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ON SOFT SETS AND GENERALIZED TOPOLOGIES IN SENSE OF CSÁSZÁR

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Abstract: The purpose of this paper is to study a generalized topological structure induced by a given soft set (F, X) over a common universe set U, and some properties of a set of parameters (say, cores) by using a base for the generalized topology. In particular, for a given soft set, we show that the set of all core parameters induces a base for the family of all image set in a soft set.

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1. Introduction

In 1999, Molodtsov introduced the concept of soft set [9] to solve complicated problems and various types of uncertainties. He introduced that a soft set is an approximate description of an object precisely consisting of two parts, namely predicate and approximate value set. Maji et al. [7] introduced several operators for soft set theory: equality of two soft sets, subset and superset of soft set, complement of a soft set, null soft set, and absolute soft set. More, new operations [2] in soft set theory were investigated by using the notions defined in [1]. In [3], Császár introduced the notion of generalized topological spaces.

For the goal of this research, we are going to study a generalized topological structure in U which is induced by a given soft set (F, X) over a common universe set U, and some properties of the set of parameters by using a base for the generalized topology. Finally, for a soft set (F, X), we show that the set

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$Co(X)$ of all core parameters in X induces a base for the generalized topology $\mathcal{F}(X)$.

2. Preliminaries

Let U be an initial universe set and E be a collection of all possible parameters with respect to U, where parameters are the characteristics or properties of objects in U. We will call E *the universe set of parameters* with respect to U.

Definition 1 ([9]). A pair (F, A) is called a *soft set* over U if F is a set-valued mapping of A into the set of all subsets of the set U, i.e.,

$$F : A \to P(U).$$

In other words, the soft set is a parameterized family of subsets of the set U. Every set $F(e)$, for $e \in A \subseteq E$, from this family may be considered as the set of e-elements of the soft set (F, A), or as the set of e-approximate elements of the soft set.

Definition 2 ([7]). Let U be an initial universe set and E be a universe set of parameters. Let (F, A) and (G, B) be soft sets over a common universe set U and $A, B \subseteq E$. Then (F, A) is a *subset* of (G, B), denoted by $(F, A)\widetilde{\subseteq}(G, B)$, if

(i) $A \subset B$;
(ii) for all $e \in A$, $F(e) \subseteq G(e)$.

(F, A) equals (G, B), denoted by $(F, A) = (G, B)$, if $(F, A)\widetilde{\subseteq}(G, B)$ and $(G, B)\widetilde{\subseteq}(F, A)$.

Definition 3 ([7]). A soft set (F, A) over U is said to be a *null soft set* denoted by Φ, if $\forall e \in A, F(e) = \emptyset$.

Definition 4 ([7]). A soft set (F, A) over U is said to be an *absolute soft set* denoted by $\tilde{A}$, if $\forall e \in A, F(e) = U$.

Let X be a nonempty set and g be a collection of subsets of X. Then g is called a *generalized topology* [3] (briefly GT) on X iff $\emptyset \in g$ and $G_i \in g$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in g$. We call the pair (X, g) a *generalized* *topological space* (briefly GTS) on X. The elements of g are called *g-open* sets and the complements are called *g-closed* sets.

3. Main Results

First, we show that a given soft set induces a generalized topology in sense of Császár in U:

Definition 5 ([6])**.** Let (F, X) be a soft set over an universe set U. For $A \subseteq X$, we define $F(A) = \cup\{F(a) : a \in A\}$.

Theorem 6. *Let (F, X) be a soft set over a universe set U. Then the family $\mathcal{F}(X) = \{F(A)|A \subseteq X\}$ is a generalized topology in U.*

Proof. (i) Since $F(\emptyset) = \emptyset$, $\emptyset \in \mathcal{F}(X)$.

(ii) For $F(A_i) \in \mathcal{F}(X)$ $(i \in I)$, $\cup_i F(A_i) = \cup_i(\cup_j(F(a_{ij}))) = F(\cup_i(\cup_j(a_{ij}))) = F(\cup_i A_i)$ for $a_{ij} \in A_i$. So, $\cup_i F(A_i) \in \mathcal{F}(X)$. □

Remark 7. (i) The null soft set and the absolute soft set are both generalized topologies in U.

