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THE HARMONIC ANALYSIS ASSOCIATED TO THE CHEREDNIK-TRIMÈCHE'S TRANSMUTATION OPERATORS ON $\mathbb{R}^d$

Khalifa Trimèche

Faculty of Sciences of Tunis
Department of Mathematics
CAMPUS, 2092 Tunis, TUNISIA

Abstract: We consider in this paper two Cherednik operators T_j^k, T_j^l, $j = 1, 2, 3, ..., d$, on $\mathbb{R}^d$, associated to the multiplicity functions k, l. First we define and study in this paper the Cherednik-Trimèche's transmutation operator U_{kl} and its dual ${}^tU_{kl}$. Next we study the Harmonic Analysis associated to these operators.

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1. Introduction

In [1] I. Cherednik has introduced a family of differential-difference operators that nowadays bear his name. These operators play a crucial role in the theory of Heckman-Opdam's hypergeometric functions, which generalize the theory of Harish-Chandra's spherical functions on Riemmann symetric spaces (see [2,3,4]).

We consider in this paper two Cherednik operators T_j^k and $T_j^l, j = 1, 2, ..., d$, on $\mathbb{R}^d$, associated to the multiplicity functions $k, l \in [0, \infty)$.

By using the Harmonic Analysis associated to the Cherednik operators (see [2,3,4,5,6,7]), given in Sections 2,3,4,5,6, we define and study in the other

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sections the Cherednik-Trimèche's transmutation operator U_{kl} and its dual ${}^tU_{kl}$, the Cherednik-Trimèche's translation operator τ_x^{kl} and its dual ${}^t\tau_x^{kl}$, the Cherednik-Trimèche's convolution product, the Cherednik-Trimèche's heat kernel $p_t^{kl}(x,y)$.

2. The Cherednik operators and their eigenfunctions

We consider $\mathbb{R}^d$ with the standard basis $\{e_j; j = 1, 2, ..., d\}$ and the inner product $\langle ., . \rangle$ for which this basis is orthonormal.

2.1. The root system

Let $\alpha \in \mathbb{R}^d \backslash \{0\}$ and $\check{\alpha} = \dfrac{2}{\|\alpha\|^2}\alpha$. We denote by

$$r_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha, \quad x \in \mathbb{R}^d, \tag{2.1}$$

the reflection on the hyperplan $H_\alpha \subset \mathbb{R}^d$ orthogonal to α. For $d = 1$, we take $\alpha = 2$.

A finite set $\mathcal{R} \subset \mathbb{R}^d \backslash \{0\}$ is called a root system if $r_\alpha \mathcal{R} = \mathcal{R}$, for all $\alpha \in \mathcal{R}$. For a given $\beta \in \mathbb{R}^d \backslash \cup_{\alpha \in \mathcal{R}} H_\alpha$, we fix the positive subsystem $\mathcal{R}_+ = \{\alpha \in \mathcal{R}, \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in \mathcal{R}$ either $\alpha \in \mathcal{R}_+$ or $-\alpha \in \mathcal{R}_+$.

The reflections $r_\alpha, \alpha \in \mathcal{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathcal{R}$. Let $\mathbb{R}^d_{reg} = \mathbb{R}^d \backslash \cup_{\alpha \in \mathcal{R}} H_\alpha$ be the set of regular elements in $\mathbb{R}^d$.

A function $k : \mathcal{R} \to [0, +\infty[$ is called a multiplicity function, if it is invariant under the action of the reflection group W. We introduce the index

$$\gamma = \gamma(\mathcal{R}) = \sum_{\alpha \in \mathcal{R}_+} k(\alpha). \tag{2.2}$$

2.2. The Cherednik operators

The Cherednik operators $T_j^k, j = 1, 2, ..., d$, on $\mathbb{R}^d$ associated with the reflection group W and the multiplicity function k, are defined for f of class C^1 on $\mathbb{R}^d$ and $x \in \mathbb{R}^d_{reg}$ by

$$T_j^k f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} \frac{k(\alpha)\alpha^j}{1 - e^{-\langle \alpha, x \rangle}} \{f(x) - f(r_\alpha x)\} - \rho_j^k f(x), \tag{2.3}$$

where

$$\rho_j^k = \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha^j, \text{ and } \alpha^j = \langle \alpha, e_j\rangle. \tag{2.4}$$

The Cherednik operators form a commutative system of differential-difference operators.

For f of class C^1 on $\mathbb{R}^d$ with compact support and g of class C^1 on $\mathbb{R}^d$, we have for $j = 1, 2, .., d$:

$$\int_{\mathbb{R}^d} T_j^k f(x)g(x)\mathcal{A}_k(x)dx = -\int_{\mathbb{R}^d} f(x)(T_j^k + S_j^k)g(x)\mathcal{A}_k(x)dx, \tag{2.5}$$

with $\mathcal{A}_k$ the weight function given by

$$\forall\, x \in \mathbb{R}^d,\ \ \mathcal{A}_k(x) = \prod_{\alpha\in\mathcal{R}_+} |2\sinh\langle\frac{\alpha}{2}, x\rangle|^{2k(\alpha)}, \tag{2.6}$$

which is W-invariant and

$$\forall\, x \in \mathbb{R}^d, S_j^k g(x) = \sum_{\alpha\in\mathcal{R}_+} k(\alpha)\alpha^j g(r_\alpha x). \tag{2.7}$$

Example 2.1. We consider for $d = 1$, the root system $\mathcal{R} = \{\pm\alpha, \pm 2\alpha\}$, with $\alpha = 2$. Here $\mathcal{R}_+ = \{\alpha, 2\alpha\}$, and the reflection group is $W = \mathbb{Z}_2$. We denote by k the multiplicity function. The Cherednik operator T_1^k is defined for f of class C^1 on $\mathbb{R}$, and x in $\mathbb{R}\backslash\{0\}$ by

$$T_1^k f(x) = \frac{d}{dx}f(x) + \left(\frac{2k(\alpha)}{1-e^{-2x}} + \frac{4k(2\alpha)}{1-e^{-4x}}\right)(f(x) - f(-x)) - \rho^k f(x) \tag{2.8}$$

with $\rho^k = k(\alpha) + 2k(2\alpha)$.

