{2n-1}\{AV(d^{(2n-3)} f_{i-(n-1)}^k, d^{(2n-3)} f_{i-(n-2)}^k) \\ &\quad - AM(d^{(2n-3)} f_{i-(n-1)}^k, d^{(2n-3)} f_{i-(n-2)}^k)\}. \end{aligned} \tag{2.10}$$

In equation (2.10), the function $AV(x,y)$ is a nonlinear average function. It can be replaced either by $MGM(x,y)$ or $PPH(x,y)$. M_{2n-1} is a parameter defined in the next section and $d^{(2n-3)}f_i$ is representing $(2n-3)^{th}$ difference for $n>1$, for example $d^{(1)}f_i = f_{i+1} - f_i$.

2.3. Lagrange Polynomials

For given n, we define the Lagrange fundamental polynomials of degree $2n-2$, at the points $-(n-1), -(n-2), \ldots, (n-1)$, as

$$L_j^{2n-2}(x) = \prod_{k=-(n-1), k\neq j}^{n-1} \frac{x-k}{j-k}, \quad j = -(n-1), -(n-2), \ldots, (n-1), \tag{2.11}$$

and Lagrange fundamental polynomials of degree $2n-3$ at the points $-(n-2), -(n-3), \ldots, (n-1)$, by

$$L_j^{2n-3}(x) = \prod_{k=-(n-2), k\neq j}^{n-1} \frac{x-k}{j-k}, \quad j = -(n-2), -(n-3), \ldots, (n-1). \tag{2.12}$$

Here we are stating three well known lemmas. Their proofs are straight forward and can be found in [7].

Lemma 2.1. *If $L_j^{2n-2}(-\frac{1}{3})$ is a Lagrange fundamental polynomial of degree $2n-2$ corresponding to the nodes $\{t\}_{-(n-1)}^{n-1}$ as defined by (2.11), then*

$$L_j^{2n-2}(-\tfrac{1}{3}) = \frac{(-1)^{n+j-1} \prod_{k=-n+2}^{n} (3k-2)}{3^{2n-2}(1+3j)(n+j-1)!(n-j-1)!}, \tag{2.13}$$
$$j = -(n-1), -(n-2), \ldots, (n-1).$$

Lemma 2.2. *If $L_j^{2n-3}(-\frac{1}{3})$ is a Lagrange fundamental polynomial of degree $2n-3$ corresponding to the nodes $\{t\}_{-(n-2)}^{n-1}$ as defined by (2.12), then*

$$L_j^{2n-3}(-\tfrac{1}{3}) = \frac{(-1)^{n+j-2} \prod_{k=-n+3}^{n} (3k-2)}{3^{2n-3}(1+3j)(n+j-2)!(n-j-1)!}, \tag{2.14}$$
$$j = -(n-2),$$
$$-(n-3), \ldots, (n-1).$$

Lemma 2.3. *If $L_j^{2n-3}(-\frac{1}{3})$ and $L_j^{2n-3}(-\frac{1}{3})$ are Lagrange fundamental polynomials defined by (2.11) and (2.12), then*

$$\psi_j = \frac{L_j^{2n-2}(-\frac{1}{3}) - L_j^{2n-3}(-\frac{1}{3})}{L_{-(n-1)}^{2n-2}(-\frac{1}{3})} = \frac{(-1)^{n+j-1}(2n-2)!}{(n+j-1)!(n-j-1)!}, \tag{2.15}$$
$$j = -(n-2), -(n-3), \dots, (n-1).$$

3. (2n-1)-Point Nonlinear Ternary Interpolating Subdivision Schemes

In this section we give the general criteria to find the mask of the linear ternary subdivision scheme S (2.8) associated with the nonlinear scheme S_{NL} (2.7) and the parameter values M_{2n-1} given in (2.10).

Theorem 3.1. *Let $\beta_j = L_j^{2n-3}(-\frac{1}{3})$ and ψ_j be defined in Lemma 2.3, then the explicit formula for the mask of associated linear scheme S of S_{NL} and the parameter M_{2n-1} are defined by:*

$$\begin{aligned} &a_{-(n-1)} = u, \\ &a_j = u\psi_j + \beta_j, \;\; j = -(n-2), -(n-3), \dots, (n-1), \\ &M_{2n-1} = 2a_{(n-1)}, \end{aligned} \tag{3.1}$$

where u is a free parameter.