(ii) In general, the family $\mathcal{F}(X)$ for a soft set (F, X) over a universe set U is not a topology as shown the next example.

Example 8. *Let $U = \{x_1, x_2, x_3, x_4\}$ and a parameter set $E = \{e_1, e_2, e_3, e_4\}$. Consider $X = E$ and a soft set (F, X) defined as the following:*

$$F(e_1) = \emptyset;\ F(e_2) = \{x_2\};\ F(e_3) = \{x_1, x_3\};\ F(e_4) = \{x_1, x_2\}.$$

(i) Let $V_1 = F(\{e_3\}) = \{x_1, x_3\}$ and $V_2 = F(\{e_4\}) = \{x_1, x_2\}$. Then $V_1 \cap V_2 = \{x_1\}$ is not in the family $\mathcal{F}(X)$.

(ii) Since $F(X) \neq U$, U is not in the family $\mathcal{F}(X)$.

We will denote $\mathcal{M}_\mu$ the union of all μ-open sets in a GTS (X, μ). In case $\mathcal{M}_\mu = X$, we will call μ *strong* [4].

Definition 9 ([5])**.** Let (F, A) be a soft set over U and A a nonempty parameter set. We say that (F, A) is a *full soft set* if $\cup_{e \in A} F(e) = U$.

Theorem 10. *Let (F, X) be a soft set over a universe set U. Then if (F, X) is full, then the generalized topology $\mathcal{F}(X)$ is strong.*

Proof. Obvious. □

Let U be an initial universe and E be the set of parameters. If a topology τ is given on the universe U, we call U a *topological unverse* [8] with a topology τ (denoted by U_τ). The member of τ is said to be *open* in U.

Let (F, A) be a soft set over a topological universe set U_τ and $A \subseteq E$. We say that (F, A) is *open* [8] if $F(a)$ is open in U for $a \in A$.

Theorem 11. *Let (F, A) be a soft set over a topological universe set U_τ and (F, A) open. Then $\mathcal{F}(X) \subseteq \tau$.*

Proof. Obvious. □

Let (X, μ) be a GTS and $\mathcal{B}$ a family of subsets in X. For each μ-open U set, U is the union of any subset of $\mathcal{B}$. Then we will call $\mathcal{B}$ *a base* for μ [3].

Theorem 12. *Let (F, X) be a soft set over a universe set U. Then the family $\mathcal{B}_X = \{F(a)|a \in X\}$ is a base for the generalized topology $\mathcal{F}(X)$ in U.*

Proof. For any element $O \in \mathcal{F}(X)$, there exists a set $A \subseteq X$ such that $F(A) = O$. Put $\mathcal{S} = \{F(a)|a \in A\}$. Then $\mathcal{S} \subseteq \mathcal{B}_X$ and $\cup\mathcal{S} = O$. So, $\mathcal{B}_X$ is a base for $\mathcal{F}(X)$ in U.

□

We will denote $|X|$ the cardinal number of the set X.

Definition 13. Let (F, X) be a soft set over a universe set U. Then for $a \in X$, a is called *core* if there is no any subset B in X satisfying the following conditions:

(i) For each $b \in B$, $F(a) \neq F(b)$;

(ii) $F(B) = F(a)$.

The class of all core elements in X is denoted by $Co(X)$.

Example 14. *Let $U = \{x_1, x_2, x_3, x_4\}$ and a parameter set $E = \{e_1, e_2, e_3, e_4\}$. Consider $X = E$ and a soft set (F, X) defined as the following:*

$$F(e_1) = \{x_1, x_3\};\ F(e_2) = \{x_2\};\ F(e_3) = \{x_1, x_2, x_3\};\ F(e_4) = \{x_3\}.$$

Then $Co(X) = \{e_1, e_2, e_4\} \neq X$.

