

A STOCHASTIC MODEL FOR HIV EPIDEMIC WITH TREATMENT AND INFLOW OF INFECTIVES

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Abstract: We present a stochastic model of the population dynamics of HIV/AIDS with treatment and inflow of infectives. Starting with a deterministic compartmental model, each of the four ordinary differential equations are stochastically perturbed. An invariant \mathcal{R}_σ similar to the basic reproduction number of an ordinary differential equation system is introduced. Under conditions which permit the existence of a disease-free equilibrium point, we prove almost sure exponential stability of the disease-free equilibrium for $\mathcal{R}_\sigma < 1$. We also investigate asymptotic behaviour of the solutions to the stochastic model around the endemic equilibrium of the underlying deterministic model. Our theoretical results are illustrated by simulations with parameters applicable to South Africa.

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1. Introduction

HIV/AIDS has become a global growing public health problem. The rate of the spread of the HIV/AIDS epidemic has reached unseemly high levels of new infections and AIDS deaths. On a global scale, the HIV epidemic has stabilized, but the greatest burden of HIV remains in sub-Saharan Africa and within this

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region, especially the women are severely affected. Scientists in general have established different aspects behind the rapid increase of HIV/AIDS in Southern Africa, and to these, we shall include biological and social aspects, economic and political issues, and migration patterns which we know also played a big role in the dynamics of HIV/AIDS in this region. HIV/AIDS affects mostly people in the economically productive age range, reducing the work-force, and thereby constraining development [21]. South Africa has the biggest HIV epidemic in the world in terms of both the incidence rate and number of people living with HIV. In 2015, an estimated 7 million people living with HIV and 7.03 million in 2016 [15]. In 2015, there were 380,000 new infections and 180,000 people died from AIDS-related causes [18]. South Africa has the largest antiretroviral treatment (ART) programme globally and these efforts have been largely financed from its own domestic resources. The country now invests more than 1.5 billion annually to run its HIV/AIDS programmes [16].

The dynamics of HIV/AIDS in the context of Southern Africa present serious challenges due to its complexities and therefore require interventions. Mathematical modeling in epidemiology has been utilized to assess the impact of the disease on the population, to identify key disease drivers and to make future projections. Parameters involved in epidemic models may not be absolutely constant, due to inhomogeneities and environmental perturbations. In particular, it is important to identify areas of uncertainty that may be crucial for control of the disease. There are different ways of dealing with random effects and uncertainty in disease modeling. One of the approaches in this regard is to introduce stochastic differential equation (*sde*) models in the population dynamics of infectious diseases. In recent years, many authors have studied *sde* epidemic models, for instance [4, 6, 9, 17, 14]. An important aspect in the study of disease models is stability of equilibrium points. Specifically, for a deterministic disease model, asymptotic stability of the disease-free equilibrium (*dfe*) means that in the long run, the disease will vanish from the population. For *sde* systems, there are different versions of the concept of stability. Many authors investigate asymptotic behaviour of stochastic systems around the equilibria of the underlying deterministic models and examples of those can be found in [5, 10, 11, 22, 23]. In the papers by [4] Dalal et al. (HIV), [6] Gray et al. (SIS), [3] Chen et al. (SIR), and [19] Witbooi (SEIR), it is proved that stochastic perturbation actually enhances stability of the disease-free equilibrium for the specific models. The current paper presents another contribution in this regard.

The aim of this paper is to study the effect of a certain type of stochastic perturbation in a population model of HIV. Our stochastic model is based on

a deterministic model which is very similar to that of Cai et al [2], and in particular we are interested in the long term behaviour and we investigate for a type of stability in mean. First we introduce the inflow of infectives in a deterministic compartmental model, and thereafter we impose the stochastic perturbation in such a manner that the total population size itself is perturbed by white noise. We introduce an analogue of the basic reproduction number and we link it to almost sure exponential stability of the *dfe*. Here we note that for an *sde* system the concept of almost sure exponential stability is very similar to global asymptotic stability when working with ordinary differential equations. We prove a result on almost sure exponential stability of the *dfe*, in the absence of inflow of infectives and with no perturbations on the class of susceptibles.

The remainder of this paper is set up as follows. In Section 2 we give some preliminaries. In Section 3 we present the model and we study the existence of global positive solutions. Section 4 covers a theorem on almost sure exponential stability of the disease-free equilibrium when there is no inflow of infectives. We present numerical simulations to illustrate the results. Section 5 deals with asymptotic behaviour of the solutions to the stochastic model around the endemic equilibrium of the underlying deterministic model. Again we provide numerical simulations to illustrate our theoretical results. In Section 6 we present some concluding remarks.

2. Preliminaries

Let us denote by \mathbb{R}_+^n (resp. \mathbb{R}_{++}^n) the set of points in \mathbb{R}^n having only non-negative (resp. strictly positive) coordinates.

Throughout this paper we assume to have a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration, $\{\mathcal{F}_t\}_{t \geq 0}$, that is right continuous and \mathcal{F}_0 containing all the subsets having measure zero.

Consider an equation of the form (1) below, for an k -dimensional Brownian motion $B(t)$ on Ω .

$$dx(t) = f(t, x)dt + g(t, x)dB(t), \quad t \geq 0. \quad (1)$$

A solution with initial value $x(0) = x_0$ is denoted by $x(t, x_0)$. Assume that $f(t, 0) = g(t, 0) = 0$ for all $t \geq 0$, so the origin point is an equilibrium of (1).

By \mathcal{L} we denote the infinitesimal generator of an equation of the form (1), see [13] of Øksendal, defined for a function $V(t, x) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^k)$.

Definition 1. (see [12]). The equilibrium $x = 0$ of the system (1) is said to be almost surely exponentially stable if for all $x_0 \in \mathbb{R}^n$,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \ln |x(t, x_0)| < 0 \text{ almost surely (a.s.).}$$

The following observation which we quote from [20] is useful when dealing with exponential stability.

Lemma 2. For $k \in \mathbb{N}$, let $X(t) = (X_1(t), X_2(t), \dots, X_k(t))$ be a bounded \mathbb{R}^k -valued function and let $(t_{0,n})$ be any increasing unbounded sequence of positive real numbers. Then there is a family of sequences $(t_{l,n})$ such that for each $l \in \{1, 2, \dots, k\}$, $(t_{l,n})$ is a subsequence of $(t_{l-1,n})$ and the sequence $X_l(t_{l,n})$ converges to the largest limit point of the sequence $X_l(t_{l-1,n})$.

Remark 3. The following inequality is applied in Section 5:

Given any finite sequence of real numbers u_1, u_2, \dots, u_n , then

$$\left(\sum_{i=1}^n u_i\right)^2 \leq n \left(\sum_{i=1}^n u_i^2\right) \quad (2)$$

3. Stochastic HIV Model

Let $W(t) = (W_0(t), W_1(t), W_2(t), W_3(t))$ be a 4-dimensional Wiener process defined on this probability space. The components of W are assumed to be mutually independent. The non-negative constants $\sigma_0, \sigma_1, \sigma_2$ and σ_3 denote the intensities of the stochastic perturbations that we shall introduce.