Remark 3.1. In the above theorem, M_{2n-1} can be replaced with twice of the any value between the weighted average of $a_1, a_2, \dots, a_{(n-1)}$ and $a_{(n-1)}$.

3.1. 3-Point Nonlinear Ternary Interpolating Subdivision Schemes

We present a special case of 3-point nonlinear ternary interpolating subdivision schemes.

Let $n = 2$ and $u = w$, then by (2.7), (2.8), (2.10) and (3.1), we get

$$\begin{aligned} f_{3i-1}^{k+1} &= wf_{i-1}^k + (\tfrac{4}{3} - 2w)f_i^k + (w - \tfrac{1}{3})f_{i+1}^k \\ &\quad +2(w - \tfrac{1}{3})\{AV(d^{(1)}f_i^k, d^{(1)}f_{i-1}^k) - AM(d^{(1)}f_i^k, d^{(1)}f_{i-1}^k)\}, \\ f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= (w - \tfrac{1}{3})f_{i-1}^k + (\tfrac{4}{3} - 2w)f_i^k + wf_{i+1}^k \\ &\quad -2(w - \tfrac{1}{3})\{AV(d^{(1)}f_i^k, d^{(1)}f_{i-1}^k) - AM(d^{(1)}f_i^k, d^{(1)}f_{i-1}^k)\}. \end{aligned} \tag{3.2}$$

This simplifies to

$$
\begin{aligned}
f_{3i-1}^{k+1} &= (2w-\tfrac{1}{3})f_{i-1}^{k} + (\tfrac{4}{3}-2w)f_{i}^{k} + 2(w-\tfrac{1}{3})AV(d^{(1)}f_{i}^{k}, d^{(1)}f_{i-1}^{k}),\\
f_{3i}^{k+1} &= f_{i}^{k},\\
f_{3i+1}^{k+1} &= (\tfrac{4}{3}-2w)f_{i}^{k} + (2w-\tfrac{1}{3})f_{i+1}^{k} - 2(w-\tfrac{1}{3})AV(d^{(1)}f_{i}^{k}, d^{(1)}f_{i-1}^{k}).
\end{aligned} \tag{3.3}
$$

By replacing the average function $AV(x,y)$ in (3.3) by $MGM(x,y)$, we get the following family of 3-point nonlinear interpolating subdivision schemes given in [4],

$$
\begin{aligned}
f_{3i-1}^{k+1} &= (2w-\tfrac{1}{3})f_{i-1}^{k} + (\tfrac{4}{3}-2w)f_{i}^{k} + 2(w-\tfrac{1}{3})MGM(d^{(1)}f_{i}^{k}, d^{(1)}f_{i-1}^{k}),\\
f_{3i}^{k+1} &= f_{i}^{k},\\
f_{3i+1}^{k+1} &= (\tfrac{4}{3}-2w)f_{i}^{k} + (2w-\tfrac{1}{3})f_{i+1}^{k} - 2(w-\tfrac{1}{3})MGM(d^{(1)}f_{i}^{k}, d^{(1)}f_{i-1}^{k}).
\end{aligned} \tag{3.4}
$$

Here $d^{(1)}f_{i}^{k} = f_{i+1}^{k} - f_{i}^{k}$. This family of nonlinear subdivision schemes is C^1 for $\frac{1}{4} < w < \frac{1}{3}$, as proved by Aslam in [4].

Similarly replacing the average function $AV(x,y)$ in (3.3) by $PPH(x,y)$, we get another family of 3-point nonlinear ternary interpolating subdivision schemes with C^1 smoothness as given in [4].

3.2. 5-Point Nonlinear Ternary Interpolating Subdivision Schemes

We are presenting two special cases of 5-point nonlinear ternary interpolating subdivision schemes below.