From Definition 13, the following lemma is easily obtained:

Lemma 15. *Let* (F, X) *be a soft set over a universe set* U*. If* a *is not core, then there is a subset* B *in* X *such that* $F(B) = F(a)$ *and* $F(a) \neq F(b)$ *for each* $b \in B$.

Theorem 16. *For a soft set* (F, X), $\mathcal{F}(Co(X)) = \{F(c)|c \in Co(X)\}$ *is a base for the generalized topology* $\mathcal{F}(X)$.

Proof. In case $Co(X) = X$: It is obvious.

In case $X \neq Co(X)$: It is sufficient to show that for every $a \in X - Co(X)$, there exists a family $\mathbf{C} \subseteq \mathbf{Co(X)}$ such that $F(a) = \cup_{c\in\mathbf{C}} F(c)$.

For the proof, assume that $|X - Co(X)| = n < |X|$.

First, take one element a in $X - Co(X)$, say a_1. Then by Lemma 15, there exist $a_{1_1}, \cdots, a_{1_l} \in X$ such that $F(a_1) \neq F(a_{1_i})$ and $F(a_1) = \cup F(a_{1_i})$, $i = 1, \cdots, l$ $(l \geq 2)$.

Without the loss of generality, suppose that there exists at least one non-core element in $\{a_{1_1}, \cdots, a_{1_l}\}$. And, take one non-core element in $\{a_{1_1}, \cdots, a_{1_l}\}$, say a_2. Then $F(a_2) \subsetneqq F(a_1)$. Since $a_2 \in X - Co(X)$, there exist $a_{2_1}, \cdots, a_{2_m} \in X$ such that $F(a_2) = \cup F(a_{2_i})$ and $F(a_{2_i}) \neq F(a_2)$ for $i = 1, \cdots, m$ $(i \geq 2)$.

Repeating this way, after the finite number $(n-1)$, we get $a_n \in X - Co(X)$ such that $F(a_1) \supsetneqq F(a_2) \supsetneqq, \cdots \supsetneqq F(a_{n-1}) \supsetneqq F(a_n)$. For $a_n \in X - Co(X)$, there exist $a_{n_1}, \cdots, a_{n_m} \in X$ such that $F(a_n) \neq F(a_{n_i})$ and $F(a_n) = \cup F(a_{n_i})$, $i = 1, \cdots, m$ $(i \geq 2)$.

However, since $F(a_1) \supsetneq F(a_2) \supsetneq, \cdots \supsetneq F(a_n)$ and $|X - Co(X)| = n < |X|$, it must be $a_{n_1}, \cdots, a_{n_m} \in Co(X)$.

For the same reason as $F(a_n)$, in the serial order $(n-1), \cdots, 2, 1$, for each a_i $(i \in \{n-1, \cdots, 2, 1\})$, there exist $a_{i_1}, \cdots, a_{i_k} \in Co(X)$ such that $F(a_i) \neq F(a_{i_j})$ and $F(a_i) = \cup F(a_{i_j})$, $j = 1, \cdots, k$ $(k \geq 2)$.

Hence, for every $a \in X - Co(X)$, $F(a) = \cup F(a_i)$ for core elements a_i $(1 \leq i \leq k)$ in X. Consequently, from Theorem 12, $\mathcal{F}(Co(X))$ is a base for the generalized topology $\mathcal{F}(X)$. □

Finally, we have the following theorem:

Theorem 17. *For a soft set* (F, X), $\mathcal{F}(Co(X)) = \{F(c)|c \in Co(X)\}$ *is the smallest base for the generalized topology* $\mathcal{F}(X)$.

Proof. In the proof of Theorem 16, we show that for every $a \in X - Co(X)$, there exists a family $\mathbf{C} \subseteq \mathbf{Co(X)}$ such that $F(a) = \cup_{c \in \mathbf{C}} F(c)$. From the fact, it follows that $\mathcal{F}(Co(X)) = \{F(c) | c \in Co(X)\}$ is the smallest base for $\mathcal{F}(X)$. □

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