

**THE ROLE OF TRANSFER FUNCTION IN
THE STUDY OF STABILITY ANALYSIS
OF FEEDBACK CONTROL SYSTEM WITH DELAY**

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Abstract: Loop delays appear obviously in several control applications. Due to loop delays, more complications arrive in feedback control systems. Loop delays cannot be avoided in a system controlled via communication networks as it decreases the stability of the system and restricts the achievable response time of the system. The transfer function is a main tool for analyzing and designing the feedback control system. It describes the system's input output behaviour. In this paper, we have examined the stability of feedback control system using transfer function.

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1. Introduction

Feedback control systems are usually mentioned as closed loop control systems. Delay is dangerous in feedback control system. The problem of time delay in the feedback loop has come into view in recent years. These delays may create the system's performance bad or even cause the system unstable. Contribution of this paper is analyzing the stability of feedback control system using transfer

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function. It is used to describe the input-output relations of systems which can be characterized by linear, time-invariant, differential equations. It is an important tool for analyzing and designing the closed loop control system. The characteristic equation of closed loop system with delay will be of the form $A(s) + B(s)e^{-\tau s} = 0$, where τ is a delay. Then the transfer function for closed loop system can be defined as $G(s) = \frac{B(s)}{A(s)}$.

The following are the properties of transfer function:

- The transfer function is expressed only for a linear time-invariant system and it is not expressed for nonlinear systems.
- Initial conditions of the LTI system should be zero.
- The transfer function is not dependent of the input of the system.

A.A. Khan et al., gave the favorable effect of time delays on the tracking performance of a control system [1]. In 2013, F.A. Salem introduced control solution of basic open loop electric DC machines, two loop current and speed control of electric machines [2]. P. Hovel described the basic concepts and suggested a summary of the time delayed feedback scheme [3]. Piriadarshani et al., discussed about the stability of neutral delay differential equation with infinite delay in 2012, [4]. Sujitha et al. analysed the stability of a second order DDE which is expressed as a special case of the one-mass system controlled over network using Lambert W function [6].

2. Preliminaries

2.1. Poles and Zeros of Transfer Function

Poles and zeros are obtained by solving the equation $A(s) = 0$ and $B(s) = 0$, respectively. It is important in knowing the locations of the poles and zeros as it disturbs the transient response by showing whether the system is stable or not. If all the coefficients of polynomials $A(s)$ and $B(s)$ are real, then the poles and zeros will be either purely real or complex conjugate pairs. We can represent the pole-zero plots by fixing the locations of poles and zeros on the complex S -plane. The stability of a feedback system can be defined directly from its transfer function. A system would be an asymptotically stable if each and every poles of the transfer function have strictly negative real part. It is marginally stable if one or more poles having zero real part. If at least one pole having positive real part then the system would be unstable.

2.2. Impulse Response

An impulse response of a system $g(t)$ is the inverse Laplace Transform of the transfer function $G(s)$ of a system given by

$$g(t) = L^{-1}(G(s)).$$

Another way of analyzing the stability of feedback control system is impulse response. A feedback control system has the following stability properties:

- A feedback control system is said to be asymptotic stable if $\lim_{t \rightarrow \infty} g(t) \rightarrow 0$.
- A feedback control system is marginally stable if $0 < \lim_{t \rightarrow \infty} g(t) < \infty$.
- A feedback control system is said to be unstable if $\lim_{t \rightarrow \infty} g(t) \rightarrow \infty$.

2.3. Long Division Method

The following steps for analyzing the stability of feedback system with Long Division method will be constructed as follows:

- Constructing the transfer function $G(s) = \frac{B(s)}{A(s)}$.
- Choosing the denominator $A(s)$.
- Creating two new polynomials $A_1(s)$ and $A_2(s)$.
- Form $\frac{A_1(s)}{A_2(s)}$.

To system to be stable, all a_1, a_2, \dots should be positive, otherwise it is unstable.

2.4. Routh-Hurwitz Method

Routh-Hurwitz stability criterion is an additional stability test for feedback control system. The number of changes in the sign in the first column of Routh array allows the roots to locate in the right half of complex S -plane otherwise the roots will be located in the left half plane.

3. Examples

In this section, we are providing examples of stability analysis of delayed resonator and second order one-mass system controlled over the network.

3.1. Delayed Resonator

The delayed resonator (DR) is an absorber tuning approach which uses partial state feedback with time delay whose governing equation of motion is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + gx(t - \tau) = 0, \quad (1)$$

where m - mass, c - damping parameter, k - stiffness parameter and $x(t)$ - displacement, g - feedback gain, τ - delay.