1. If $n=3$, $u = w - \frac{4}{81}$,and $\delta f_i = d^{(3)}f_i = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i$, then by (2.7), (2.8), (2.10) and (3.1), we get

$$
\begin{aligned}
f_{3i-1}^{k+1} &= (w-\tfrac{4}{81})f_{i-2}^{k} + (-4w+\tfrac{10}{27})f_{i-1}^{k} + (6w+\tfrac{20}{27})f_{i}^{k}\\
&\quad +(-4w-\tfrac{5}{81})f_{i+1}^{k} + wf_{i+2}^{k} + 2w\{AV(\delta f_{i-2}^{k}, \delta f_{i-1}^{k})\\
&\quad -AM(\delta f_{i-2}^{k}, \delta f_{i-1}^{k})\},\\
f_{3i}^{k+1} &= f_{i}^{k},\\
f_{3i+1}^{k+1} &= wf_{i-2}^{k} + (-4w-\tfrac{5}{81})f_{i-1}^{k} + (6w+\tfrac{20}{27})f_{i}^{k}\\
&= (-4w+\tfrac{10}{27})f_{i+1}^{k} + (w-\tfrac{4}{81})f_{i+2}^{k} - 2w\{AV(\delta f_{i-2}^{k}, \delta f_{i-1}^{k})\\
&\quad -AM(\delta f_{i-2}^{k}, \delta f_{i-1}^{k})\}.
\end{aligned} \tag{3.5}
$$

This simplifies to

$$
\begin{aligned}
f_{3i-1}^{k+1} &= (2w-\tfrac{4}{81})f_{i-2}^{k} + (\tfrac{10}{27}-6w)f_{i-1}^{k} + (6w+\tfrac{20}{27})f_{i}^{k}\\
&\quad +(-2w-\tfrac{5}{81})f_{i+1}^{k} + 2wAV(\delta f_{i-2}^{k}, \delta f_{i-1}^{k}),\\
f_{3i}^{k+1} &= f_{i}^{k},\\
f_{3i+1}^{k+1} &= (-2w-\tfrac{5}{81})f_{i-1}^{k} + (6w+\tfrac{20}{27})f_{i}^{k} + (\tfrac{10}{27}-6w)f_{i+1}^{k}\\
&\quad +(2w-\tfrac{4}{81})f_{i+2}^{k} - 2wAV(\delta f_{i-2}^{k}, \delta f_{i-1}^{k}).
\end{aligned} \tag{3.6}
$$

We will show the convergence of this family of nonlinear ternary interpolating schemes in the next section.

2. As per our Remark 3.1, if we replace $M_5 = 2w$ with $M_5 = \frac{1}{27} - 3w$ in (3.5), then we get

$$\begin{aligned}
f_{3i-1}^{k+1} &= (w - \tfrac{4}{81})f_{i-2}^k + (-4w + \tfrac{10}{27})f_{i-1}^k + (6w + \tfrac{20}{27})f_i^k \\
&= (-4w - \tfrac{5}{81})f_{i+1}^k + wf_{i+2}^k + (\tfrac{1}{27} - 3w)\{AV(\delta f_{i-2}^k, \delta f_{i-1}^k) \\
&\quad -AM(\delta f_{i-2}^k, \delta f_{i-1}^k)\}, \\
f_{3i}^{k+1} &= f_i^k, \\
f_{3i+1}^{k+1} &= wf_{i-2}^k + (-4w - \tfrac{5}{81})f_{i-1}^k + (6w + \tfrac{20}{27})f_i^k \\
&\quad +(-4w + \tfrac{10}{27})f_{i+1}^k + (w - \tfrac{4}{81})f_{i+2}^k \\
&\quad -(\tfrac{1}{27} - 3w)\{AV(\delta f_{i-2}^k, \delta f_{i-1}^k) - AM(\delta f_{i-2}^k, \delta f_{i-1}^k)\}.
\end{aligned} \tag{3.7}$$

The above equation can be simplified to the the following subdivision scheme of [5] which is C^2 for $\frac{1}{324} < w < \frac{1}{162}$.

