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A SUBDIVISION ALGORITHM FOR $\vec{h}$-BÉZIER VOLUMES USING TRIVARIATE $\vec{h}$-BLOSSOMING

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Abstract: We extend the definition of h-blossoming, introduced by Simeonov, Zafiris, and Goldman, to the trivariate polynomials and we define the $\vec{h}$- Bézier volumes. We derive a subdivision algorithm for $\vec{h}$-Bézier volumes and illustrate it on examples using *Mathematica*.

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1. Introduction and Summary

The classical Bernstein polynomials, Bézier curves, surfaces, and volumes have found applications in many areas of numerical analysis, approximation theory, computer aided geometric design, and various other fields of applied and computational mathematics. The quantum q-analogues of Bernstein basis functions were defined and studied by Oruç and Phillips in [7]-[11], and their h-analogues were introduced and explored by Stancu in [15, 16] and by Goldman and Barry in [2, 3, 4]. For a survey of Bernstein polynomial basis, see [1] by Farouki, and for Bézier volumes, we refer to [12] by Samuelčík.

The quantum q- and h-blossoming was introduced by Simeonov, Zafiris, and Goldman in [13, 14]. Its importance is in quantum blossoming representation of quantum Bézier curves, surfaces, and splines, which is used to derive efficient

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algorithms for recursive evaluation, degree elevation, subdivision, and other identities and properties. Some of these identities and properties were derived using mathematical induction in [6] by Jegdić, Larson, and Simeonov.

In this paper we extend the univariate h-blossoming defined by Simeonov, Zafiris, and Goldman in [13]. Its two-dimensional version was studied in [5] by Jegdić. The paper is organized as follows. In §2 we define trivariate $\vec{h}$-Bernstein basis functions and $\vec{h}$-Bézier volumes, and we show the effect of the parameter $\vec{h}$ on an example. We use the recurrence relations for $\vec{h}$-Bernstein basis functions to derive an analogue of the de Casteljau evaluation algorithm. In §3 we define $\vec{h}$-blossoming, we derive recursive evaluation algorithms, and we state several results regarding $\vec{h}$-blossoming and $\vec{h}$-Bézier volumes. In §4 we derive a subdivision algorithm for $\vec{h}$-Bézier volumes and we illustrate it on examples using *Mathematica*.

2. Definition of $\vec{h}$-Bézier Volumes

We recall the definition of the h-Bernstein basis functions over an interval $[a, b]$

$$B_k^n(t; [a, b]; h) := \binom{n}{k} \frac{\prod_{i=0}^{k-1}(t - a + ih) \prod_{i=0}^{n-k-1}(b - t + ih)}{\prod_{i=0}^{n-1}(b - a + ih)},$$

$k = 0, \ldots, n$, where the parameter h is such $b - a + ih \neq 0$ for all $i = 0, \ldots, n-1$.

Definition 1. The trivariate $\vec{h}$-Bernstein basis functions of degree m in t_1, n in t_2, and p in t_3, over a rectangular solid $\mathcal{S} := [a, b] \times [c, d] \times [e, f]$, with $\vec{h} = (h_1, h_2, h_3) \in \mathbb{R}^3$, are given by

$$B_{j,k,l}^{m,n,p}(t_1, t_2, t_3; \mathcal{S}; \vec{h}) := B_j^m(t_1; [a, b]; h_1)\, B_k^n(t_2; [c, d]; h_2)\, B_l^p(t_3; [e, f]; h_3),$$

where $j = 0, \ldots, m$, $k = 0, \ldots, n$, and $l = 0, \ldots, p$, and the parameter $\vec{h}$ is such that $b - a + \alpha h_1 \neq 0$ (for $\alpha = 0, \ldots, m-1$), $d - c + \beta h_2 \neq 0$ (for $\beta = 0, \ldots, n-1$), and $f - e + \gamma h_3 \neq 0$ (for $\gamma = 0, \ldots, p-1$).