In Delayed Resonator the delay is introduced due to the feedback control law. The characteristic equation of this feedback control system is

$$ms^2 + cs + k + ge^{-s\tau} = 0. \quad (2)$$

It can be expressed in the general form of $A(s) + B(s)e^{-s\tau} = 0$. The transfer function can be represented as $G(s) = \frac{B(s)}{A(s)}$. Here

$$G(s) = \frac{g}{ms^2 + cs + k},$$

where feedback gain $g = \sqrt{(cw)^2 + (mw^2 - k)^2}$ and $w > \sqrt{\frac{k}{m}}$. The poles are given by

$$s_1 = \frac{(-c) + \sqrt{c^2 - 4*mk}}{2*m} \quad \text{and} \quad s_2 = \frac{(-c) - \sqrt{c^2 - 4*mk}}{2*m}.$$

Case i.

Suppose $m = 50kg$; $c = 2 * 10^3 kg/s$; $k = 2 * 10^7 N/m$. Take $w = 650$ and hence $g = 1,719,193.12$. The transfer function for the system described in (1) is

$$G(s) = \frac{1,719,193.12}{50 * s^2 + (2 * 10^3) * s + (2 * 10^7)}.$$

The poles are $s_1 = -0.2000 + 6.3214i$ and $s_2 = -0.2000 - 6.3214i$. As s_1 and s_2 have negative real part, the above single-degree -of-freedom(SDOF) system is stable.

Case ii.

The following Figure 1 shows that the impulse response of the delayed resonator.

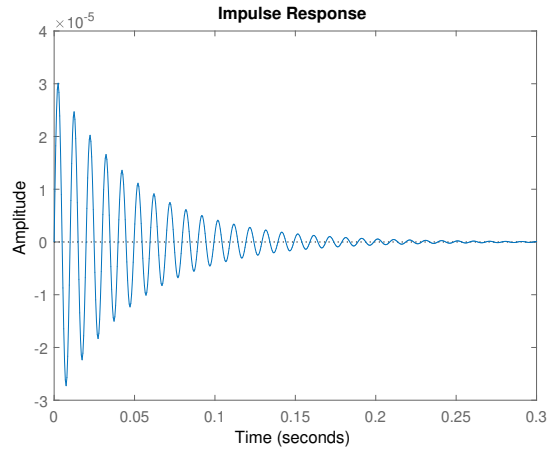


Fig. 1. Impulse response for delayed resonator

From the above figure as $\lim_{t \rightarrow \infty} g(t) \rightarrow 0$, the system represented by (1) is asymptotically stable.

Case iii.

Here $A(s) = ms^2 + cs + k$. By the long division method, formulate two polynomials:

$$A_1(s) = ms^2 + k \quad \text{and} \quad A_2(s) = cs,$$

$$\frac{A_1(s)}{A_2(s)} = \frac{m}{c}s + \frac{1}{\frac{c}{k}s},$$

where $a_1 = \frac{m}{c}$ and $a_2 = \frac{c}{k}$. For $m = 50kg$; $c = 2 * 10^3 kg/s$; $k = 2 * 10^7 N/m$, $a_1 = 0.025$ and $a_2 = 0.0001$ are positive, then the roots of $A(s)$ lie in the left half plane.

The system represented by (1) is stable.

Case iv.

Consider $m = 50kg$; $c = 2 * 10^3 kg/s$; $k = 2 * 10^7$. The characteristic polynomial is $50s^2 + (2 * 10^3)s + (2 * 10^7)$. Using the Routh-Hurwitz method, a Routh array is constructed as

s^2	50	$2 * 10^7$
s^1	$2 * 10^3$	0
s^0	$4 * 10^{10}$	

As there is no change in the sign in the first column of Routh array all the roots will be placed in the left half plane and confirms the stability.

Values of a_n and b_n	Poles s_1 and s_2	Nature of the System
$a_n = 3 > 0, b_n = 2 > 0$	-2, -1	Stable
$a_n = 1 > 0, b_n = -2 < 0$	1, -2	Unstable
$a_n = -3 < 0, b_n = 2 > 0$	2, 1	Unstable
$a_n = -1 < 0, b_n = -2 < 0$	2, -1	Unstable

Table 1: Stability Analysis based on the nature of Poles

3.2. Second Order System Controlled over the Network

Consider the second order system controlled over the network whose governing equation is

$$\ddot{y}(t) + a_n \dot{y}(t) + b_n y(t) = u(t), \quad (3)$$

where a_n and b_n are real numbers.

By control law with controller gain K is given by

$$u(t) = Ky(t - \tau). \quad (4)$$

Equation (3) becomes

$$\ddot{y}(t) + a_n \dot{y}(t) + b_n y(t) - Ky(t - \tau) = 0, \quad (5)$$

whose characteristic equation is

$$s^2 + a_n s + b_n - K_p e^{-s\tau} = 0.$$

Case i.

The transfer function for (3.3) is

$$G(s) = \frac{-K_p}{s^2 + a_n s + b_n}. \quad (6)$$

Here $A(s) = s^2 + a_n s + b_n$ and $B(s) = -K_p$, and the poles are

$$s_1 = \frac{-a_n + \sqrt{a_n^2 - 4b_n}}{2} \quad \text{and} \quad s_2 = \frac{-a_n - \sqrt{a_n^2 - 4b_n}}{2}.$$

Case ii.