We assume a homogeneously mixing population of size $N(t)$ at time t . The total population $N(t)$ is subdivided into the classes of susceptible individuals $S(t)$, asymptomatic phase of HIV $I(t)$, symptomatic phase $J(t)$, and the AIDS patients $A(t)$. The term μK is the recruitment rate of susceptibles into the population, μ being the birth rate which is assumed to coincide with the average mortality rate by natural causes. The disease-induced mortality rate is denoted by δ . The parameters β_1 and β_2 denote the probabilities of disease transmission per contact by an infective in the asymptomatic and the symptomatic phase respectively. For an individual, c is the average number of contacts with others per unit time. By k_1 and k_2 we denote the transfer rates from the asymptomatic phase I to the symptomatic phase J and from the symptomatic phase to the A -class, respectively. The parameter α is the rate of transfer from the

symptomatic phase J to the asymptomatic phase I due to treatment. The parameters Q_1, Q_2 denote the rates of inflow of infectives into the asymptomatic class and into the symptomatic class respectively.

Based on the assumptions above, we present the following stochastic model:

$$\begin{aligned} dS &= [\mu K - c(\beta_1 I + \beta_2 J)S - \mu S] dt + \sigma_0 S dW_0(t), \\ dI &= [Q_1 + c(\beta_1 I + \beta_2 J)S - (\mu + k_1)I + \alpha J] dt + \sigma_1 I dW_1(t), \\ dJ &= [Q_2 + k_1 I - (\mu + k_2 + \alpha)J] dt + \sigma_2 J dW_2(t), \\ dA &= [k_2 J - (\mu + \delta)A] dt + \sigma_3 A dW_3(t). \end{aligned} \quad (3)$$

We now show that solutions of (3) exist globally and are positive.

Theorem 4. *For model (3) and any initial value $(S(0), I(0), J(0), A(0)) \in \mathbb{R}_{++}^4$, there is a unique solution $(S(t), I(t), J(t), A(t))$ on $t \geq 0$ which remains in \mathbb{R}_{++}^4 with probability one.*

Proof. Note that the coefficients of the system (3) are locally Lipschitz continuous. Thus there exists a unique local solution on $t \in [0, \tau_{\text{en}})$, where τ_{en} is the explosion time. We need to show that this solution is global; that is, $\tau_{\text{en}} = \infty$ a.s.

Let $m_0 > 0$ be sufficiently large so that $S(0), I(0), J(0)$, and $A(0)$ belong to the interval $[1/m_0, m_0]$. For each integer $m \geq m_0$, define a sequence of stopping times by

$$\begin{aligned} \tau_m &= \inf \left\{ t \in [0, \tau_{\text{en}}) : S(t) \notin \left(\frac{1}{m}, m \right) \text{ or } I(t) \notin \left(\frac{1}{m}, m \right) \text{ or } \right. \\ &\quad \left. J(t) \notin \left(\frac{1}{m}, m \right) \text{ or } A(t) \notin \left(\frac{1}{m}, m \right) \right\}. \end{aligned} \quad (4)$$

Now since τ_m is nondecreasing, the following limit exists:

$$\tau_\infty = \lim_{m \rightarrow \infty} \tau_m,$$

and $\tau_\infty \leq \tau_{\text{en}}$ (a.s.). In what follows we prove that $\tau_\infty = \infty$ a.s. If this statement is violated, then there exists $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon. \quad (5)$$

Thus, there is an integer $m_1 \geq m_0$ such that

$$\mathbb{P}\{\tau_m \leq T\} \geq \epsilon \quad \forall m \geq m_1.$$

Choose $a_0 > 0$ sufficiently small in order to have $a_0 c \beta < \mu$ and $a_0 c \beta b < \mu$.

Consider the function V_1 defined by

$$V_1(S, I, J, A) = \left(S - a_0 - a_0 \ln \frac{S}{a_0} \right) + (I - 1 - \ln I) \\ + (J - 1 - \ln J) + (A - 1 - \ln A).$$

Note that each of the four bracketed terms are non-negative while $(S, I, J, A) \in \mathbb{R}_{++}^4$. By applying Itô's formula we have

$$dV_1(S, I, J, A) = \mathcal{L}V_1 dt + (S - a_0)\sigma_0 dW_0(t) + (I - 1)\sigma_1 dW_1(t) \\ + (J - 1)\sigma_2 dW_2(t) + (A - 1)\sigma_3 dW_3(t), \quad (6)$$

where

$$\begin{aligned} \mathcal{L}V_1 &= \left[\left(1 - \frac{a_0}{S}\right) (\mu K - c(\beta_1 I + \beta_2 J)S - \mu S) \right] \\ &+ \left[\left(1 - \frac{1}{I}\right) (Q_1 + c(\beta_1 I + \beta_2 J)S - (\mu + k_1)I + \alpha J) \right] \\ &+ \left[\left(1 - \frac{1}{J}\right) (Q_2 + k_1 I - (\mu + k_2 + \alpha)J) \right] \\ &+ \left[\left(1 - \frac{1}{A}\right) (k_2 J - (\mu + \delta)A) \right] + \frac{1}{2} (a_0 \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2) \\ &\leq \mu K - \mu(I + J) + a_0 c(\beta_1 I + \beta_2 J) + \mu(3 + a_0) + k_1 + k_2 \\ &\quad + \alpha + \delta + Q_1 + Q_2 + \frac{1}{2} (a_0 \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2). \end{aligned}$$

Note that by the choice of a_0 we have:

$$a_0 c \beta_1 I - \mu I = I(a_0 c \beta_1 - \mu) < 0$$

and

$$a_0 c \beta_2 J - \mu J = J(a_0 c \beta_2 - \mu) < 0.$$

Therefore,

$$\mathcal{L}V_1 \leq C,$$

where

$$C = \mu(K + (3 + a_0)) + k_1 + k_2 + \alpha + \delta + Q_1 + Q_2 + \frac{1}{2} (a_0 \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

is a constant.