We use the recurrence relations of the h-Bernstein basis functions, as in [5] and [13], to obtain the following relations in the trivariate case:

$$B_{0,0,0}^{0,0,0}(t_1, t_2, t_3; \mathcal{S}; \vec{h}) = 1,$$

and if $j = 1, \dots, m-1$, $k = 1, \dots, n-1$, and $l = 1, \dots, p-1$, we have

$$\begin{aligned} B^{m,n,p}_{j,k,l}(t_1, t_2, t_3; \mathcal{S}; \vec{h}) = {} & E_1 \times B^{m-1,n-1,p-1}_{j-1,k-1,l-1} + E_2 \times B^{m-1,n-1,p-1}_{j-1,k-1,l} \\ & + E_3 \times B^{m-1,n-1,p-1}_{j-1,k,l-1} + E_4 \times B^{m-1,n-1,p-1}_{j-1,k,l} \\ & + E_5 \times B^{m-1,n-1,p-1}_{j,k-1,l-1} + E_6 \times B^{m-1,n-1,p-1}_{j,k-1,l} \\ & + E_7 \times B^{m-1,n-1,p-1}_{j,k,l-1} + E_8 \times B^{m-1,n-1,p-1}_{j,k,l}, \end{aligned}$$

where the polynomials on the right hand side are in variables t_1, t_2, and t_3, over $\mathcal{S}$, with respect to parameter $\vec{h}$, and the expressions $E_1, \dots, E_8$ are given by

$$\frac{t_1 - a + (j-1)h_1}{b - a + (m-1)h_1} \frac{t_2 - c + (k-1)h_2}{d - c + (n-1)h_2} \frac{t_3 - e + (l-1)h_3}{f - e + (p-1)h_3},$$

$$\frac{t_1 - a + (j-1)h_1}{b - a + (m-1)h_1} \frac{t_2 - c + (k-1)h_2}{d - c + (n-1)h_2} \frac{f - t_3 + (p-l-1)h_3}{f - e + (p-1)h_3},$$

$$\frac{t_1 - a + (j-1)h_1}{b - a + (m-1)h_1} \frac{d - t_2 + (n-k-1)h_2}{d - c + (n-1)h_2} \frac{t_3 - e + (l-1)h_3}{f - e + (p-1)h_3},$$

$$\frac{t_1 - a + (j-1)h_1}{b - a + (m-1)h_1} \frac{d - t_2 + (n-k-1)h_2}{d - c + (n-1)h_2} \frac{f - t_3 + (p-l-1)h_3}{f - e + (p-1)h_3},$$

$$\frac{b - t_1 + (m-j-1)h_1}{b - a + (m-1)h_1} \frac{t_2 - c + (k-1)h_2}{d - c + (n-1)h_2} \frac{t_3 - e + (l-1)h_3}{f - e + (p-1)h_3},$$

$$\frac{b - t_1 + (m-j-1)h_1}{b - a + (m-1)h_1} \frac{t_2 - c + (k-1)h_2}{d - c + (n-1)h_2} \frac{f - t_3 + (p-l-1)h_3}{f - e + (p-1)h_3},$$

$$\frac{b - t_1 + (m-j-1)h_1}{b - a + (m-1)h_1} \frac{d - t_2 + (n-k-1)h_2}{d - c + (n-1)h_2} \frac{t_3 - e + (l-1)h_3}{f - e + (p-1)h_3},$$

$$\frac{b - t_1 + (m-j-1)h_1}{b - a + (m-1)h_1} \frac{d - t_2 + (n-k-1)h_2}{d - c + (n-1)h_2} \frac{f - t_3 + (p-l-1)h_3}{f - e + (p-1)h_3},$$

respectively.