The impulse response of the above system (3) is given below.

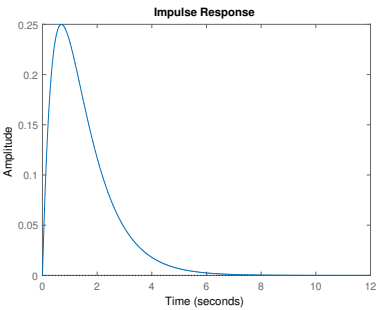


Fig. 2. Impulse response of second order system controlled over the network for $a = 3 > 0, b = 2 > 0$

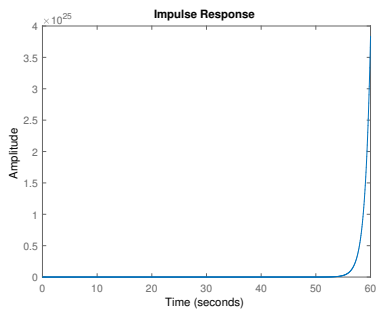


Fig. 3. Impulse response of second order system controlled over the network for $a = 1 > 0, b = -2 < 0$

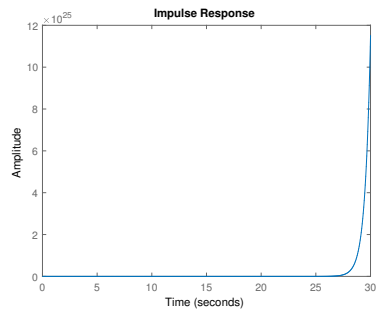


Fig. 4. Impulse response of second order system controlled over the network for $a = -3 < 0, b = 2 > 0$

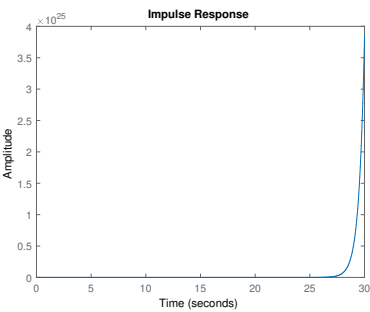


Fig. 5. Impulse response of second order system controlled over the network for $a = -1 < 0, b = -2 < 0$

From the above figures, we have the following table.

Values of a_n and b_n	$\lim_{t \rightarrow \infty} g(t)$	Nature of the System
$a_n = 3 > 0, b_n = 2 > 0$	tends to zero	Stable
$a_n = 1 > 0, b_n = -2 < 0$	tends to infinity	Unstable
$a_n = -3 < 0, b_n = 2 > 0$	tends to infinity	Unstable
$a_n = -1 < 0, b_n = -2 < 0$	tends to infinity	Unstable

Table 2: Stability Analysis using Impulse Response

Case iii.

From (6), $A(s) = s^2 + a_n s + b_n$, by Long division method we formulate two polynomials

$$A_1(s) = s^2 + b_n \text{ and } A_2(s) = a_n s,$$

$$\text{Then, } \frac{A_1(s)}{A_2(s)} = \frac{1}{a_n} s + \frac{1}{\frac{a_n}{b_n} s},$$

where $a_1 = \frac{1}{a_n}$ and $a_2 = \frac{a_n}{b_n}$.

Case iv.

By the Routh-Hurwitz method, let us construct the Routh array in the following table.

From all the above four cases it is well understood that the system described in (3) is stable only for positive values of a_n and b_n .

Values of a_n and b_n	Values of a_1 and a_2	Nature of the System
$a_n = 3 > 0, b_n = 2 > 0$	$a_1 = 0.3333, a_2 = 1.5$	Stable
$a_n = 1 > 0, b_n = -2 < 0$	$a_1 = 1, a_2 = -0.5$	Unstable
$a_n = -3 < 0, b_n = 2 > 0$	$a_1 = -0.3333, a_2 = -1.5$	Unstable
$a_n = -1 < 0, b_n = -2 < 0$	$a_1 = -1, a_2 = 0.5$	Unstable

Table 3: Stability Analysis using Long Division Method

Values of a_n and b_n	Routh array	Nature of the System
$a_n = 3 > 0, b_n = 2 > 0$	s^2 1 2 s^1 3 0 s^0 2	Stable
$a_n = 1 > 0, b_n = -2 < 0$	s^2 1 -2 s^1 1 0 s^0 -2	Unstable
$a_n = -3 < 0, b_n = 2 > 0$	s^2 1 2 s^1 -3 0 s^0 2	Unstable
$a_n = -1 < 0, b_n = -2 < 0$	s^2 1 -2 s^1 -1 0 s^0 -2	Unstable

Table 4: Stability Analysis using Routh-Hurwitz Method

4. Conclusion

The analytical stability of a feedback control system such as delayed resonator and second order system controlled over the network is analysed with the help of transfer function using various kinds of methods. Moreover the theory can be extended to higher order delay differential equation representing dynamical system.

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