Integrating both sides of (6) from 0 to $\tau_m \wedge T$ yields

$$\int_0^{\tau_m \wedge T} dV_1(S(s), I(s), J(s), A(s)) \leq \int_0^{\tau_m \wedge T} C ds + H(\tau_m \wedge T),$$

where

$$\begin{aligned} H(s) &= \int_0^s (S(u) - a_0) \sigma_0 dW_0(u) + \int_0^s (I(u) - 1) \sigma_1 dW_1(u) \\ &\quad + \int_0^s (J(u) - 1) \sigma_2 dW_2(u) + \int_0^s (A(u) - 1) \sigma_3 dW_3(u). \end{aligned}$$

Note that $H(s)$ is a mean zero martingale process and therefore by taking expectation we have

$$\begin{aligned} \mathbb{E}V_1(S(\tau_m \wedge T), I(\tau_m \wedge T), J(\tau_m \wedge T), A(\tau_m \wedge T)) \\ \leq V_1(S(0), I(0), J(0), A(0)) + CT. \end{aligned}$$

For each $m \geq m_1$, let $\Omega_m = \{\omega \in \Omega : \tau_m(\omega) < T\}$. From equation (5), we have that $\mathbb{P}(\Omega_m) \geq \epsilon$ for each $m > m_1$. For every $\nu \in \Omega_m$, we have

$$\{S(\tau_m, \nu), I(\tau_m, \nu), J(\tau_m, \nu), A(\tau_m, \nu)\} \cap \{m, 1/m\} \neq \emptyset.$$

Consequently,

$$V_1(S(\tau_m \wedge T), I(\tau_m \wedge T), J(\tau_m \wedge T), A(\tau_m \wedge T)) \geq G_m,$$

where

$$G_m = \min_{u \in \{1, a_0\}} \left\{ m - u - u \ln \frac{m}{u}, \frac{1}{m} - u - u \ln \frac{1}{um} \right\} > 0.$$

Then we obtain

$$\begin{aligned} &V_1(S(0), I(0), J(0), A(0)) + CT \\ &\geq \mathbb{E}(1_{\Omega_m} V_1(S(\tau_m \wedge T), I(\tau_m \wedge T), J(\tau_m \wedge T), A(\tau_m \wedge T))) \\ &\geq \epsilon G_m, \end{aligned}$$

where 1_{Ω_m} is the indicator function of Ω_m . Letting $m \rightarrow \infty$ leads to the contradiction $\infty = V_1(S(0), I(0), J(0), A(0)) + CT < \infty$. Therefore, the solution of model (3) is positive and will not explode in finite time, with probability one. This completes the proof. \square

4. Almost sure exponential stability

We investigate the behaviour of the system (3) under small perturbations, with $\sigma_0 = 0$, and $Q_1 = Q_2 = 0$. In this case the disease-free equilibrium $E_0 = (K, 0, 0, 0)$ exists. The basic reproduction number of the underlying deterministic model is very similar to that in [2] and is given as

$$\mathcal{R}_0 = \frac{cK[\beta_1(\mu + k_2 + \alpha) + \beta_2 k_1]}{(\mu + k_1)(\mu + k_2) + \alpha\mu}.$$

The following subset Φ of sample paths will be of interest:

$$\Phi = \{\omega \in \Omega \mid (S(t, \omega), I(t, \omega), J(t, \omega), A(t, \omega)) \in \mathbb{R}_{++}^4 \text{ for all } t \geq 0\}.$$

From Theorem 4 it follows that $\mathbb{P}(\Omega \setminus \Phi) = 0$. In the remainder of this section we assume that sample paths are restricted to Φ .

Proposition 5. *If $(S(0), I(0), J(0), A(0)) \in \mathbb{R}_{++}^4$, then almost surely, $S(t) \leq K$ for all $t > 0$.*

Proof. Given any path (in Φ), then

$$\frac{d(S - K)}{dt} = -\mu(S - K) - c(\beta_1 I + \beta_2 J)S \leq -\mu(S - K).$$

Therefore $S(0) < K$ implies that $S(t) < K$ for all $t > 0$. □

The following numbers will play a key role in our study of exponential stability. Let $\xi_0, \xi_1, \xi_2, \xi_3$ and ξ_4 be non-negative numbers, chosen as follows:

$$\begin{aligned} \xi_1 &= \beta_1(\mu + k_2 + \alpha) + \beta_2 k_1, \quad \xi_2 = \beta_1 \alpha + \beta_2(\mu + k_1), \\ \xi_4 &= (\mu + k_1)(\mu + k_2) + \mu \alpha. \end{aligned}$$

The numbers ξ_0 and ξ_3 will be chosen later. For now we just bear in mind that they are both non-negative.

We continue by preparing notation and concepts for our theorem on almost sure exponential stability. Recall that we work with sample paths in Φ . This implies in particular that if $Z(t)$ is defined as below, then $Z(t) > 0$ for all $t \geq 0$. Thus we define

$$Z(t) = \xi_0(K - S(t)) + \xi_1 I(t) + \xi_2 J(t) + \xi_3 A(t) \tag{7}$$

and let

$$V_2(t) = \ln Z(t).$$

For a stochastic process $\{x(t)\}_{t \geq 0}$ we write

$$\langle x \rangle_t = \frac{1}{t} \int_0^t x(s) ds.$$

Proposition 6. *The disease free equilibrium of system (3) is almost surely exponentially stable if*

$$\limsup_{t \rightarrow \infty} \langle \mathcal{L}V_2(X) \rangle_t < 0 \quad (\text{a.s.}).$$

Proof. We start off by noting that

$$V_2(X(t)) = V_2(X(0)) + \int_0^t \mathcal{L}V_2(X(u)) du + M_t,$$

where

$$M_t = \int_0^t \left(\frac{\sigma_0 S(u)}{z(X(u))} + \frac{\sigma_1 I(u)}{z(X(u))} + \frac{\sigma_2 J(u)}{z(X(u))} + \frac{\sigma_3 A(u)}{z(X(u))} \right) dW(u).$$

The strong law of large numbers for local martingales, see [12, p12] for instance, implies that

$$\lim_{t \rightarrow \infty} \frac{1}{t} M_t = 0 \quad (\text{a.s.}).$$

Also, we observe that

$$\lim_{t \rightarrow \infty} \frac{1}{t} V_2(X(0)) = 0.$$

Therefore

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} V_2(X(t)) &= \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathcal{L}V_2(X(u)) du \\ &= \limsup_{t \rightarrow \infty} \langle \mathcal{L}V_2(X) \rangle_t \quad (\text{a.s.}). \end{aligned}$$

This completes the proof. □

We now calculate $\mathcal{L}V_2$:

$$\begin{aligned} \mathcal{L}V_2 &= -\frac{\xi_0}{Z} [\mu K - c(\beta_1 I + \beta_2 J)S - \mu S] + \frac{\xi_1}{Z} [c(\beta_1 I + \beta_2 J)S \\ &\quad - (\mu + k_1)I + \alpha J] + \frac{\xi_2}{Z} [k_1 I - (\mu + k_2 + \alpha)J] + \frac{\xi_3}{Z} [k_2 J \\ &\quad - (\mu + \delta)A] - \frac{1}{2} \left[\left(\frac{\xi_1 \sigma_1 I}{Z} \right)^2 + \left(\frac{\xi_2 \sigma_2 J}{Z} \right)^2 + \left(\frac{\xi_3 \sigma_3 A}{Z} \right)^2 \right]. \end{aligned}$$

By Lemma 2 we can find, for every sample path $w \in \Phi$, a sequence t_n which is increasing and unbounded, such that

$$\limsup_{t \rightarrow \infty} \mathcal{L}\langle V_2(w) \rangle_t = \lim_{n \rightarrow \infty} \mathcal{L}\langle V_2(w) \rangle_{t_n},$$

and for which we can define the following limits:

$$s = \lim_{n \rightarrow \infty} \langle S \rangle_{t_n}, \quad i = \lim_{n \rightarrow \infty} \left\langle \frac{I}{Z} \right\rangle_{t_n}, \quad j = \lim_{n \rightarrow \infty} \left\langle \frac{J}{Z} \right\rangle_{t_n}, \quad a = \lim_{n \rightarrow \infty} \left\langle \frac{A}{Z} \right\rangle_{t_n},$$

and

$$q = \lim_{n \rightarrow \infty} \left\langle \frac{K - S}{Z} \right\rangle_{t_n}.$$

In particular, we note that $\xi_0 q + \xi_1 i + \xi_2 j + \xi_3 a = 1$ and $\xi_0 q, \xi_1 i, \xi_2 j, \xi_3 a \in [0, 1]$.