Definition 2. The $\vec{h}$-Bézier volume of degree m in t_1, n in t_2, and p in t_3, over a solid $\mathcal{S} := [a, b] \times [c, d] \times [e, f]$, where $\vec{h} := (h_1, h_2, h_3) \in \mathbb{R}^3$, with control points $P_{j,k,l}$, $j = 0, \dots, m$, $k = 0, \dots, n$, and $l = 0, \dots, p$, is defined by

$$P(t_1, t_2, t_3) := \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} P_{j,k,l} B^{m,n,p}_{j,k,l}(t_1, t_2, t_3; \mathcal{S}; \vec{h}).$$

Example 3. We investigate the effect of the parameter $\vec{h}$ on an $\vec{h}$-Bézier volume that is linear in t_1, quadratic in t_2, and linear in t_3. The control points are set to

$P_{0,0,0}(0,0,0)$	$P_{0,1,1}(0,1/2,1)$	$P_{1,0,0}(1,0,0)$	$P_{1,1,1}(1,1/2,1)$
$P_{0,0,1}(0,0,1)$	$P_{0,2,0}(-1,1,0)$	$P_{1,0,1}(1,0,1)$	$P_{1,2,0}(0,1,0)$
$P_{0,1,0}(0,1/2,0)$	$P_{0,2,1}(-1,1,1)$	$P_{1,1,0}(1,1/2,0)$	$P_{1,2,1}(0,1,1)$

implying $P_1(t_1,t_2,t_3) = t_1 - t_2(t_2+h_2)/(1+h_2)$, $P_2(t_1,t_2,t_3) = t_2$, and $P_3(t_1,t_2,t_3) = t_3$. Since the given $\vec{h}$-Bézier volume is linear in t_1 and in t_3, $P(t_1,t_2,t_3)$ depends only on the parameter h_2. In Figure 1 we plot $P(t_1,t_2,t_3)$ for h_2 taking values of 0, 0.29, 2.01, and 1000.21, respectively.

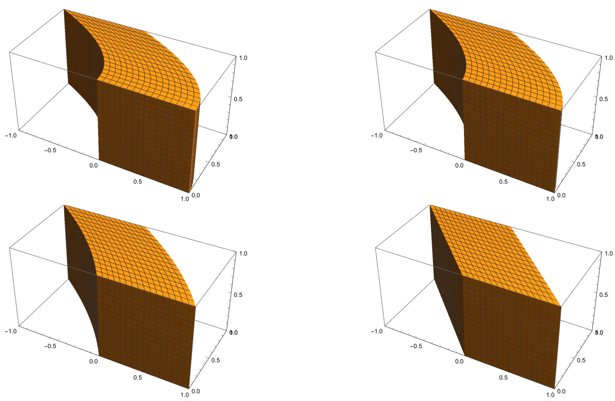

Figure 1: The effect of the parameter $\vec{h}$.

Using the above recurrence relation, we derive the $\vec{h}$-de Casteljau evaluation algorithm as follows. Define

$$P^{0,0,0}_{j,k,l}(t_1,t_2,t_3) = P_{j,k,l},$$

for $j = 0,\ldots,m$, $k = 0,\ldots,n$, and $l = 0,\ldots,p$. If $\alpha = 1,\ldots,m$, $\beta = 1,\ldots,n$, and $\gamma = 1,\ldots,p$, define recursively

$$\begin{aligned} P^{\alpha,\beta,\gamma}_{j,k,l}(t_1,t_2,t_3) &= E_1 \times P^{\alpha-1,\beta-1,\gamma-1}_{j,k,l} + E_2 \times P^{\alpha-1,\beta-1,\gamma-1}_{j+1,k,l} \\ &+ E_3 \times P^{\alpha-1,\beta-1,\gamma-1}_{j,k+1,l} + E_4 \times P^{\alpha-1,\beta-1,\gamma-1}_{j+1,k+1,l} \\ &+ E_5 \times P^{\alpha-1,\beta-1,\gamma-1}_{j,k,l+1} + E_6 \times P^{\alpha-1,\beta-1,\gamma-1}_{j+1,k,l+1} \end{aligned}$$

$$+ E_7 \times P^{\alpha-1,\beta-1,\gamma-1}_{j,k+1,l+1} + E_8 \times P^{\alpha-1,\beta-1,\gamma-1}_{j+1,k+1,l+1},$$

for $j = 0, \ldots, m-\alpha$, $k = 0, \ldots, n-\beta$, and $l = 0, \ldots, p-\gamma$, where the expressions $E_1, \ldots, E_8$ are given by