If we put $k_1 = k(\alpha)+k(2\alpha)$, $k_2 = k(2\alpha)$, the operator T_1^k takes the following form

$$T_1^k f(x) = \frac{d}{dx}f(x) + (k_1\coth(x) + k_2\tanh(x))\,(f(x) - f(-x)) - \rho^k f(-x), \tag{2.9}$$

with $\rho^k = k_1 + k_2$.

2.3. The Opdam-Cherednik's kernel

We denote by $G_\lambda^k, \lambda \in \mathbb{C}^d$, the eigenfunction of the operators $T_j^k, j = 1, 2, .., d$. It is the unique analytic function on $\mathbb{R}^d$ which satisfies the differential-difference system

$$\begin{cases} T_j^k G_\lambda^k(x) &= i\lambda_j G_\lambda^k(x), \quad j=1,2,..,d,\ x\in\mathbb{R}^d, \\ G_\lambda^k(0) &= 1. \end{cases} \tag{2.10}$$

It is called the Opdam-Cherednik's kernel.

Remarks 2.1. For $k=0$, we have for all $x\in\mathbb{R}^d$, $G_\lambda^k(x)=e^{i\langle\lambda,x\rangle}$.

The functions G_λ^k possess the following properties:

i) For all $\lambda\in\mathbb{C}^d$, the function $x\mapsto G_\lambda^k(x)$ is of class C^∞ on $\mathbb{R}^d$.

ii) For all $x\in\mathbb{R}^d$, the function $\lambda\mapsto G_\lambda^k(x)$ is entire on $\mathbb{C}^d$.

iii) For all $x\in\mathbb{R}^d$ and $\lambda\in\mathbb{C}^d$, we have

$$\overline{G_\lambda^k(x)} = G_{-\bar\lambda}^k(x). \tag{2.11}$$

iv) For all $x\in\mathbb{R}^d$ and $\lambda\in\mathbb{C}^d$, we have

$$|G_\lambda^k(x)| \leq G_{i\Im m(\lambda)}^k(x). \tag{2.12}$$

v) For all $x\in\mathbb{R}^d$ and $\lambda\in\mathbb{R}^d$, we have

$$|G_\lambda^k(x)| \leq |W|^{1/2}. \tag{2.13}$$

vi) Let p and q be polynomials of degree m and n. Then, there exists a positive constant M such that for all $\lambda\in\mathbb{C}^d$ and $x\in\mathbb{R}^d$, we have

$$|p(\frac{\partial}{\partial\lambda})q(\frac{\partial}{\partial x})G_\lambda^k(x)| \leq M(1+\|x\|)^m(1+\|\lambda\|)^n F_0^k(x)e^{-\max_{w\in W} Im\langle w\lambda,x\rangle}, \tag{2.14}$$

where

$$\forall x\in\mathbb{R}^d,\quad F_0^k(x)=\frac{1}{|W|}\sum_{w\in W} G_0^k(wx). \tag{2.15}$$

Example 2.2. We consider for $d=1$, the root system $\mathcal{R}=\{\pm\alpha,\pm2\alpha\}$, with $\alpha=2$, the reflection group $W=\mathbb{Z}_2$, the multiplicity function k, the parameters k_1,k_2, and the Cherednik operator T_1^k, given in Example 2.1.

The Opdam-Cherednik's kernel is given by

$$\forall x\in\mathbb{R}, \forall\lambda\in\mathbb{C},\ G_\lambda^k(x)=\varphi_\lambda^{(k_1-\frac{1}{2},k_2-\frac{1}{2})}(x)+\frac{1}{i\lambda-\rho^k}\frac{d}{dx}\varphi_\lambda^{(k_1-\frac{1}{2},k_2-\frac{1}{2})}(x), \tag{2.16}$$

where $\varphi_\lambda^{(a,b)}(x)$ is the Jacobi function of index (a,b) given by

$$\varphi_\lambda^{(a,b)}(x) = \ {}_2F_1(\frac{1}{2}(\rho+i\lambda), \frac{1}{2}(\rho-i\lambda); \alpha+1; -(\sinh(x))^2),$$

with ${}_2F_1$ the hypergeometric function of Gauss and $\rho = a+b+1$.