$$\begin{aligned}
&f_{3i-1}^{k+1} = (-\tfrac{w}{2} - \tfrac{5}{162})f_{i-2}^k + (\tfrac{1}{3} - w)f_{i-1}^k + (6w + \tfrac{20}{27})f_i^k \\
&+(-7w - \tfrac{2}{81})f_{i+1}^k + (\tfrac{5}{2}w - \tfrac{3}{162})f_{i+2}^k + (\tfrac{1}{27} - 3w)AV(\delta f_{i-2}^k, \delta f_{i-1}^k), \\
&f_{3i}^{k+1} = f_i^k, \\
&f_{3i+1}^{k+1} = (\tfrac{5}{2}w - \tfrac{3}{162})f_{i-2}^k + (-7w - \tfrac{2}{81})f_{i-1}^k + (6w + \tfrac{20}{27})f_i^k \\
&+(\tfrac{1}{3} - w)f_{i+1}^k + (-\tfrac{w}{2} - \tfrac{5}{162})f_{i+2}^k - (\tfrac{1}{27} - 3w)AV(\delta f_{i-2}^k, \delta f_{i-1}^k).
\end{aligned} \tag{3.8}$$

4. Convergence Analysis of a Family of 5-Point Nonlinear Subdivision Schemes

In this section we prove the convergence of 5-point nonlinear ternary interpolating subdivision scheme S_{NL} given in (3.6). The convergence for 7-point or 9-point ternary nonlinear interpolating subdivision schemes can be proved following the similar steps.

The 5-point nonlinear scheme S_{NL} (3.6) when expressed in the form of equation (2.7), gives us associated 5-point linear interpolating scheme S:

$$\begin{aligned}
f_{3i-1}^{k+1} &= (w - \tfrac{4}{81})f_{i-2}^k + (-4w + \tfrac{10}{27})f_{i-1}^k + (6w + \tfrac{20}{27})f_i^k \\
&\quad +(-4w - \tfrac{5}{81})f_{i+1}^k + wf_{i+2}^k, \\
f_{3i}^{k+1} &= f_i^k, \\
f_{3i+1}^{k+1} &= wf_{i-2}^k + (-4w - \tfrac{5}{81})f_{i-1}^k + (6w + \tfrac{20}{27})f_i^k \\
&\quad +(-4w + \tfrac{10}{27})f_{i+1}^k + (w - \tfrac{4}{81})f_{i+2}^k,
\end{aligned} \tag{4.1}$$

and the associated nonlinear function F

$$\begin{aligned} &F(\delta f^k)_{3i-1} = 2w\{AV(\delta f^k_{i-2}, \delta f^k_{i-1}) - AM(\delta f^k_{i-2}, \delta f^k_{i-1})\},\\ &F(\delta f^k)_{3i} = 0,\\ &F(\delta f^k)_{3i+1} = -2w\{AV(\delta f^k_{i-2}, \delta f^k_{i-1}) - AM(\delta f^k_{i-2}, \delta f^k_{i-1})\}. \end{aligned} \tag{4.2}$$

The linear subdivision scheme S given in (4.1) is C^2 continuous for $\frac{1}{324} < w < \frac{1}{162}$ as proved by Zheng etc. [11].

We present the following lemmas about the linear subdivision schemes S and the nonlinear function F given above.

Lemma 4.1. *The subdivision scheme S defined in (4.1) satisfies the following inequalities:*

1. $|\delta S(f^k)_{3i-1}| \le (\frac{11}{81} - 8w)||\delta f^k||_\infty$ *for* $0 < w < \frac{1}{81}$,
2. $|\delta S(f^k)_{3i}| \le (\frac{3}{81} + 12w)||\delta f^k||_\infty$ *for* $w > 0$,
3. $|\delta S(f^k)_{3i+1}| \le (\frac{11}{81} - 8w)||\delta f^k||_\infty$ *for* $0 < w < \frac{1}{81}$.

For the proof of this lemma, we refer to Proposition 1 in [5].