$$\frac{b - t_1 + (m-j-\alpha)h_1}{b-a+(m-\alpha)h_1}\frac{d-t_2+(n-k-\beta)h_2}{d-c+(n-\beta)h_2}\frac{f-t_3+(p-l-\gamma)h_3}{f-e+(p-\gamma)h_3},$$

$$\frac{t_1-a+jh_1}{b-a+(m-\alpha)h_1}\frac{d-t_2+(n-k-\beta)h_2}{d-c+(n-\beta)h_2}\frac{f-t_3+(p-l-\gamma)h_3}{f-e+(p-\gamma)h_3},$$

$$\frac{b - t_1 + (m-j-\alpha)h_1}{b-a+(m-\alpha)h_1}\frac{t_2-c+kh_2}{d-c+(n-\beta)h_2}\frac{f-t_3+(p-l-\gamma)h_3}{f-e+(p-\gamma)h_3},$$

$$\frac{t_1-a+jh_1}{b-a+(m-\alpha)h_1}\frac{t_2-c+kh_2}{d-c+(n-\beta)h_2}\frac{f-t_3+(p-l-\gamma)h_3}{f-e+(p-\gamma)h_3},$$

$$\frac{b - t_1 + (m-j-\alpha)h_1}{b-a+(m-\alpha)h_1}\frac{d-t_2+(n-k-\beta)h_2}{d-c+(n-\beta)h_2}\frac{t_3-e+lh_3}{f-e+(p-\gamma)h_3},$$

$$\frac{t_1-a+jh_1}{b-a+(m-\alpha)h_1}\frac{d-t_2+(n-k-\beta)h_2}{d-c+(n-\beta)h_2}\frac{t_3-e+lh_3}{f-e+(p-\gamma)h_3},$$

$$\frac{b - t_1 + (m-j-\alpha)h_1}{b-a+(m-\alpha)h_1}\frac{t_2-c+kh_2}{d-c+(n-\beta)h_2}\frac{t_3-e+lh_3}{f-e+(p-\gamma)h_3},$$

$$\frac{t_1-a+jh_1}{b-a+(m-\alpha)h_1}\frac{t_2-c+kh_2}{d-c+(n-\beta)h_2}\frac{t_3-e+lh_3}{f-e+(p-\gamma)h_3},$$

respectively. It is easily shown by induction on m, n, and p that

$$P^{m,n,p}_{0,0,0}(t_1,t_2,t_3) = P(t_1,t_2,t_3).$$

3. Definition of $\vec{h}$-Blossoming for Trivariate Polynomials

In this section we define an $\vec{h}$-blossom for polynomials in three variables.

Definition 4. Given a polynomial $P(t_1, t_2, t_3)$ of degree m in t_1, n in t_2, and p in t_3, the $\vec{h}$-blossom of $P(t_1, t_2, t_3)$, where $\vec{h} = (h_1, h_2, h_3) \in \mathbb{R}^3$, is a polynomial

$$p(u_1, \ldots, u_m; v_1, \ldots, v_n; w_1, \ldots, w_p; \vec{h})$$

which satisfies the following properties

- *symmetry:* for any permutations σ_1, σ_2, and σ_3 of the sets $\{1,\dots,m\}$, $\{1,\dots,n\}$, and $\{1,\dots,p\}$, respectively,

$$\begin{aligned}&p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h})\\&=p(u_{\sigma_1(1)},\dots,u_{\sigma_1(m)};v_{\sigma_2(1)},\dots,v_{\sigma_2(n)};w_{\sigma_3(1)},\dots,w_{\sigma_3(p)};\vec{h}),\end{aligned}$$