3. The intertwining operator V_k and its dual ${}^tV_k$

Notation. We denote by

- $\mathcal{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$. Its topology is defined by the semi-norms

$$q_{n,K}(\varphi) = \sup_{\substack{|\mu|\le n\\ x\in K}} |D^\mu \varphi(x)|,$$

where K is a compact of $\mathbb{R}^d$, $n \in \mathbb{N}$, and

$$D^\mu = \frac{\partial^{|\mu|}}{\partial x_1^{\mu_1}...\partial x_d^{\mu_d}}\,, \quad \mu = (\mu_1, ..., \mu_d) \in \mathbb{N}^d, \quad |\mu| = \sum_{i=1}^d \mu_i.$$

- $\mathcal{D}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$, with compact support. We have

$$\mathcal{D}(\mathbb{R}^d) = \bigcup_{a>0} \mathcal{D}_a(\mathbb{R}^d),$$

where $\mathcal{D}_a(\mathbb{R}^d)$ is the space of C^∞-functions on $\mathbb{R}^d$ with support in the closed ball $B(0,a)$ of center 0 and radius a. The topology of $\mathcal{D}_a(\mathbb{R}^d)$ is defined by the semi-norms

$$p_n(\psi) = \sup_{\substack{0\le|\mu|\le n\\ x\in B(0,a)}} |D^\mu \varphi(x)|, \quad n \in \mathbb{N}.$$

The space $\mathcal{D}(\mathbb{R}^d)$ is equipped with the inductive limit topology.

- $\mathcal{S}(\mathbb{R}^d)$ the classical Schwartz space on $\mathbb{R}^d$. Its topology is defined by the semi-norms

$$Q_{\ell,n}(f) = \sup_{\substack{0\le|\mu|\le n\\ x\in\mathbb{R}^d}} (1+\|x\|)^\ell |D^\mu f(x)|, \quad n, l \in \mathbb{N}.$$

- $\mathcal{S}_2(\mathbb{R}^d)$ the generalized Schwartz space of C^∞-functions on $\mathbb{R}^d$ such that for $\ell, n \in \mathbb{N}$, we have

$$P_{n,l}(f) = \sup_{\substack{0\le|\mu|\le n\\ x\in\mathbb{R}^d}} (1+\|x\|)^\ell (F_0^k(x))^{-1} |D^\mu f(x)| < +\infty,$$

where $F_0^k(x)$ is the function given by the relation (2.15). It is topologized by means of the semi-norms $P_{n,l},\ n,l \in \mathbb{N}$.

- $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$. It is the topological dual of $\mathcal{D}(\mathbb{R}^d)$.

- $\mathcal{E}'(\mathbb{R}^d)$ the space of distributions on $\mathbb{R}^d$ with compact support. It is the topological dual of $\mathcal{E}(\mathbb{R}^d)$.

Definition 3.1. i) The intertwining operator V_k is the unique linear topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself satisfying the transmutations relations

$$\forall x \in \mathbb{R}^d, T_j^k V_k(g)(x) = V_k(\frac{\partial}{\partial y_j} g)(x), j = 1, 2..., d, \tag{3.1}$$

and the relation

$$V_k(g)(0) = g(0). \tag{3.2}$$

ii) The dual ${}^tV_k$ of the operator V_k is defined by the following duality relation

$$\int_{\mathbb{R}^d} {}^tV_k(f)(y)g(y)dy = \int_{\mathbb{R}^d} V_k(g)(x)f(x)\mathcal{A}_k(x)dx. \tag{3.3}$$

with f in $\mathcal{D}(\mathbb{R}^d)$ and g in $\mathcal{E}(\mathbb{R}^d)$.

Proposition 3.1. *i) The operator ${}^tV_k$ is a linear topological isomorphism from*

-$\mathcal{D}(\mathbb{R}^d)$ onto itself,

-$\mathcal{S}_2(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$,

satisfying the transmutation relations

$$\forall y \in \mathbb{R}^d,\ \ {}^tV_k((T_j^k + S_j^k)f)(y) = \frac{\partial}{\partial y} {}^tV_k(f)(y), \tag{3.4}$$

where S_j^k is the operator on $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) given by the relation (2.7).

ii) The dual ${}^tV_k^{-1}$ of the operator V_k^{-1} satisfies the following duality relation

$$\int_{\mathbb{R}^d} {}^tV_k^{-1}(f)(y)g(y)\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} V_k^{-1}(g)(x)f(x)dx, \tag{3.5}$$

with f in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) and g in $\mathcal{E}(\mathbb{R}^d)$.

iii) For all f in $\mathcal{D}(\mathbb{R}^d)$ we have

$$Supp f \subset B(0,a) \Rightarrow Supp {}^tV_k(f) \subset B(0,a), \tag{3.6}$$

where $B(0,a)$ is the closed ball of center 0 and radius $a > 0$.

Remarks 3.1. From the relations (2.3),(3.1),(3.4) we deduce that the operators V_0 and ${}^tV_0$ are the identity operators.

4. The hypergeometric Fourier transform associated with the Cherednik operators

Notation. For $a > 0$, we denote by $PW(\mathbb{C}^d)_a$ (resp. $\mathcal{PW}(\mathbb{C}^d)_a$) the spaces of functions h which are entire on $\mathbb{C}^d$ and satisfying

$\forall\, m \in \mathbb{N}, s_m(h) = \sup_{\lambda\in\mathbb{C}^d}(1 + \|\lambda\|)^m e^{a\|Im\lambda\|}|h(\lambda)| < \infty.$
(resp. $\exists m \in \mathbb{N}, \sigma_m(h) = \sup_{\lambda\in\mathbb{C}^d}(1 + \|\lambda\|)^{-m} e^{a\|Im\lambda\|}|h(\lambda)| < \infty.$)

Their topologies is given by the semi-norms $s_m, m \in \mathbb{N}$, (resp. $\sigma_m, m \in \mathbb{N}$).