Lemma 4.2. *The function F defined in (4.2) satisfies the following inequalities:*

1. $|\delta F(\delta f^k)_{3i-1}| \le 6w||\delta f^k||_\infty,$ *for* $w > 0$,
2. $|\delta F(\delta f^k)_{3i}| \le 12w||\delta f^k||_\infty,$ *for* $w > 0$,
3. $|\delta F(\delta f^k)_{3i+1}| \le 6w||\delta f^k||_\infty,$ *for* $w > 0$.

Proof. Since

$$\delta F(\delta f^k)_{3i-1} = F(\delta f^k)_{3i+2} - 3F(\delta f^k)_{3i+1} + 3F(\delta f^k)_{3i} - F(\delta f^k)_{3i-1},$$

therefore,

$$\begin{aligned} \delta F(\delta f^k)_{3i-1} = 2w\{&AV(\delta f^k_{i-1}, \delta f^k_i) - AM(\delta f^k_{i-1}, \delta f^k_i) + 3AV(\delta f^k_{i-2}, \delta f^k_{i-1})\\ &-3AM(\delta f^k_{i-2}, \delta f^k_{i-1}) + 0 - AV(\delta f^k_{i-2}, \delta f^k_{i-1}) + AM(\delta f^k_{i-2}, \delta f^k_{i-1})\}. \end{aligned} \tag{4.3}$$

Which simplifies to

$$\begin{aligned} \delta F(\delta f^k)_{3i-1} = &2w\{AV(\delta f^k_{i-1}, \delta f^k_i) - AM(\delta f^k_{i-1}, \delta f^k_i)\}\\ &+4w\{AV(\delta f^k_{i-2}, \delta f^k_{i-1}) - AM(\delta f^k_{i-2}, \delta f^k_{i-1})\}. \end{aligned} \tag{4.4}$$

Since we are using the function $AV(x, y)$ as a replacement for $MGM(x, y)$ in (2.1) or $PPH(x, y)$ in (2.6), therefore, it satisfies properties (2.2) - (2.5). By using property (2.5), we have

$$|\delta F(\delta f^k)_{3i-1}| \leq 6w||\delta f^k||_\infty \text{ for } w > 0. \tag{4.5}$$

The proofs of the other two inequalities in Lemma 4.2 are very similar and straight forward. □

To prove the convergence of nonlinear scheme S_{NL}, we recall the following result from ([1], [2], [3]).

Theorem 4.3. *For F, S defined in (2.7) and a linear operator $\delta : l^\infty(Z) \to l^\infty(Z)$, if $\exists M > 0$ such that $\forall f \in l^\infty(Z)$*

$$||F(f)||_\infty \leq M||f||_\infty, \tag{4.6}$$

and $\exists\ c < 1$ such that

$$||\delta S(f) + \delta F(\delta f)||_\infty \leq c||\delta f||_\infty, \tag{4.7}$$

then the subdivision scheme S_{NL} is uniformly convergent. Moreover, if S is C^α convergent then, for all sequence $f \in l^\infty(Z)$, $S^\infty_{NL}(f)$ is at least C^β with $\beta = \min(\alpha, -\log_2(c))$.

Proof. Since we are proving the convergence of 5-point ternary subdivision scheme 3.4, therefore, associated linear subdivision scheme S is given in (4.1), F is given in (4.2) and $\delta f_i = f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i$. In order to prove conditions (4.6) and (4.7) of Theorem (4.3), we have to consider each of them separately at the points $3i - 1$, $3i$ and $3i + 1$.

Now to prove equation (4.6), we consider $F(f)$ at the point $3i - 1$ as defined in (4.2), and by using the fact (2.5), we have,

$$|F(f)_{3i-1}| \leq 2w||f||_\infty, \text{ for } w > 0. \tag{4.8}$$

At the point $3i$,

$$|F(f)_{3i}| = 0 \tag{4.9}$$

and similarly, at the point $3i + 1$, by using property (2.5),

$$|F(f)_{3i+1}| \leq 2w||g||_\infty, \text{ for } w > 0. \tag{4.10}$$

Let $M = 2w$, then $M > 0$ for $w > 0$. Therefore, from equations (4.8), (4.9) and (4.10), we get

$$||F(f)||_\infty \leq M||f||_\infty. \tag{4.11}$$

Which proves equation (4.6)

To prove equation (4.7), we again need to consider all three cases.