- *multi-affine:*

$$\begin{aligned}&p(u_1,\dots,(1-\alpha)u_k+\alpha u_k',\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h})\\&=(1-\alpha)p(u_1,\dots,u_k,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h})\\&+\alpha p(u_1,\dots,u_k',\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h}),\end{aligned}$$

$$\begin{aligned}&p(u_1,\dots,u_m;v_1,\dots,(1-\beta)v_k+\beta v_k',\dots,v_n;w_1,\dots,w_p;\vec{h})\\&=(1-\beta)p(u_1,\dots,u_m;v_1,\dots,v_k,\dots,v_n;w_1,\dots,w_p;\vec{h})\\&+\beta p(u_1,\dots,u_m;v_1,\dots,v_k',\dots,v_n;w_1,\dots,w_p;\vec{h}),\end{aligned}$$

  and

$$\begin{aligned}&p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,(1-\gamma)w_k+\gamma w_k',\dots,w_p;\vec{h})\\&=(1-\gamma)p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_k,\dots,w_p;\vec{h})\\&+\gamma p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_k',\dots,w_p;\vec{h}),\end{aligned}$$

- *$\vec{h}$-diagonal:*

$$\begin{gathered}p(t_1,\dots,t_1-(m-1)h_1;t_2,\dots,t_2-(n-1)h_2;\\t_3,\dots,t_3-(p-1)h_3;\vec{h})=P(t_1,t_2,t_3).\end{gathered}$$

The following three-dimensional results follow in the same way as the one-dimensional results in [13] and two-dimensional results in [5]. For convenience, we state them here.

Theorem 5. *(Existence and uniqueness of the $\vec{h}$-blossom)*
For every polynomial $P(t_1,t_2,t_3)$ of degree m in t_1, n in t_2, and p in t_3, there exists a unique $\vec{h}$-blossom

$$p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h}).$$

Theorem 6. *($\vec{h}$-Recursive evaluation algorithms)*
Let $P(t_1, t_2, t_3)$ be a polynomial of degree m in t_1, n in t_2, and p in t_3, with $\vec{h}$-blossom $p(u_1, \ldots, u_m; v_1, \ldots, v_n; w_1, \ldots, w_p; \vec{h})$. There exist $m!n!p!$ affine invariant, recursive evaluation algorithms for $P(t_1, t_2, t_3)$ defined recursively as follows. Let σ_1, σ_2, and σ_3 be permutations of $\{1, \ldots, m\}$, $\{1, \ldots, n\}$, and $\{1, \ldots, p\}$, respectively. Define

$$\begin{aligned} P^{0,0,0}_{j,k,l} = p(&a - jh_1, \ldots, a - (m-1)h_1, b, b - h_1, \ldots, b - (j-1)h_1; \\ &c - kh_2, \ldots, c - (n-1)h_2, d, d - h_2, \ldots, d - (k-1)h_2; \\ &e - lh_3, \ldots, e - (p-1)h_3, f, f - h_3, \ldots, f - (l-1)h_3; \vec{h}), \end{aligned}$$

where $j = 0, \ldots, m$, $k = 0, \ldots, n$, and $l = 0, \ldots, p$. For $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, n-1$, and $\gamma = 0, \ldots, p-1$, define recursively

$$\begin{aligned} &P^{\alpha+1,\beta+1,\gamma+1}_{j,k,l}(t_1, t_2, t_3) \\ &\qquad := (1 - A_{\alpha,j})(1 - B_{\beta,k})(1 - C_{\gamma,l}) P^{\alpha,\beta,\gamma}_{j,k,l}(t_1, t_2, t_3) \\ &\qquad + A_{\alpha,j}(1 - B_{\beta,k})(1 - C_{\gamma,l}) P^{\alpha,\beta,\gamma}_{j+1,k,l}(t_1, t_2, t_3) \\ &\qquad + (1 - A_{\alpha,j}) B_{\beta,k}(1 - C_{\gamma,l}) P^{\alpha,\beta,\gamma}_{j,k+1,l}(t_1, t_2, t_3) \\ &\qquad + A_{\alpha,j} B_{\beta,k}(1 - C_{\gamma,l}) P^{\alpha,\beta,\gamma}_{j+1,k+1,l}(t_1, t_2, t_3) \\ &\qquad + (1 - A_{\alpha,j})(1 - B_{\beta,k}) C_{\gamma,l} P^{\alpha,\beta,\gamma}_{j,k,l+1}(t_1, t_2, t_3) \\ &\qquad + A_{\alpha,j}(1 - B_{\beta,k}) C_{\gamma,l} P^{\alpha,\beta,\gamma}_{j+1,k,l+1}(t_1, t_2, t_3) \\ &\qquad + (1 - A_{\alpha,j}) B_{\beta,k} C_{\gamma,l} P^{\alpha,\beta,\gamma}_{j,k+1,l+1}(t_1, t_2, t_3) \\ &\qquad + A_{\alpha,j} B_{\beta,k} C_{\gamma,l} P^{\alpha,\beta,\gamma}_{j+1,k+1,l+1}(t_1, t_2, t_3), \end{aligned}$$

