International Journal of Applied Mathematics

Volume 31 No. 5 2018, 585-594
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v31i5.4

THE VECTOR FIELDS ACROSS THE TANGENT BUNDLE TO A SPINNING 2-SPHERE

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Abstract: Assuming a spinning 2-sphere in the Euclidean 3-space that is observed by a frame at any exterior point, we deduce the Coriolis effect, which asserts that the rotational directions of the flows on the Earth surface are opposite across the Equator. By the continuity of the angular momentum, we project the vector of angular momentum onto all tangent planes to the sphere. We arrive at elliptical orbits and the spiral motions toward the plug point of any domestic sink.

AMS Subject Classification: 76U05, 37C10, 53A35
Key Words: coriolis effect, angular momentum projection, rotation directions, Earth spin, vortex flows, current helicity

1. Introduction

This paper examines the effect of a spinning 2-sphere S^2 on the local tangent planes $\{T_pS^2 \mid p \in S^2\}$. An extensive literature research has produced null results similar to our treatment here, with the closest being [3], where the idea of projection of angular momentum was invoked, an idea that incidentally *is* ubiquitous in quantum mechanics, as the Planck constant $\hbar$ is angular momentum and the projection operator is a standard apparatus (cf. [4], also see, e.g., [7]). We also note that our analysis here is not about a vector field over S^2

Received: May 31, 2018

(as in [10], also cf. [11], p. 468, 476); rather we analyze a tangent bundle to a spinning sphere. To be sure, our approach here is that of a Gaussian intrinsic geometry; we deduce the existence of a spinning S^2, which is observable only extrinsically, by analyzing the vector field on T_pS^2 locally (cf. e.g., [13], where extrinsic features such as centrifugal forces were factors in the analysis).

Our motivation here is to demystify the Coriolis effect as noted by the French engineer Gaspard Gustav de Coriolis (1792-1843) that fluids rotate on Earth counterclockwise in the northern hemisphere and clockwise in the southern hemisphere, by giving a mathematical derivation of this effect as based on the spin of Earth and its projections onto $\left\{T_pS^2 \mid p \in S^2 = \text{Earth surface}\right\}$. To our knowledge, our treatment here has never been made before; the bulk of the literature has been about how the Coriolis effect alters the wave equations in fluid dynamics (see, e.g. [5], [8]), which in particular include oceanic and atmospheric motions (see, e.g. [9] , [12]), and how it affects molecular and mechanical motions (see, e.g., [1], [10]) as well as how it has far-reaching consequences in cosmic rotating black holes (see [2]) or even in flight performances of pilots due to optical disorientation (see, e.g. [6]).

Section 2 below presents our analysis, which begins with a local T_pS^2 and ends with a global spinning S^2; we present dynamical systems on $\left\{T_pS^2 \mid p \in S^2\right\}$ as well as their rotating orbits, and finally a domestic sink, which is subject to Earth gravity. Section 3 concludes with a summary remark.

2. Analysis

Consider in a reference frame Φ_p a circle $S^1 \subset \mathbb{R}^2_{(x,y)}$ spinning relative to the z-axis with angular frequency $\omega > 0$; i.e.,

$$S^1_\omega = \{(x, y, z) = (\cos\omega t, \sin\omega t, 0) \mid \forall t \in \mathbb{R}\}_{\Phi_p},$$

with (the conserved) angular momentum

$$\mathbf{l} = \mathbf{r}(t) \times \mathbf{P}(t) = \mathbf{e}_z \left(kg \cdot \frac{m^2}{s}\right)$$

for any particle of mass 1 kilogram rotating around the origin O_{Φ_p} in a distance of $r(t)$ meters to O_{Φ_p} with an instantaneous velocity of magnitude $r^{-1}(t)$ meters/second and in the direction of the tangent to the orbit, which includes the present case of a circle with $r(t) \equiv 1$, with

$$\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y \quad \text{at} \quad t = 0.$$

Consider *at* $t = 0$ a rotation of $\mathbf{e}_x$ from $(1, 0, 0)$ to $(\cos\alpha, \sin\alpha, 0)$, $\alpha \in [0, \pi]$; then we have

$$(\cos\alpha, \sin\alpha, 0) \times \mathbf{e}_y = \cos\alpha \cdot \mathbf{e}_z, \tag{1}$$

with $\|\cos\alpha \cdot \mathbf{e}_z\| \leq 1$; equality holds if and only if $\alpha = 0$ or π.