At the point $3i-1$:

$$|\delta S(f^k)_{3i-1} + \delta F(\delta f^k)_{3i-1}| \leq |\delta S(f^k)_{3i-1}| + |\delta F(\delta f^k)_{3i-1}|.$$

By Lemma 4.1, Part 1 and Lemma 4.2, Part 1, for $0 < w < \frac{1}{81}$, we get

$$|\delta S(f^k)_{3i-1} + \delta F(\delta f^k)_{3i-1}| \leq (\frac{11}{81} - 8w)||\delta f^k||_\infty + 6w||\delta f^k||_\infty,$$

$$|\delta S(f^k)_{3i-1} + \delta F(\delta f^k)_{3i-1}| \leq (\frac{11}{81} - 2w)||\delta f^k||_\infty \text{ for } 0 < w < \frac{1}{81}. \tag{4.12}$$

At the point $3i$:

$$|\delta S(f^k)_{3i} + \delta F(\delta f^k)_{3i}| \leq |\delta S(f^k)_{3i}| + |\delta F(\delta f^k)_{3i}|.$$

By Lemma 4.1, Part 2 and Lemma 4.2, Part 2, for $0 < w < \frac{1}{81}$, we get

$$|\delta S(f^k)_{3i} + \delta F(\delta f^k)_{3i}| \leq (\frac{3}{81} + 12w)||\delta f^k||_\infty + 12w||\delta f^k||_\infty,$$

$$|\delta S(f^k)_{3i} + \delta F(\delta f^k)_{3i}| \leq (\frac{1}{27} + 24w)||\delta f^k||_\infty \text{ for } 0 < w < \frac{1}{81}. \tag{4.13}$$

At the point $3i+1$:

$$|\delta S(f^k)_{3i+1} + \delta F(\delta f^k)_{3i+1}| \leq |\delta S(f^k)_{3i+1}| + |\delta F(\delta f^k)_{3i+1}|.$$

By Lemma 4.1, Part 3 and Lemma 4.2, Part 3, for $0 < w < \frac{1}{81}$, we get

$$|\delta S(f^k)_{3i+1} + \delta F(\delta f^k)_{3i+1}| \leq (\frac{11}{81} - 8w)||\delta f^k||_\infty + 6w||\delta f^k||_\infty,$$

$$|\delta S(f^k)_{3i+1} + \delta F(\delta f^k)_{3i+1}| \leq (\frac{11}{81} - 2w)||\delta f^k||_\infty \text{ for } 0 < w < \frac{1}{81}. \tag{4.14}$$

Let $c = \max\{\frac{11}{81} - 2w, \frac{1}{27} + 24w\}$, then for $0 < w < \frac{1}{81}$, we get $0 < c < 1$. Therefore, from (4.12), (4.13) and (4.14), we have

$$||\delta S(f^k) + \delta F(\delta f^k)||_\infty \leq c||\delta f^k||_\infty \tag{4.15}$$

for $c < 1$ with $0 < w < \frac{1}{81}$.

This proves equation (4.7) and consequently proves that our class of nonlinear 5-point ternary interpolating subdivision schemes S_{NL} given in (3.4) is uniformly convergent for $0 < w < \frac{1}{81}$.

It is noted that c as given in (4.15) can be restricted as $\frac{5}{39} < c < \frac{5}{27}$ for $\frac{1}{324} < w < \frac{1}{162}$. Which gives $\beta = 2$ as defined in Theorem 4.3 and hence proves that S_{NL}^{∞} is C^2 for $\frac{1}{324} < w < \frac{1}{162}$. □

5. Numerical Results

We are presenting curves generated by nonlinear and associated linear interpolating subdivision schemes. The graphs on the left side of Figure 1 are generated by 3-points schemes at $w = 0.251$. Both of the 3-point schemes are C^1 smooth. The graphs on the right side of Figure 1 are generated by 5-point nonlinear and associated linear schemes with $w = 1/320$. Both of the 5-point schemes are C^2 smooth. It is evident that the nonlinear schemes reduce the Gibbs phenomenon significantly.