- We consider the spaces $PW(\mathbb{C}^d)$ (resp. $\mathcal{PW}(\mathbb{C}^d)$) of the entire functions on $\mathbb{C}^d$ which are rapidly decreasing (resp. slowly increasing) and of exponential type. We have

$$PW(\mathbb{C}^d) = \cup_{a>0}PW(\mathbb{C}^d)_a \text{ (resp. } \mathcal{PW}(\mathbb{C}^d) = \cup_{a>0}\mathcal{PW}(\mathbb{C}^d)_a.$$

They are equipped with the inductive limit topology.

Definition 4.1. The hypergeometric Fourier transform $\mathcal{H}^k$ is defined for all function f in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by

$$\forall\, \lambda \in \mathbb{C}^d, \quad \mathcal{H}^k(f)(\lambda) = \int_{\mathbb{R}^d} f(x)G^k_\lambda(x)\mathcal{A}_k(x)dx. \tag{4.1}$$

Theorem 4.1. *The hypergeometric Fourier transform $\mathcal{H}^k$ is a topological isomorphism from*

- $\mathcal{D}(\mathbb{R}^d)$ onto $PW(\mathbb{C}^d)$,

- $\mathcal{S}_2(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$.

The inverse transform $(\mathcal{H}^k)^{-1}$ is given by

$$\forall\, x \in \mathbb{R}^d, (\mathcal{H}^k)^{-1}(h)(x) = \int_{\mathbb{R}^d} h(\lambda)G^k_\lambda(-x)\mathcal{C}_k(\lambda)d\lambda, \tag{4.2}$$

where for all $\lambda \in \mathbb{C}^d$,

- For $k \in (0, \infty)$

$$\mathcal{C}_k(\lambda) = c \prod_{\alpha\in\mathcal{R}_+} \frac{\Gamma(-i\langle\lambda, \check{\alpha}\rangle + \frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha))\Gamma(i\langle\lambda, \check{\alpha}\rangle + k(\frac{\alpha}{2}) + k(\alpha) + 1)}{\Gamma(-i\langle\lambda, \check{\alpha}\rangle + \frac{1}{2}\Gamma(\frac{\alpha}{2}))\Gamma(i\langle\lambda, \check{\alpha}\rangle + \frac{1}{2}\Gamma(\frac{\alpha}{2}) + 1)}, \tag{4.3}$$

with c a normalising constant.

- *For $k = 0$,*

$$\mathcal{C}_k(\lambda) = 1. \tag{4.4}$$

Definition 4.2. The hypergeometric Fourier transform $\mathcal{H}^k$ is defined for S in $\mathcal{E}'(\mathbb{R}^d)$ by

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{H}^k(S)(\lambda) = \langle S, G^k_\lambda \rangle. \tag{4.5}$$

Theorem 4.2. *The transform $\mathcal{H}^k$ is a topological isomorphism from $\mathcal{E}'(\mathbb{R}^d)$ onto $\mathcal{PW}(\mathbb{C}^d)$.*

5. The hypergeometric translation operator and its dual and the hypergeometric convolution product associated with the Cherednik operators

5.1. The hypergeometric translation operator and its dual

Definition 5.1. The hypergeometric translation operator $\mathcal{T}^k_x$, $x \in \mathbb{R}^d$, is defined on $\mathcal{E}(\mathbb{R}^d)$ by

$$\forall\, y \in \mathbb{R}^d, \mathcal{T}^k_x(f)(y) = (V_k)_x((V_k)_y[V_k^{-1}(f)(x+y)]. \tag{5.1}$$

Proposition 5.1. *The operator $\mathcal{T}^k_x$, $x \in \mathbb{R}^d$, satisfies the following properties:*

i) For all $x \in \mathbb{R}^d$, the operator $\mathcal{T}^k_x$, is continuous from $\mathcal{E}(\mathbb{R}^d)$ into itself.

ii) For all f in $\mathcal{E}(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$, we have

$$\mathcal{T}^k_x(f)(0) = f(x), \text{ and } \mathcal{T}^k_x(f)(y) = \mathcal{T}^k_y(f)(x). \tag{5.2}$$

iii) For all $x, y \in \mathbb{R}^d$, and $\lambda \in \mathbb{C}^d$, we have the product formula

$$\mathcal{T}^k_x(G^k_\lambda)(y) = G^k_\lambda(x) G^k_\lambda(y), \tag{5.3}$$

where G^k_λ, the opdam-cherednick kernel given by (2.10).

Definition 5.2. For each $x \in \mathbb{R}^d$, the dual of the hypergeometric translation operator $\mathcal{T}^k_x$, is the operator ${}^t\mathcal{T}^k_x$ defined on $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by

$$\forall\, y \in \mathbb{R}^d, {}^t\mathcal{T}^k_x(f)(y) = (V_k)_x({}^tV_k^{-1})_y[{}^tV_k(f)(y-x)]. \tag{5.4}$$

Proposition 5.2. *We give in the following the properties of the operator* ${}^t\mathcal{T}_x^k$*:*

i) For all $x \in \mathbb{R}^d$*, the operator* ${}^t\mathcal{T}_x^k$ *is continuous from*