Next consider a rotation of $\mathbf{e}_x$ from $(1,0,0)$ to $(\cos\alpha, 0, \sin\alpha)$, $\alpha \in [0,\pi]$; then we have

$$(\cos\alpha, 0, \sin\alpha) \times \mathbf{e}_y = (-\sin\alpha, 0, \cos\alpha) \equiv \mathbf{e}_{z^*},$$

if we project $\mathbf{e}_{z^*}$ onto $\mathbf{e}_z$, then we obtain (cf. [8])

$$\Pi_z(\mathbf{e}_{z^*}) = (0, 0, \cos\alpha) = \cos\alpha \cdot \mathbf{e}_z. \tag{2}$$

Comparing Equation (1) with Equation (2), we have

$$\Pi_z(\mathbf{e}_{z^*}) = \cos\alpha \cdot \mathbf{e}_z = (\cos\alpha, \sin\alpha, 0) \times \mathbf{e}_y,$$

or the projection of $\mathbf{e}_{z^*}$ onto $\mathbf{e}_z$ is indistinguishable from an altered cross product $(\cos\alpha, \sin\alpha, 0) \times \mathbf{e}_y$ on the tangent plane T to a sphere S^2 at $p \in S^2$, T_pS^2, where

$$S^2 = \left\{ \begin{array}{c} (x,y,z) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \\ \mid \phi \in [0,\pi], \theta \in [0, 2\pi) \end{array} \right\}_{\Phi_q}$$

spins relative to the z-axis in a reference frame Φ_q $\forall q \in \mathbb{R}^3 - r \cdot S^2$, $\forall r \in [0,1]$; i.e., S^2 is observed by any exterior point $q \in \mathbb{R}^3 - B^3$ spinning as

$$S^2_\omega = \left\{ \begin{array}{c} (x,y,z) = (\sin\phi\cos\omega t, \sin\phi\sin\omega t, \cos\phi) \\ \mid \phi \in [0,\pi] \end{array} \right\}_{\Phi_q},$$

on which

$$\begin{aligned} p &= (\sin\phi\cos\omega t, \sin\phi\sin\omega t, \cos\phi) \text{ with} \\ \phi_q &= -\alpha_p \in [0,\pi]. \end{aligned}$$

By the continuous dependence of $(\cos\alpha_p \cdot \mathbf{e}_z)$ on $\alpha_p \equiv -\phi_q$, we see that the spin of S^2_ω yields an angular momentum $\cos\alpha \cdot \mathbf{e}_z$ on T_pS^2. We thus rotate $T_{(0,0,1)}S^2$ to T_pS^2 by an angle ϕ along any given $\theta \in [0, 2\pi)$, i.e.,

$$\begin{aligned} &\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}_{\Phi_q} \\ = &\begin{pmatrix} x \\ \cos\phi \cdot y \\ \sin\phi \cdot y \end{pmatrix}_{\Phi_q} \end{aligned} \tag{3}$$

and project this rotated plane (3) onto T_pS^2 by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \cos\phi \cdot y \\ \sin\phi \cdot y \end{pmatrix}_{\Phi_q} = \begin{pmatrix} x \\ \cos\phi \cdot y \\ 0 \end{pmatrix}_{\Phi_p},$$

so that

$$\begin{pmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{pmatrix}_{\Phi_q} \in T_{(0,0,1)}S^2$$

varies continuously to

$$\begin{pmatrix} \cos\omega t \\ \cos\phi \cdot \sin\omega t \\ 0 \end{pmatrix}_{\Phi_p} \in T_pS^2, \quad \forall t \in \mathbb{R}, \tag{4}$$

with orbit equal to an ellipse (cf. [3])

$$\cos^2\phi \cdot x^2(t) + y^2(t) = \cos^2\phi,$$

and the dynamical system over T_pS^2 being a (velocity) vector field $\mathbf{v}_p(x, y) \equiv (\dot{x}, \dot{y})$ that equals, $\forall\phi \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$,

$$\begin{pmatrix} 0 & -\omega\sec\phi \\ \omega\cos\phi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}_p \tag{5}$$

$$= \begin{pmatrix} -\omega\sec\phi \cdot y \\ \omega\cos\phi \cdot x \end{pmatrix}_p; \tag{6}$$

at $\phi = \frac{\pi}{2}$, the motion (4) degenerates to $\{(\cos\omega t, 0, 0) \mid t \in \mathbb{R}\}$, resulting in the orbit $[-1, 1] \subset \mathbb{R}_x$. Otherwise, $\forall\phi \in \left(\frac{\pi}{2}, \pi\right]$ the rotation on T_pS^2 is clockwise, i.e.,

$$\begin{pmatrix} \cos\omega t \\ \cos\phi \cdot \sin\omega t \\ 0 \end{pmatrix}_{\Phi_p} = \begin{pmatrix} \cos(-\omega)t \\ |\cos\phi| \cdot \sin(-\omega)t \\ 0 \end{pmatrix}_{\Phi_p}.$$