6. Conclusion

In this article, we proposed a general formula for $(2n-1)$-point nonlinear ternary interpolating subdivision schemes. It is shown that 3-point and 5-point nonlinear ternary subdivision schemes developed in ([4], [5]) are special cases to our proposed schemes. Convergence of another special case (5-point nonlinear ternary interpolating subdivision scheme) is proved and it is shown that it is at least C^2 continuous. Numerical results are presented to show the performance of nonlinear over linear subdivision schemes.

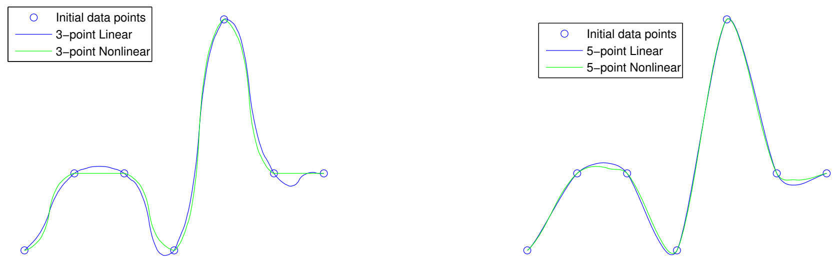

Figure 1: On the left, initial control points and the curves generated by 3-point nonlinear ternary interpolating subdivision scheme (5th level) in (3.4) and its associated 3-point linear ternary interpolating subdivision scheme with $w = 0.251$. On the right, same initial control points and the curves generated by 5-point nonlinear ternary interpolating subdivision scheme (5th level) in (3.8) and its associated 5-point linear ternary interpolating subdivision scheme with $w = 1/320$.

References

[1] S. Amat, K. Dadourian and J. Liandrat, On a nonlinear subdivision scheme avoiding gibbs oscillations and converging towards C^s functions with $s > 1$, *Mathematics of Computation*, **80**, No 274 (2011), 959-971.

[2] S. Amat, K. Dadourian and J. Liandrat, On a nonlinear 4-point ternary and interpolatory multiresolution scheme elimenating the Gibbs phenomenon, *International J. of Numerical Analysis and Modeling*, **7**, No 2 (2010), 261-280.

[3] F. Arandiga, R. Donat, M. Santagueda, The PCHIP subdivision scheme, *Applied Mathematics and Computation*, **272**, No 1 (2016), 28-40.

[4] M. Aslam, C^1 continuity of 3-point nonlinear ternary interpolating subdivision schemes, *Internatonal J. of Numerical Methods and Applications*, **14**, No 2 (2015), 119-132; http://dx.doi.org/10.17654/IJNMADec2015_119_132.

[5] M. Aslam, A family of 5-point nonlinear ternary interpolating subdivision schemes with c^2 smoothness, *Mathematical and Computational Applications*, **23**, No 2 (2018), 18; doi:10.3390/mca23020018.

[6] M. Aslam and W. Abeysinghe, Odd-Ary approximating subdivision schemes and rs strategy for irregular dense initial data, *ISRN Mathematical Analysis* (2012); DOI:10.5402/2012/745096.

[7] M. Aslam, G. Mustafa and A. Ghaffar, $(2n-1)$-point ternary approximating and interpolating subdivision schemes, *J. of Applied Mathematics* (2011); doi:10.1155/2011/832630.

[8] N. Aspert, Non-linear subdivision of univariate signals and discrete surfaces, *EPFL Thesis* (2003).

[9] A. Cohen, N. Dyn and B. Matei, Quasilinear subdivision schemes with applications to ENO interpolation, *Applied and Computational Harmonic Analysis*, **15**, No 2 (2003), 89-116.

[10] S. Harizanov, Stability of nonlinear subdivision schemes and multiresolutions, *Master's Thesis*, Jacobs University, Bremen (2007).

[11] H. Zheng, M. Hu, G. Peng, Constructing (2n-1)-point ternary interpolatory subdivision schemes by using variation of constants, *IEEE Proc. of the Intern. Conf. on Computational Intelligence and Software Engineering* (2009), 1-4.