- $\mathcal{D}(\mathbb{R}^d)$ *into itself,*
- $\mathcal{S}_2(\mathbb{R}^d)$ *into itself.*

ii) The operator ${}^t\mathcal{T}_x^k$*,* $x \in \mathbb{R}^d$*, is related to the operator* $\mathcal{T}_x^k$*,* $x \in \mathbb{R}^d$*, by the following relation*

$$\int_{\mathbb{R}^d} \mathcal{T}_x^k(g)(y) f(y) \mathcal{A}_k(y) dy = \int_{\mathbb{R}^d} g(z) {}^t\mathcal{T}_x^k(f)(z) \mathcal{A}_k(z) dz, \tag{5.5}$$

with g *in* $\mathcal{E}(\mathbb{R}^d)$*, and* f *in* $\mathcal{D}(\mathbb{R}^d)$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)$*).*

iii) For all f *in* $\mathcal{D}(\mathbb{R}^d)$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)$*) and* $x \in \mathbb{R}^d$*, we have*

$$\forall \lambda \in \mathbb{R}^d, \mathcal{H}^k({}^t\mathcal{T}_x^k(f))(\lambda) = G_\lambda^k(x) \mathcal{H}^k(f)(\lambda). \tag{5.6}$$

Thus from the relation (4.2) we have

$$\forall y \in \mathbb{R}^d, {}^t\mathcal{T}_x^k(f)(y) = \int_{\mathbb{R}^d} G_\lambda^k(x) G_\lambda^k(-y) \mathcal{H}^k(f)(\lambda) \mathcal{C}_k(\lambda) d\lambda. \tag{5.7}$$

iv) For all f *in* $\mathcal{D}(\mathbb{R}^d)$ *with support in the closed ball* $B(0, a)$ *of center* o *and radius* $a > 0$*, and* $x \in \mathbb{R}^d$*, we have*

$$Supp {}^t\mathcal{T}_x^k(f) \subset B(0, a + \|x\|). \tag{5.8}$$

5.2. The hypergeometric convolution product

Definition 5.3. The hypergeometric convolution product $f *_k g$ of the functions f, g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) is defined by

$$\forall\, x \in \mathbb{R}^d, f *_k g(x) = \int_{\mathbb{R}^d} {}^t\mathcal{T}_x^k(f)(y) g(y) \mathcal{A}_k(y) dy. \tag{5.9}$$

Proposition 5.3. *The convolution product* $*_k$ *satisfies the following properties:*

i) For all f, g *in* $\mathcal{D}(\mathbb{R}^d)$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)$*) the function* $f *_k g$ *belongs to* $\mathcal{D}(\mathbb{R}^d)$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)$*).*

ii) For all f, g *in* $\mathcal{D}(\mathbb{R}^d)$ *(resp.* $\mathcal{S}_2(\mathbb{R}^d)$*), we have*

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^k(f *_k g)(\lambda) = \mathcal{H}^k(f)(\lambda) \mathcal{H}^k(g)(\lambda). \tag{5.10}$$

iii) This convolution product is commutative and associative.
iv) For all f, g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$), we have

$$ {}^tV_k(f *_k g) = {}^tV_k(f) * {}^tV_k(g), \tag{5.11} $$

where $*$ is the classical convolution product on $\mathbb{R}^d$.

6. The heat kernel associated to the Cherednik operators

Definition 6.1. Let $t > 0$. The heat kernel $p_t^k(x,y)$ associated with the Cherednik operators, is defined for all $x, y \in \mathbb{R}^d$, by

$$ p_t^k(x,y) = \int_{\mathbb{R}^d} e^{-t(\|\lambda\|^2+\|\rho^k\|^2)} G_\lambda^k(x) G_\lambda^k(-y) \mathcal{C}_k(\lambda) d\lambda. \tag{6.1} $$

Notation. We denote by:
- H_k the heat operator associated with the Cherednik operator given by

$$ H_k = \mathcal{L}_k - \frac{\partial}{\partial t} - \|\rho^k\|^2, \tag{6.2} $$

where $\mathcal{L}_k$ is the Heckman-Opdam Laplacian defined for f of class C^2 on $\mathbb{R}^d$ by

$$ \mathcal{L}_k f = \sum_{j=1}^d (T_j^k)^2 (f). \tag{6.3} $$

- $E_t^k, t > 0$, the fundamental solution of the operator H_k given by

$$ \forall\, x \in \mathbb{R}^d, \quad E_t^k(x) = p_t^k(x,0). \tag{6.4} $$

Proposition 6.1. *i) For all $t > 0$, the function E_t^k belongs to $\mathcal{S}_2(\mathbb{R}^d)$.*
ii) For all $t > 0$, we have

$$ \forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^k(E_t^k)(\lambda) = e^{-t(\|\lambda\|^2+\|\rho^k\|^2)}. \tag{6.5} $$

iii) The function $(x,t) \to E_t^k(x)$ is strictly positive on $\mathbb{R}^d \times (0,\infty)$.
iv) For all $t > 0$, we have

$$ \int_{\mathbb{R}^d} E_t^k(x) \mathcal{A}_k(x) dx = 1. \tag{6.6} $$

v) We have

$$ H_k E_t^k(x) = 0, \quad \text{on } \mathbb{R}^d \times (0,\infty). \tag{6.7} $$

Proposition 6.2. *i) For all $t > 0$ and $x \in \mathbb{R}^d$, the function $y \to p_t^k(x, y)$ belongs to $\mathcal{S}_2(\mathbb{R}^d)$.*

ii) For all $t > 0$ and $x, y \in \mathbb{R}^d$, we have

$$p_t^k(x, y) = {}^t\mathcal{T}_x^k(E_t^k)(y). \tag{6.8}$$

iii) The function $p_t^k(x, y)$ is strictly positive on $\mathbb{R}^d \times \mathbb{R}^d \times (0, \infty)$.