We remark that the essence of the reversal of the rotation directions across $\phi = \frac{\pi}{2}$ is that the spinning of the z-axis in S^2_ω, when viewed at $0 \leq \phi < \frac{\pi}{2}$, is counterclockwise, but when viewed at $\frac{\pi}{2} < \phi \leq \pi$, is clockwise. Equivalently stated, the fact that (x, y)-plane is oriented from x to y (i.e., counterclockwise) is due to the fact that the "outward normal" to the (x, y)-plane is defined to be the z-axis, so that $\det(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) = 1 > 0$, but at $\phi = \pi$ the outward normal to $T_{(0,0,-1)}S^2$ is $-z$ so that $T_{(0,0,-1)}S^2$ is oriented from y to x and thus $\det(\mathbf{e}_y, \mathbf{e}_x, -\mathbf{e}_z) = 1 > 0$, maintaining the consistency of the orientation of S^2. In this connection, we add an incidental note that for the following selected points $p = (x, y, z) \in S^2$, the individual associated outward normal to T_pS^2 and its orientation are:

$$\begin{aligned} p &= (1,0,0) \Longrightarrow \det(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z) = 1, \\ &\quad \text{where the outward normal is } \mathbf{e}_x, \\ p &= (0,1,0) \Longrightarrow \det(\mathbf{e}_y, \mathbf{e}_z, \mathbf{e}_x) = 1, \\ p &= (-1,0,0) \Longrightarrow \det(-\mathbf{e}_x, \mathbf{e}_z, \mathbf{e}_y) = 1, \\ p &= (0,-1,0) \Longrightarrow \det(-\mathbf{e}_y, \mathbf{e}_x, \mathbf{e}_z) = 1, \\ p &= (0,0,1) \Longrightarrow \det(\mathbf{e}_z, \mathbf{e}_x, \mathbf{e}_y) = 1, \text{ and} \\ p &= (0,0,-1) \Longrightarrow \det(-\mathbf{e}_z, \mathbf{e}_y, \mathbf{e}_x) = 1. \end{aligned}$$

That is, for any orientable manifold M such as S^2 there exist exactly two possible choices to assign an orientation to M (with the physical analogue of the "right-hand rule" or the "left-hand rule") by virtue of the fact that there exists exactly two equivalence classes of basis for any vector space, the class with a positive determinant and the class with a negative determinant. Our analysis here has to do with a spinning z-axis and its projection onto the outward normal vectors to T_pS^2 $\forall p \in S^2$.

Now let S^2 be the surface of Earth and $p \in S^2$ be the point of the plug of any household sink. By the gravity of Earth, the above dynamical system (5) is altered to

$$\left\{ \begin{array}{c} \dot{x} = -ax - \omega \sec\phi \cdot y \\ \dot{y} = \omega \cos\phi \cdot x - ay \end{array} \right\},$$

where $a > 0$ is a function of the construct of the sink as well as the contained liquid. Then the matrix

$$\begin{pmatrix} -a & -\omega \sec\phi \\ \omega \cos\phi & -a \end{pmatrix}, \quad \phi \neq \frac{\pi}{2},$$

yields eigenvalues

$$\lambda = -a \pm \omega i,$$

leading to the general solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}_p = c_1 e^{-at} \begin{pmatrix} \cos\omega t \\ \cos\phi \cdot \sin\omega t \end{pmatrix}$$

$$
\begin{aligned}
&+c_2 e^{-at}\begin{pmatrix} \sin\omega t \\ -\cos\phi\cdot\cos\omega t \end{pmatrix}, \\
\forall c_1, c_2 \quad &\in \quad \mathbb{R},
\end{aligned} \tag{7}
$$

exhibiting a spiral motion toward p.

Thus any reference frame T_pS^2 can deduce its $|\phi|$ on S^2_ω by the ratio

$$
|\cos\phi| = \frac{\min\|(x(t), y(t))\|}{\max\|(x(t), y(t))\|}
$$

of the ellipse on T_pS^2. That is, one knows that Earth is spinning as well as the latitude he/she is on simply by observing the sink flows (albeit with the need of a measuring instrument of great precision); i.e., we have contributed to a Gaussian intrinsic geometry.

3. Summary Remark

It has been said that around the Earth equator are tourists' attractions that contrast the sink flow directions between those to the north and those to the south of the equator. Our analysis here shows that unless the sinks are specially constructed, it is practically impossible to make such a demonstration, since the above Equation (7) shows that at $\phi \approx \frac{\pi}{2}$,

$$
\begin{aligned}
c_1 e^{-at}\begin{pmatrix} \cos\omega t \\ \cos\phi\cdot\sin\omega t \end{pmatrix} \quad &\approx \quad c_1 e^{-at}\begin{pmatrix} \cos\omega t \\ (\pm\epsilon)\cdot\sin\omega t \end{pmatrix}, \\
\text{with } \epsilon \quad &\approx \quad 0,
\end{aligned}
$$

indicating that the sinks would have their flows nearly radially drawn to the plug points. Thus our analysis here has clarified this myth. Otherwise we have provided a mathematical derivation of the Coriolis effect, showing the exact effect of a spinning 2-sphere S^2_ω on T_pS^2 $\forall p \in S^2_\omega$: the dynamical system on T_pS^2 with its general solutions and rotating orbits. As mentioned in the Introduction, the Coriolis effect has wide-ranging implications, from nuclear to molecular, to meteorological, to geological, to rotating black holes, and to flight pilots. As such, our results here should also be of relevance to these diverse fields. Mathematically since the Coriolis effect enters into the wave equations in fluid dynamics, our analysis may serve as an added aspect. Overall we have contributed an intrinsic geometry that one can potentially deduce being on a spinning 2-sphere and the existing latitude by the elliptical orbit of a local flow motion. Future studies may extend to relativistic considerations.

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