We define $F(\xi)$ as:

$$F(\xi) = F(\xi_0, \xi_1, \xi_2, \xi_3) = \limsup_{t \rightarrow \infty} \langle \mathcal{L}V_2 \rangle_t. \quad (8)$$

Then $F(\xi)$ takes the form:

$$\begin{aligned} F(\xi) = & \xi_0 [-\mu q + c(\beta_1 i + \beta_2 j)s] + \xi_1 [c(\beta_1 i + \beta_2 j)s - (\mu + k_1)i \\ & + \alpha j] + \xi_2 [k_1 i - (\mu + k_2 + \alpha)j] + \xi_3 [k_2 j - (\mu + d)a] \\ & - \frac{1}{2} [(\xi_1 \sigma_1 i)^2 + (\xi_2 \sigma_2 j)^2 + (\xi_3 \sigma_3 a)^2]. \end{aligned} \quad (9)$$

An invariant \mathcal{R}_σ of the model (3).

Let us define a function $h : [0, 1] \rightarrow \mathbb{R}$ as follows:

$$h(u) = \frac{\xi_1 \xi_2 (\sigma_1 u)^2 + \sigma_2^2 (1 - u)^2}{2 \beta_1 \xi_2 u + \beta_2 \xi_1 (1 - u)}. \quad (10)$$

Then h is continuous and positive. Therefore h has a minimum, which we shall denote by h_* . Note that $h_* > 0$. In the final theorem we use the following number \mathcal{R}_σ , which we define to be:

$$\mathcal{R}_\sigma = \frac{cK[\beta_1(\mu + k_2 + \alpha) + \beta_2 k_1]}{(\mu + k_1)(\mu + k_2) + \alpha\mu + h_*}. \quad (11)$$

Theorem 7. *If $\mathcal{R}_\sigma < 1$, then restricted to the subset Φ , I and J almost surely converge exponentially to 0.*

Proof. For ξ_1 , ξ_2 and ξ_4 as above (and for $\xi_0 = \xi_3 = 0$), we define $Z_0 = \xi_1 I + \xi_2 J$ and $V_0 = \ln Z_0$. It suffices to prove that $\limsup_{t \rightarrow \infty} \langle \mathcal{L}V_0 \rangle_t < 0$. Also, we let $F_0 = F(0, \xi_1, \xi_2, 0)$. We need to prove that $F_0 < 0$.

From (9) we simplify to have

$$F_0 < \xi_1 cK(\beta_1 i + \beta_2 j) - \xi_4(\beta_1 i + \beta_2 j) - \frac{1}{2} [(\xi_1 \sigma_1 i)^2 + (\xi_2 \sigma_2 j)^2]. \quad (12)$$

Now we note that

$$(\xi_1 \sigma_1 i)^2 + (\xi_2 \sigma_2 j)^2 = \frac{(\xi_1 \sigma_1 i)^2 + (\xi_2 \sigma_2 j)^2}{\beta_1 i + \beta_2 j} (\beta_1 i + \beta_2 j),$$

and since $\xi_2 j = 1 - \xi_1 i$, we have

$$F_0 < cK\xi_1(\beta_1 i + \beta_2 j) - \xi_4(\beta_1 i + \beta_2 j) - h(\xi_1 i)(\beta_1 i + \beta_2 j).$$

This leads to the inequality below:

$$\begin{aligned} F_0 &< cK\xi_1(\beta_1 i + \beta_2 j) - \xi_4(\beta_1 i + \beta_2 j) - h_*(\beta_1 i + \beta_2 j) \\ &= \xi_4(\mathcal{R}_\sigma - 1)(\beta_1 i + \beta_2 j) < 0. \end{aligned}$$

□

We now prove the main theorem.

Theorem 8. *If $\mathcal{R}_\sigma < 1$, then the disease-free equilibrium is almost surely exponentially stable.*

Proof. The proof is by contradiction. From Theorem 7 we know that $\lim_{t \rightarrow \infty} I(t) = 0$ (a.s) and $\lim_{t \rightarrow \infty} J(t) = 0$ (a.s). Let us now suppose, contrary to the claim of this theorem, that for some subset Θ of Φ with $\mathbb{P}(\Theta) > 0$, on Θ we have:

$$\lim_{t \rightarrow \infty} [(K - S(t)) + A(t)] \neq 0. \quad (13)$$

Now let Z be as in (7) and $F(\xi)$ as in (9). In particular we choose $\xi_0 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = 1$. Then in view of (13) and by the definition of i and j , on Θ we have $i = 0$ (a.s) and $j = 0$ (a.s). Thus, from (9) it follows that

$$F(\xi) \leq -\mu q - (\mu + \delta)a - \frac{1}{2}(\sigma_3 a)^2 \quad (\text{a.s}).$$

Therefore, $F < 0$ (a.s). Then by Proposition 6 it follows that on Θ , we have that $\lim_{t \rightarrow \infty} (K - S(t)) = 0$ (a.s) and $\lim_{t \rightarrow \infty} A(t) = 0$ (a.s). This is a contradiction, and it completes the proof. □

A case study of HIV/AIDS in South Africa.

The parameters such as c, α, k_1, k_2 found in [1, 8, 7] are applicable to Southern Africa. In [7] for instance, the average number of sexual partners per given time denoted by c has been assigned values ranging from 1 to 2 for a specific case. In our case we take $c = 3$ in order to avoid addressing a problem that is simpler than the actual one. We expect the following inequality holds $\beta_1 < \beta_2$, knowing that the probability of disease transmission in the symptomatic phase far exceeds that of the asymptomatic phase. In the year 2016, the life expectancy in South Africa was estimated at 62.4 years, see for instance in [15]. The mortality rate μ is simply the inverse of the life expectancy given by $\frac{1}{62.4} \text{ yr}^{-1}$. The disease induced mortality rate δ is found in [15]. The parameter K is the size of the population when disease-free and does not complicate our task. We estimate values for the rates of inflow of infectives Q_1 and Q_2 since they are not easily obtainable. The parameter values of the model are given in the table below.