iv) For all $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} p_t^k(x, y)\mathcal{A}_k(y)dy = 1. \tag{6.9}$$

v) For all $y \in \mathbb{R}^d$, the function $(x, t) \to p_t^k(x, y)$ satisfies

$$H_k p_t^k(x, y) = 0, \text{ on } \mathbb{R}^d \times (0, \infty). \tag{6.10}$$

Remark 6.1. To give the new results of this paper, we consider a second multiplicity function $l : \mathcal{R} \to [0, \infty[$. We consider also the cherednik operators T_j^l, $j = 1, 2, ..., d$, the Opdam-cherednik's kernel G_λ^l, the transmutation operators V_l and its dual ${}^tV_l$, the hypergeometric Fourier transform $\mathcal{H}^l$, the hypergeometric translation operator $\mathcal{T}_x^l$ and its dual ${}^t\mathcal{T}_x^l$, the fundamental solution E_t^l of the operator H_l and of the heat kernel $p_t^l(x.y)$.

7. The Cherednik-Trimèche's transmutation operator U_{kl} and its dual ${}^tU_{kl}$

Definition 7.1. The Cherednik-Trimèche's transmutation operator U_{kl} is defined on $\mathcal{E}(\mathbb{R}^d)$ by

$$\forall\, x \in \mathbb{R}^d, U_{kl}(f)(x) = V_k \circ V_l^{-1}(f)(x). \tag{7.1}$$

By using the properties of the transmutation operators V_k and V_l given in Section 3, we obtain the following properties of the operator U_{kl}.

Theorem 7.1. *i) For $k = l$, we have*

$$U_{kl} = Id. \tag{7.2}$$

ii) The operator U_{kl} is the unique topological isomorphism from $\mathcal{E}(\mathbb{R}^d)$ onto itself satisfying the condition

$$U_{kl}(f)(0) = f(0). \tag{7.3}$$

iii) The inverse operator U_{kl}^{-1} is given for f in $\mathcal{E}(\mathbb{R}^d)$ by

$$\forall\, x \in \mathbb{R}^d, \quad U_{kl}^{-1}(f)(x) = V_l \circ V_k^{-1}(f)(x) = U_{lk}(f)(x). \tag{7.4}$$

iv) The operator U_{kl} satisfies for all f in $\mathcal{E}(\mathbb{R}^d)$ the following transmutation relations

$$\forall\, x \in \mathbb{R}^d, \quad T_j^k(U_{kl}(f))(x) = U_{kl}(T_j^l(f))(x), \; j = 1, 2, ..., d. \tag{7.5}$$

v) We have

$$\forall\, \lambda \in \mathbb{C}^d, \forall\, x \in \mathbb{R}^d, U_{kl}(G_\lambda^l)(x) = G_\lambda^k(x). \tag{7.6}$$

vi) We have

$$U_{kl}(1) = 1. \tag{7.7}$$

Definition 7.2. The dual of the Cherednik-Trimèche's transmutation operator U_{kl} is the operator ${}^tU_{kl}$ defined on $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by

$$\forall\, y \in \mathbb{R}^d, {}^tU_{kl}(g)(y) = {}^tV_l^{-1} \circ {}^tV_k(g)(y). \tag{7.8}$$

The properties of the operator ${}^tV_k$ and ${}^tV_l$, given in Section 3, imply the following properties of the operator ${}^tU_{kl}$.

Theorem 7.2. *i) For $k = l$, we have*

$${}^tU_{kl} = Id. \tag{7.9}$$

ii) The operator ${}^tU_{kl}$ is a topological isomorphism from $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) onto itself. iii) The inverse operator ${}^tU_{kl}^{-1}$ is given for g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by

$$\forall\, y \in \mathbb{R}^d, \quad {}^tU_{kl}^{-1}(g)(y) = {}^tV_k^{-1} \circ {}^tV_l(g)(y) = {}^tU_{lk}(g)(y). \tag{7.10}$$

iv) The operator ${}^tU_{kl}$ satisfies for all g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) the following transmutation relation

$$\forall\, y \in \mathbb{R}^d, \quad {}^tU_{kl}((T_j^k + S_j^k)f))(y) = (T_j^l + S_j^l)({}^tU_{kl}(g)(y). \tag{7.11}$$

Proposition 7.1. *The operators U_{kl} and ${}^tU_{kl}$ are related for f in $\mathcal{E}(\mathbb{R}^d)$ and g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by the following duality relation*

$$\int_{\mathbb{R}^d} U_{kl}(f)(x)g(x)\mathcal{A}_k(x)dx = \int_{\mathbb{R}^d} {}^tU_{kl}(g)(y)f(y)\mathcal{A}_l(y)dy. \tag{7.12}$$

Corollary 7.1. *The operators ${}^tU_{kl}$ satisfys for all g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$ the following expression:*

$$\forall\, y \in \mathbb{R}^d, \quad {}^tU_{kl}(g)(y) = (\mathcal{H}^l)^{-1} \circ \mathcal{H}^k(g)(y). \tag{7.13}$$

Proof. From the relations (7.12),(7.6), we have

$$\forall\, \lambda \in \mathbb{R}^d, \mathcal{H}^k(g)(\lambda) = \mathcal{H}^l({}^tU_{kl}(g))(\lambda).$$