Parameters	Value	Source
α	0.33	[1]
k_1	0.125	[8]
k_2	0.1	[1]
c	3	cf. [7]
μ	$\frac{1}{62.5}$	[15]
δ	0.279	[15]

Table 1: The parameters values in this table are fixed

Regarding the initial conditions, we start off with the year 2016 in order to do our projection. According to the South African 2016 mid-year population estimate [15], the total population which we denote by $N(t_{16}) = S(t_{16}) + I(t_{16}) + J(t_{16}) + A(t_{16})$, and where t_{16} is the time on 25 August 2016, was 55.91 million. An estimated 7.03 million of the total population were infected with HIV/AIDS in 2016. This means that the classes $I(t_{16})$, $J(t_{16})$ and $A(t_{16})$ add up to 7.03 million. We shall then use the parameters listed in Table 1 to find a suitable equilibrium point to split the numbers between the classes $I(t_{16})$, $J(t_{16})$ and $A(t_{16})$. We vary the values of β_1 and β_2 in order to vary the value of the basic reproduction number.

Let us denote the force of infection by

$$\lambda = c(\beta_1 I + \beta_2 J).$$

We note that with inflow of infectives, we find the following equilibrium values for I and J :

$$I = \frac{\alpha (\lambda + \mu) Q_2 + (\alpha + \mu + k_2) (K \lambda \mu + (\lambda + \mu) Q_1)}{(\lambda + \mu) ((\mu + k_1) (\mu + k_2) + \alpha \mu)}$$

and

$$J = \frac{1}{(\alpha + \mu + k_2)} [Q_2 + k_1 I].$$

This consideration leads us to assign initial values to I_0 and J_0 , and thus our initial state is taken as:

$$S_0 = 48.88, I_0 = 5.22, \quad J_0 = 1.46, \quad A_0 = 0.344.$$

We present some simulations in order to illustrate the analytical results of stochastic model (3) and the underlying deterministic system. For simplicity we use one common value for σ_1, σ_2 and σ_3 (call it σ) while $\sigma_0 = 0$. In each graph we show trajectories of $J(t)$ for the stochastic model and of $J(t)$ for the underlying deterministic model with respect to time in years.

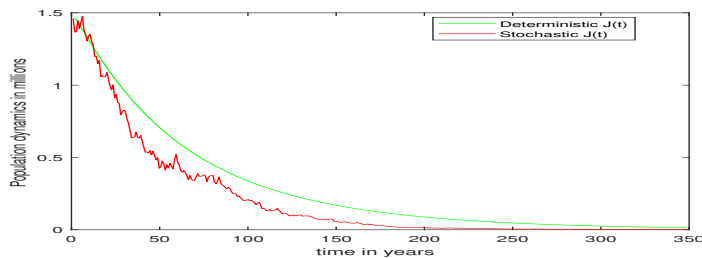


Figure 1: Improving stability in the case $\mathcal{R}_\sigma < 1$ and $\sigma = 0.05$.

Chosen values: $\beta_1 = 0.000176, \beta_2 = 0.00037, \sigma = 0.05$.

Calculated values: $\mathcal{R}_0 = 0.967734, \mathcal{R}_\sigma = 0.9350$.

In Figure 1 we choose $\beta_1 = 0.000176, \beta_2 = 0.00037$ and $\sigma = 0.05$. In this case \mathcal{R}_σ is found to be less than 1. Theorem 7 assures us that the disease-free equilibrium is almost surely exponentially stable. Indeed the graph shows that over time, the state of the system converges to disease-free equilibrium. Figure 2 shows that for small values of the perturbation parameter there is convergence to disease-free equilibrium for a bigger range of values of the basic reproduction number of the underlying deterministic model.

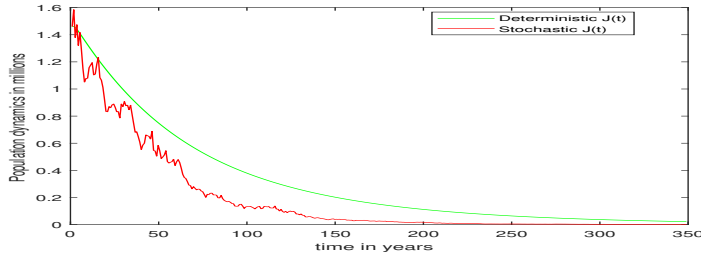


Figure 2: Stability obtained beyond $\mathcal{R}_0 < 1$ while $\mathcal{R}_\sigma < 1$.

Chosen values: $\beta_1 = 0.000186$, $\beta_2 = 0.00039$, $\sigma = 0.06$.

Calculated values: $\mathcal{R}_0 = 1.022$, $\mathcal{R}_\sigma = 0.9702$.

5. Asymptotic behaviour around the endemic equilibrium

We investigate the asymptotic behaviour around the endemic equilibrium of the underlying deterministic model system.

Before stating the main theorem, let us first define the positive numbers below:

$$\begin{aligned} D_1 &= 1 + \frac{2\mu + k_2}{\alpha}, \quad D_2 = \mu + \frac{(\mu + k_1)(2\mu + k_2)}{\alpha}, \\ D_3 &= 2\mu + \frac{(\mu + k_1)(2\mu + k_2)}{\alpha} + \frac{\mu(2\mu + k_2)}{\alpha}, \\ D_4 &= \frac{2D_3}{c\beta_1}. \end{aligned} \quad (14)$$

Theorem 9. *Let $(S(t), I(t), J(t), A(t))$ be the solution of system (3) with any initial value $(S(0), I(0), J(0), A(0)) \in \mathbb{R}_{++}^4$. Let $E^* = (S^*, I^*, J^*, A^*)$ be an endemic equilibrium point of the underlying deterministic model. If $\mathcal{R}_0 > 1$, and the following condition is satisfied:*

$$2(\mu + \delta) - k_2 > 0,$$

then the solution of model (3) has the property:

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t [(S(\tau) - S^*)^2 + (I(\tau) - I^*)^2 + (J(\tau) - J^*)^2 + (A(\tau) - A^*)^2] d\tau \leq \frac{D_0}{\theta}, \quad (15)$$

where

$$\theta = 2 \min \{ \mu D_1, D_2, \mu, 2(\mu + \delta) - k_2 \},$$

and

$$D_0 = K^2(\sigma_0^2 D_1 + \sigma_1^2 D_1 + \sigma_2^2 + 2\sigma_3^2) + \frac{1}{2} D_4 (S^* \sigma_0^2 + 2I^* \sigma_1^2 + A_2 J^* \sigma_2^2).$$

Proof. We note that at the point $E^* = (S^*, I^*, J^*, A^*)$ we have

$$\begin{aligned} \mu K + Q_1 + Q_2 &= \mu S^* + \mu I^* + (\mu + k_2) J^* \\ \mu K + Q_2 &= \mu S^* + (\mu + k_1) I^* - \alpha J^* \\ (\mu + \delta) &= k_2 \frac{J^*}{A^*}. \end{aligned} \tag{16}$$

Consider the following function

$$V_3(S, I, J, A) = V_4 + V_5 + V_6 + V_7$$

where

$$\begin{aligned} V_4 &= [(S - S^*) + (I - I^*) + (J - J^*)]^2, \\ V_5 &= \frac{(2\mu + k_2)}{\alpha} [(S - S^*) + (I - I^*)]^2, \quad V_6 = 2(A - A^*)^2, \end{aligned}$$

and

$$\begin{aligned} V_7 &= D_4 \left(S - S^* - S^* \ln \frac{S}{S^*} \right) + 2D_4 \left(I - I^* - I^* \ln \frac{I}{I^*} \right) \\ &\quad + A_2 D_4 \left(J - J^* - J^* \ln \frac{J}{J^*} \right) \end{aligned}$$

with

$$A_2 = \frac{2\alpha J^* + c\beta_2 J^* S^*}{k_1 I^*}.$$

Then,

$$\begin{aligned} \int_0^t dV_3(S, I, J, A) &= \int_0^t \mathcal{L}V_3 du + R_t \\ &= \int_0^t [\mathcal{L}V_4 + \mathcal{L}V_5 + \mathcal{L}V_6 + \mathcal{L}V_7] du + R_t \end{aligned}$$

where

$$\begin{aligned}
R_t = & \int_0^t 2[(S - S^*) + (I - I^*) + (J - J^*)](\sigma_0 S dW_0(u) \\
& + \sigma_1 I dW_1(u) + \sigma_2 J dW_2(u)) + \int_0^t 2 \frac{(2\mu + k_2)}{\alpha} [(S - S^*) \\
& + (I - I^*)](\sigma_0 S dW_0(u) + \sigma_1 I dW_1(u)) \\
& + \int_0^t D_4(S - S^*) \sigma_0 dW_0(u) + \int_0^t 2D_4(I - I^*) \sigma_1 dW_1(u) \\
& + \int_0^t A_2 D_4(J - J^*) \sigma_2 dW_2(u) + \int_0^t 4(A - A^*) \sigma_3 dW_3(u).
\end{aligned}$$

We expand the $\mathcal{L}V_i$ terms as follows:

$$\begin{aligned}
\mathcal{L}V_4 = & 2[(S - S^*) + (I - I^*) + (J - J^*)] \\
& \times [-\mu(S - S^*) - \mu(I - I^*) - (\mu + k_2)(J - J^*)] \\
& + (\sigma_0^2 S^2 + \sigma_1^2 I^2 + \sigma_2^2 J^2), \\
\mathcal{L}V_5 = & 2 \frac{(2\mu + k_2)}{\alpha} [(S - S^*) + (I - I^*)] \\
& \times [-\mu(S - S^*) - (\mu + k_1)(I - I^*) + \alpha(J - J^*)], \\
& + \frac{(2\mu + k_2)}{\alpha} (\sigma_0^2 S^2 + \sigma_1^2 I^2),
\end{aligned}$$

$$\mathcal{L}V_6 = 4(A - A^*)[k_2(J - J^*) - (\mu + \delta)(A - A^*)] + 2\sigma_3^2 A^2,$$

$$\begin{aligned}
\mathcal{L}V_7 = & \mu S^* D_4 \left(2 - \frac{S}{S^*} - \frac{S^*}{S} \right) - D_4 \left(1 - \frac{S^*}{S} \right) \\
& \times c\beta_1(IS - I^*S^*) + 2D_4 \left(2 - \frac{I}{I^*} - \frac{I^*}{I} \right) Q_1 \\
& - D_4 \left(1 - \frac{S^*}{S} \right) c\beta_2(JS - J^*S^*) + 2D_4 c\beta_1(I - I^*)(S - S^*) \\
& + 2D_4 \left(1 - \frac{I^*}{I} \right) c\beta_2 \left(JS - J^*S^* \frac{I}{I^*} \right) \\
& + 2D_4 \alpha \left(1 - \frac{I^*}{I} \right) \left(J - J^* \frac{I}{I^*} \right) + A_2 D_4 \left(1 - \frac{J^*}{J} \right) \\
& \times \left(k_1 I - k_1 I^* \frac{J}{J^*} \right) + A_2 D_4 \left(2 - \frac{J}{J^*} - \frac{J^*}{J} \right) Q_2
\end{aligned}$$

$$+\frac{1}{2}D_4(S^*\sigma_0^2+2I^*\sigma_1^2+A_2J^*\sigma_2^2).$$

Let us compute $\mathcal{L}V_4, \mathcal{L}V_5$ and $\mathcal{L}V_6$ in detail:

$$\begin{aligned}\mathcal{L}V_4 &= -2\mu(S-S^*)^2 - 4\mu(S-S^*)(I-I^*) - 2(2\mu+k_2) \\ &\quad \times (S-S^*)(J-J^*) - 2(\mu+k_2)(J-J^*)^2 \\ &\quad + (\sigma_0^2S^2 + \sigma_1^2I^2 + \sigma_2^2J^2) \\ \mathcal{L}V_5 &= -2\mu\frac{(2\mu+k_2)}{\alpha}(S-S^*)^2 - 2\frac{(\mu+k_1)(2\mu+k_2)}{\alpha}(I-I^*)^2 \\ &\quad - 2\left(\frac{(\mu+k_1)(2\mu+k_2)}{\alpha} + \mu\frac{(2\mu+k_2)}{\alpha}\right)(S-S^*)(I-I^*) \\ &\quad + 2(2\mu+k_2)(S-S^*)(J-J^*) + 2(2\mu+k_2)(I-I^*)(J-J^*) \\ &\quad + \frac{(2\mu+k_2)}{\alpha}(\sigma_0^2S^2 + \sigma_1^2I^2) \\ \mathcal{L}V_6 &= 4k_2(A-A^*)(J-J^*) - 4(\mu+\delta)(A-A^*)^2 + 2\sigma_3^2A^2.\end{aligned}$$

Thus we have

$$\begin{aligned}\mathcal{L}V_3 &= -2\mu D_1(S-S^*)^2 - 2D_2(I-I^*)^2 - 2(\mu+k_2)(J-J^*)^2 \\ &\quad - 2D_3(S-S^*)(I-I^*) + 4k_2(A-A^*)(J-J^*) \\ &\quad - 4(\mu+\delta)(A-A^*)^2 + \sigma_0^2S^2D_1 + \sigma_1^2I^2D_1 + \sigma_2^2J^2 \\ &\quad + 2\sigma_3^2A^2 + \mathcal{L}V_7,\end{aligned}\tag{17}$$

where D_1, D_2 and D_3 are as in (14).