We deduce (7.13) from this relation and Theorem 4.2. □

8. The Cherednik-Trimèche's translation operator and its dual

Definition 8.1. For $x, y \in \mathbb{R}^d$, the Cherednik-Trimèche's translation operator $\mathcal{T}_x^{kl}$ is defined for all f in $\mathcal{E}(\mathbb{R}^d)$ by

$$\mathcal{T}_x^{kl}(f)(y) = (V_l)_x((V_k)_y[V_k^{-1}(f)(x+y)]. \tag{8.1}$$

Definition 8.2. For all $x, y \in \mathbb{R}^d$, the dual of the Cherednik-Trimèche's translation operator ${}^t\mathcal{T}_x^{kl}$ is defined for all g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by

$${}^t\mathcal{T}_x^{kl}(g)(z) = (V_l)_x({}^tV_k^{-1})_z[{}^tV_k(g)(z-x)]. \tag{8.2}$$

From the properties of the operator V_l, V_k and ${}^tV_k$ given in Section 3, we obtain the following properties of the operators $\mathcal{T}_x^{kl}$ and ${}^t\mathcal{T}_x^{kl}$.

Proposition 8.1. *i) For all f in $\mathcal{E}(\mathbb{R}^d)$, the function $\mathcal{T}_x^{kl}(f)(y)$ is of class C^∞ on $\mathbb{R}^d$ with respect to the variables x and y. ii) For all g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) the function ${}^t\mathcal{T}_x^{kl}(g)(z)$ is of class $\mathbb{R}^d$ with respect to the variables x, and belongs to $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) with respect to the variable z. More precisely if the support of g is contained in the ball of center 0 and radius $a > 0$, and $x \in \mathbb{R}^d$, we have have*

$$Supp\, {}^t\mathcal{T}_x^{kl}(g)(z) \subset B(0, a + \|x\|), \tag{8.3}$$

iii) For all f in $\mathcal{E}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have

$$\mathcal{T}_x^{kl}(f)(0) = V_l \circ V_k^{-1}(f)(x). \tag{8.4}$$

iv) For all g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) and $z \in \mathbb{R}^d$, we have

$$^t\mathcal{T}_0^{kl}(g)(z) = g(z). \tag{8.5}$$

v) For all $x, y \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}^d$, we have

$$\mathcal{T}_x^{kl}(G_\lambda^k)(y) = G_\lambda^l(x)G_\lambda^k(y). \tag{8.6}$$

Proposition 8.2. *The operators $\mathcal{T}_x^{kl}$ and $^t\mathcal{T}_x^{kl}$ are related for all f in $\mathcal{E}(\mathbb{R}^d)$ and g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) by the following duality relation:*

$$\int_{\mathbb{R}^d} \mathcal{T}_x^{kl}(f)(y)g(y)\mathcal{A}_k(y)dy = \int_{\mathbb{R}^d} f(z)^t\mathcal{T}_x^{kl}(g)(z)\mathcal{A}_k(z)dz. \tag{8.7}$$

Corollary 8.1. *For all $x \in \mathbb{R}^d$, the dual $^t\mathcal{T}_x^{kl}$ of the Cherednik-Trimèche's translation operator, satisfies for all g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) the following relations:*

$$i) \quad \forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{H}^k(^t\mathcal{T}_x^{kl}(g))(\lambda) = G_\lambda^l(x)\mathcal{H}^k(g)(\lambda), \tag{8.8}$$

$$ii) \quad \forall\, z \in \mathbb{R}^d, \quad ^t\mathcal{T}_x^{kl}(g)(z) = \int_{\mathbb{R}^d} G_\lambda^l(x)G_\lambda^k(-z)\mathcal{H}^k(g)(\lambda)\mathcal{C}_k(\lambda)d\lambda. \tag{8.9}$$

Proof. i) By applying thr relation (8.7) with $f(z) = G_\lambda^k(z), z \in \mathbb{R}^d$, $\lambda \in \mathbb{R}^d$, we obtain

$$\int_{\mathbb{R}^d} G_\lambda^k(z)^t\mathcal{T}_x^{kl}(g)(z)\mathcal{A}_k(z)dz = \int_{\mathbb{R}^d} \mathcal{T}_x^{kl}(G_\lambda^k)(y)g(y)\mathcal{A}_k(y)dy.$$

We deduce relation (8.8) from this relation and relations (8.6),(4.1).

ii) We deduce relation (8.9) from relations (8.8),(4.2),(2,14) and Theorem 4.2. □

9. The Cherednik-Trimèche's convolution product

Definition 9.1. The Cherednik-Trimèche's convolution product of the functions f, g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) is the function$f *_{kl} g$ defined by

$$\forall\, y \in \mathbb{R}^d, \quad f *_{kl} g(y) = \int_{\mathbb{R}^d} {}^t\mathcal{T}_x^{kl}(f)(y) g(x) \mathcal{A}_k(x) dx. \tag{9.1}$$

Proposition 9.1. *i) For all functions f, g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$) the function $f *_{kl} g$ belongs to $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$).*

ii) For all f, g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$), we have the following relations:

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{H}^k(f *_{kl} g)(\lambda) = \mathcal{H}^l(g *_{kl} f)(\lambda), \tag{9.2}$$

$$\forall\, \lambda \in \mathbb{R}^d, \quad \mathcal{H}^k(f *_{kl} g)(\lambda) = \mathcal{H}^k(f)(\lambda)\mathcal{H}^l(g)(\lambda). \tag{9.3}$$

Proposition 9.2. *For all functions f, g in $\mathcal{D}(\mathbb{R}^d)$ (resp. $\mathcal{S}_2(\mathbb{R}^d)$), we have*

$$\forall\, y \in \mathbb{R}^d, \quad {}^tV_k(f *_{kl} g)(y) = {}^tV_k(f) * {}^tV_l(g)(y), \tag{9.4}$$

where $*$ is the classical convolution product of functions on $\mathbb{R}^d$.