Applying the inequality (2), we observe that:

$$2(A-A^*)(J-J^*) \leq (A-A^*)^2 + (J-J^*)^2.$$

Now from (17) we obtain the inequality:

$$\begin{aligned}\mathcal{L}V_3 &\leq -2\mu D_1(S-S^*)^2 - 2D_2(I-I^*)^2 - 2\mu(J-J^*)^2 \\ &\quad - 2(2(\mu+\delta)-k_2)(A-A^*)^2 + K^2(\sigma_0^2D_1 + \sigma_1^2D_1 + \sigma_2^2 + 2\sigma_3^2) \\ &\quad - 2D_3(S-S^*)(I-I^*) + \mathcal{L}V_7 \leq \Lambda,\end{aligned}\tag{18}$$

where

$$\Lambda = \mathcal{L}V_7 - 2D_3(S-S^*)(I-I^*) + K^2(\sigma_0^2D_1 + \sigma_1^2D_1 + \sigma_2^2 + 2\sigma_3^2).$$

Then

$$\begin{aligned}
\Lambda = & \mu S^* D_4 \left(2 - \frac{S}{S^*} - \frac{S^*}{S} \right) - D_4 \left(1 - \frac{S^*}{S} \right) c\beta_1 (IS - I^* S^*) \\
& + 2D_4 \left(2 - \frac{I}{I^*} - \frac{I^*}{I} \right) Q_1 - D_4 \left(1 - \frac{S^*}{S} \right) c\beta_2 (JS - J^* S^*) \\
& + 2D_3 (I - I^*) (S - S^*) + 2D_4 \left(1 - \frac{I^*}{I} \right) \\
& \times c\beta_2 \left(JS - J^* S^* \frac{I}{I^*} \right) + 2D_4 \alpha \left(1 - \frac{I^*}{I} \right) \left(J - J^* \frac{I}{I^*} \right) \\
& + A_2 D_4 \left(1 - \frac{J^*}{J} \right) \left(k_1 I - k_1 I^* \frac{J}{J^*} \right) \\
& + A_2 D_4 \left(2 - \frac{J}{J^*} - \frac{J^*}{J} \right) Q_2 + \frac{1}{2} D_4 (S^* \sigma_0^2 + 2I^* \sigma_1^2 + A_2 J^* \sigma_2^2) \\
& + K^2 (\sigma_0^2 D_1 + \sigma_1^2 D_1 + \sigma_2^2 + 2\sigma_3^2).
\end{aligned}$$

Letting $\frac{S}{S^*} = x$, $\frac{I}{I^*} = y$, $\frac{A}{A^*} = z$, it follows that

$$\begin{aligned}
\Lambda = & S^* (D_4 \mu + 2D_3 I^*) \left(2 - \frac{1}{x} - x \right) + 2D_4 \alpha J^* \left(2 - \frac{z}{y} - \frac{y}{z} \right) \\
& + D_4 S c\beta_2 J^* \left(3 - \frac{xz}{y} - \frac{1}{x} - \frac{y}{z} \right) + 2D_4 \left(2 - \frac{1}{y} - y \right) Q_1 \\
& + \frac{1}{2} D_4 (S^* \sigma_0^2 + 2I^* \sigma_1^2 + A_2 J^* \sigma_2^2) + A_2 D_4 \left(2 - \frac{1}{z} - z \right) Q_2 \\
& + K^2 (\sigma_0^2 D_1 + \sigma_1^2 D_1 + \sigma_2^2 + 2\sigma_3^2).
\end{aligned}$$

Note that since the arithmetic mean is greater than or equal to the geometric mean, it follows that

$$\frac{1}{y} + y \geq 2, \quad \frac{1}{x} + x \geq 2, \quad \frac{z}{y} + \frac{y}{z} \geq 2, \quad \frac{1}{x} + \frac{xz}{y} + \frac{y}{z} \geq 3.$$

We now have

$$\begin{aligned}
\Lambda \leq & \frac{1}{2} D_4 (S^* \sigma_0^2 + 2I^* \sigma_1^2 + A_2 J^* \sigma_2^2) \\
& + K^2 (\sigma_0^2 D_1 + \sigma_1^2 D_1 + \sigma_2^2 + 2\sigma_3^2).
\end{aligned} \tag{19}$$

Substituting (18) into (17), it follows that

$$\mathcal{LV}_3 \leq -2\mu D_1 (S - S^*)^2 - 2D_2 (I - I^*)^2 - 2\mu (J - J^*)^2$$

$$\begin{aligned}
& -2(2(\mu + \delta) - k_2)(A - A^*)^2 \\
& + K^2(\sigma_0^2 D_1 + \sigma_1^2 D_1 + \sigma_2^2 + 2\sigma_3^2) \\
& + \frac{1}{2} D_4 (S^* \sigma_0^2 + 2I^* \sigma_1^2 + A_2 J^* \sigma_2^2).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^t dV_3 \leq & \int_0^t [-2\mu D_1 (S - S^*)^2 - 2D_2 (I - I^*)^2 - 2\mu (J - J^*)^2 \\
& - 2(2(\mu + \delta) - k_2)(A - A^*)^2 \\
& + K^2(\sigma_0^2 D_1 + \sigma_1^2 D_1 + \sigma_2^2 + 2\sigma_3^2) \\
& + \frac{1}{2} D_4 (S^* \sigma_0^2 + 2I^* \sigma_1^2 + A_2 J^* \sigma_2^2)] du + R_t.
\end{aligned}$$

We take expectation and note that $\mathbb{E}[R_t] = 0$. Thus we obtain

$$\begin{aligned}
0 \leq & \mathbb{E}[V_3(S(t), I(t), J(t), A(t))] \leq \mathbb{E}[V_3(S(0), I(0), J(0), A(0))] \\
& + \mathbb{E} \int_0^t [-2\mu D_1 (S - S^*)^2 - 2D_2 (I - I^*)^2 - 2\mu (J - J^*)^2 \\
& - 2(2(\mu + \delta) - k_2)(A - A^*)^2 + D_0] du,
\end{aligned} \tag{20}$$

which gives

$$\begin{aligned}
& \mathbb{E} \int_0^t \left[2\mu D_1 (S(u) - S^*)^2 + 2D_2 (I(u) - I^*)^2 + 2\mu (J(u) - J^*)^2 \right. \\
& \left. + 2(2(\mu + \delta) - k_2)(A(u) - A^*)^2 \right] du \\
& \leq \mathbb{E}[V_3(S(0), I(0), J(0), A(0))] + D_0 t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t \left[2\mu D_1 (S(u) - S^*)^2 + 2D_2 (I(u) - I^*)^2 \right. \\
& \left. + 2\mu (J(u) - J^*)^2 + 2(2(\mu + \delta) - k_2)(A(u) - A^*)^2 \right] du \leq D_0.
\end{aligned}$$

We take θ as in the formulation of Theorem 9, and then it follows that:

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} \frac{1}{t} \mathbb{E} \int_0^t \left[(S(\tau) - S^*)^2 + (I(\tau) - I^*)^2 + (J(\tau) - J^*)^2 \right. \\
& \left. + (A(\tau) - A^*)^2 \right] d\tau \leq \frac{D_0}{\theta}.
\end{aligned}$$

This completes the proof. \square

Remark 10. Theorem 9 states that for small values of the perturbation parameters the solutions of the stochastic system (3) will eventually stay very close to the endemic equilibrium of the underlying deterministic model.

We present numerical simulations in order to illustrate Theorem 9 with parameter values given in Table 1.

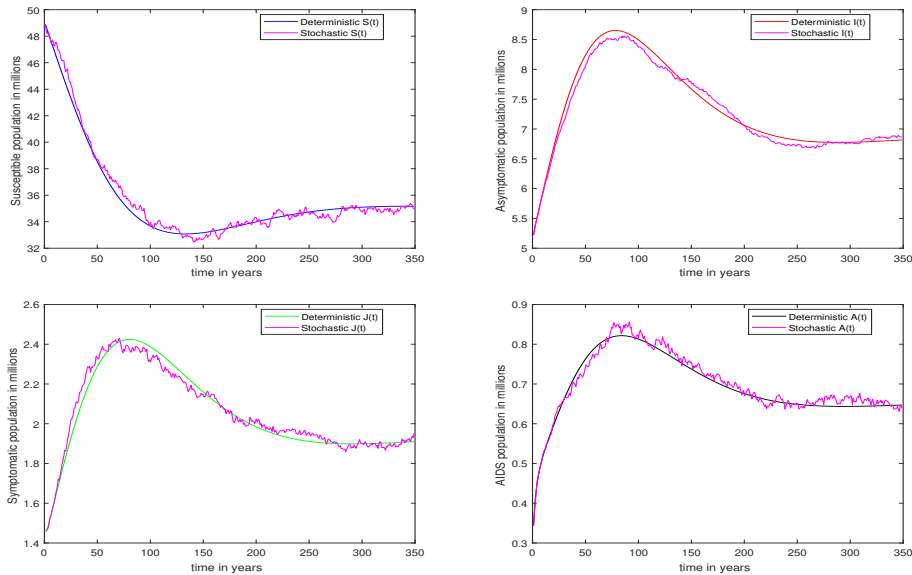


Figure 3: The dynamics of system (3) without the rates of inflow of infectives

Chosen values: $\beta_1 = 0.0002904, \beta_2 = 0.00061, Q_1 = Q_2 = 0, \sigma_0 = 0.005, \sigma_1 = 0.004, \sigma_2 = 0.009, \sigma_3 = 0.01$.
 Calculated value: $\mathcal{R}_0 = 1.595, \lambda = 0.009527, S^* = 35.04, I^* = 6.88, J^* = 1.92, A^* = 0.65$.

In Fig. 3 the basic reproduction number \mathcal{R}_0 is bigger than one and the stochastic solutions remain close to the endemic solutions of the underlying deterministic model. We observe a similar pattern in these graphs. In Fig. 4 all the parameters and their values have remained unchanged, except that the rates of inflow of infectives Q_1, Q_2 now are taken as positive. In the case of the underlying deterministic model, the rates of inflow of infectives lead to increasing the values of the force of infection λ, I, J and A while decreasing the

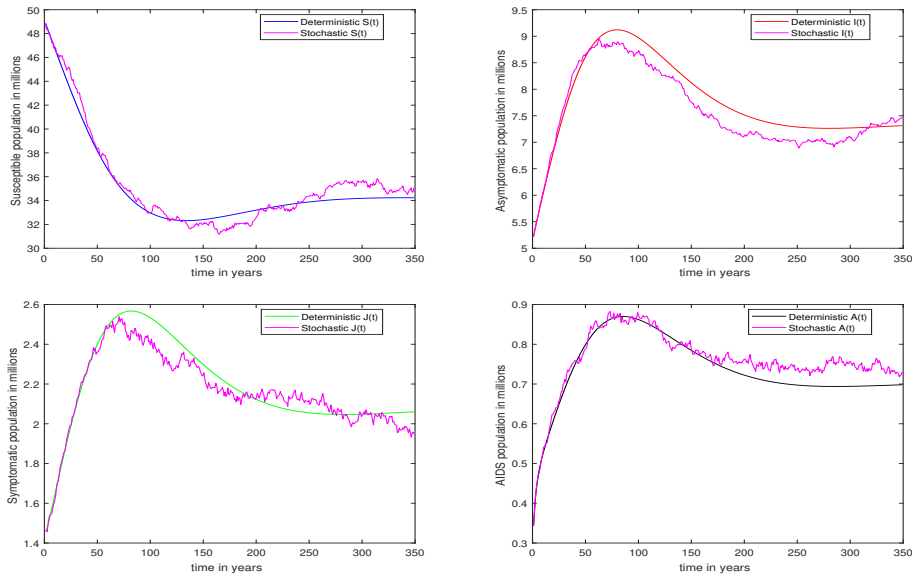


Figure 4: The dynamics of system (3) with the rates of inflow of infectives

Chosen values: $\beta_1 = 0.0002904, \beta_2 = 0.00061, Q_1 = Q_2 = 0.005, \sigma_0 = 0.005, \sigma_1 = 0.004, \sigma_2 = 0.009, \sigma_3 = 0.01$.

Calculated value: $\mathcal{R}_0 = 1.595, \lambda = 0.01022, S^* = 34.12, I^* = 7.37, J^* = 2.077, A^* = 0.70$.

value of S . In the stochastic case, the fluctuation in each graph is higher than the fluctuations in Fig. 4 due to the rates of inflow of infectives. In some simulations not shown here, it is found that a strong perturbation leads to a strong divergence. In this case, we would not expect to see the stochastic solutions be close to the endemic solutions of the underlying deterministic system.

6. Concluding remarks

We have presented an *sde* model of HIV, which we showed to have well-behaved solutions. In the special case that we have no inflow of infected individuals into the system and $\sigma_0 = 0$, Theorem 7 describes convergence to disease-free equilibrium. In particular, the theorem asserts that for sufficiently small values of the

perturbation parameter, stability of the disease-free equilibrium is obtained for a bigger range of values of the basic reproduction number \mathcal{R}_0 of the deterministic model, i.e., beyond the range $\mathcal{R}_0 < 1$. This is sufficiently significant that it can be observed in simulations. The almost sure exponential stability is a fairly strong type of stability, it being a stochastic version of global asymptotic stability. For the public health authorities it is comforting to know that the presence of minor stochasticity on their model will not be a hindrance if eradication strategies should be launched. With respect to the general model, we have been able to describe the long-term behaviour of solutions in comparison with that of the deterministic model, in Theorem 9. The theorem asserts that asymptotically the stochastic solutions stay within a certain bound from the (non-trivial) equilibrium point of the underlying deterministic model. This is very well observed in simulations. Further it is also investigated that the positive flow of infectives could affect the stability and also lead the dynamics of the model system from stable to the unstable situation. Our *sde* model has revealed some new phenomena and is useful when planning intervention strategies.

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