Proof. We deduce relation (9.4) from relations (9.1) and (8.2). □

10. The Cherednik-Trimèche's heat kernel

Definition 10.1. For all $x, y \in \mathbb{R}^d$ and $t \in (0, \infty)$, we define the Cherednik-Trimèche's heat kernel $p_t^{kl}(x, y)$ by

$$p_t^{kl}(x, y) = {}^t\mathcal{T}_x^{kl}(E_t^k)(y), \tag{10.1}$$

where $E_t^k(x)$ is the fundamental solution of the operator H_k given by the relation (6.4).

Proposition 10.1. *i) For all $x \in \mathbb{R}^d$, $t \in (0, \infty)$ and $\lambda \in \mathbb{R}^d$, we have*

$$\mathcal{H}^k(p_t^{kl}(x, .))(\lambda) = e^{-t(\|\lambda\|^2 + \|\rho^k\|^2)} G_\lambda^l(x). \tag{10.2}$$

ii) For all $x, y \in \mathbb{R}^d$ and $t \in (0, \infty)$, we have

$$p_t^{kl}(x, y) = \int_{\mathbb{R}^d} e^{-t(\|\lambda\|^2 + \|\rho^k\|^2)} G_\lambda^l(x) G_\lambda^k(-y) \mathcal{C}_k(\lambda) d\lambda. \tag{10.3}$$

Proof. i) From the relations (10.1),(8.7) we have

$$\forall(x,t)\in\mathbb{R}^d\times(0,\infty),\lambda\in\mathbb{R}^d,\mathcal{H}^k(p_t^{kl}(x,.))(\lambda)=G_\lambda^l(x)\mathcal{H}^k(E_t^k)(\lambda).$$

We deduce relation (10.2) from relation (6.5).

ii) The relations (10.1), (10.2), (8.9) (8.10) imply relation (10.3).

□

Proposition 10.2. *For $x\in\mathbb{R}^d$ and $t\in(0,\infty)$, we have*

$$\forall y\in\mathbb{R}^d,\quad {}^tU_{kl}(p_t^{kl}(x,.))(y)=e^{-t(\|\rho^k\|^2-\|\rho^l\|^2)}p_t^l(x,y). \tag{10.4}$$

Proof. From the relations (7.13),(10.2), we have for all $y\in\mathbb{R}^d$,

$$\begin{aligned}{}^tU_{kl}(p_t^{kl}(x,.))(y) &= (\mathcal{H}^l)^{-1}[e^{-t(\|\lambda\|^2+\|\rho^k\|^2)}G_\lambda^l(x)](y)\\ &= e^{-t(\|\rho^k\|^2-\|\rho^l\|^2)}(\mathcal{H}^l)^{-1}[e^{-t(\|\lambda\|^2+\|\rho^l\|^2)}G_\lambda^l(x)](y).\end{aligned}$$

By using the relation (4.2) we obtain for all $y\in\mathbb{R}^d$,

$${}^tU_{kl}(p_t^{kl}(x,.))(y)=e^{-t(\|\rho^k\|^2-\|\rho^l\|^2)}\int_{\mathbb{R}^d}e^{-t(\|\lambda\|^2+\|\rho^l\|^2)}G_\lambda^l(x)G_\lambda^l(-y)\mathcal{C}_l(\lambda)d\lambda.$$

Thus the relation (6.1) implies

$$\forall y\in\mathbb{R}^d,\quad {}^tU_{kl}(p_t^{kl}(x,.))(y)=e^{-t(\|\rho^k\|^2-\|\rho^l\|^2)}p_t^l(x,y).$$

□

In coming papers we plan to study:

1. The Harmonic Analysis associated to the Cherednik-Trimèche's transmutation operators on $\mathbb{R}^d$ in the W-invariant case.

2. Applications of the Harmonic Analysis associated to the Cherednik-Trimèche's operators on $\mathbb{R}^d$.

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References

[1] I. Cherednik, A unification of Knizhnik-Zamolod-dchnikov equations and Dunkl operators via affine Hecke algebras, *Invent. Math. Res.*, **106** (1991), 411-432.

[2] G.J. Heckman, E.M. Opdam, Root systems and hypergeometric functions I, *Compositio Math.*, **64** (1987), 329-352.

[3] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, *Acta Math.*, **175** (1995), 75-121.

[4] B. Schapira, Contribution to the hypergeometric function theory of Heckman and Opdam; sharp estimates, Schwartz spaces, heat kernel, *Geom. Funct. Anal.*, **18** (2008), 222-250.

[5] K. Trimèche, The trigonometric Dunkl intertwining operator and its dual associated with the Cherednik operators and the Heckman Opdam theory, *Adv. Pure Appl. Math.*, **1** (2010), 293-323.

[6] K. Trimèche, Harmonic analysis associated with the Cherednik operators and the Heckman-Opdam theory, *Adv. Pure Appl. Math.*, **2** (2011), 23-46.

[7] K. Trimèche, The positivity of the hypergeometric translation operators associated to the Cherednik operators and the Heckman-Opdam theory attached to the root system of type BC_2, *International Journal of Applied Mathematics*, **29**, No 6 (2016), 687-715; doi: 10.12732/ijam.v29i6.4; available at: http://www.diogenes.bg/ijam/index.html.