where $j = 0, \ldots, m - \alpha - 1$, $k = 0, \ldots, n - \beta - 1$, and $l = 0, \ldots, p - \gamma - 1$. Here,

$$A_{j,\alpha} = \frac{t_1 - a - (\sigma_1(j+1) - 1 - \alpha - j)h_1}{b - a + jh_1},$$

$$B_{k,\beta} = \frac{t_2 - c - (\sigma_2(k+1) - 1 - \beta - k)h_2}{d - c + kh_2},$$

and

$$C_{l,\gamma} = \frac{t_3 - e - (\sigma_3(l+1) - 1 - \gamma - l)h_3}{f - e + lh_3}.$$

Then for $\alpha = 0, \ldots, m$, $\beta = 0, \ldots, n$, and $\gamma = 0, \ldots, p$, we have

$$P^{\alpha,\beta,\gamma}_{j,k,l}(t_1, t_2, t_3) =$$

$$
\begin{aligned}
&p(a-(\alpha+j)h_1,\dots,a-(m-1)h_1,b,b-h_1\dots,b-(j-1)h_1,\\
&t_1-(\sigma_1(1)-1)h_1,\dots,t_1-(\sigma_1(\alpha)-1)h_1;\\
&c-(\beta+k)h_2,\dots,c-(n-1)h_2,d,d-h_2,\dots,d-(k-1)h_2,\\
&t_2-(\sigma_2(1)-1)h_2,\dots,t_2-(\sigma_2(\beta)-1)h_2;\\
&e-(\gamma+l)h_3,\dots,e-(p-1)h_3,f,f-h_3,\dots,f-(l-1)h_3,\\
&t_3-(\sigma_3(1)-1)h_3,\dots,t_3-(\sigma_3(\gamma)-1)h_3;\vec{h}),
\end{aligned}
$$

where $j=0,\dots,m-\alpha$, $k=0,\dots,n-\beta$, *and* $l=0,\dots,p-\gamma$. *In particular,*

$$
\begin{aligned}
P^{m,n,p}_{0,0,0}(t_1,t_2,t_3) &= p(t_1-(\sigma_1(1)-1)h_1,\dots,t_1-(\sigma_1(m)-1)h_1;\\
&\qquad t_2-(\sigma_2(1)-1)h_2,\dots,t_2-(\sigma_2(n)-1)h_2;\\
&\qquad t_3-(\sigma_3(1)-1)h_3,\dots,t_3-(\sigma_3(p)-1)h_3;\vec{h})\\
&= P(t_1,t_2,t_3).
\end{aligned}
$$

The next result shows that every polynomial volume is an $\vec{h}$-Bézier volume over any solid.

Theorem 7. *Let* $P(t_1,t_2,t_3)$ *be a polynomial of degree* m *in* t_1, n *in* t_2, *and* p *in* t_3, *with* $\vec{h}$*-blossom*

$$p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h}).$$

Then

$$
\begin{aligned}
P(t_1,t_2,t_3) = &\sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p}\\
&p(a-jh_1,\dots,a-(m-1)h_1,b,b-h_1,\dots,b-(j-1)h_1;\\
&\quad c-kh_2,\dots,c-(n-1)h_2,d,d-h_2,\dots,d-(k-1)h_2;\\
&\quad e-lh_3,\dots,e-(p-1)h_3,f,f-h_3,\dots,f-(l-1)h_3;\vec{h})\\
&\quad \times B^{m,n,p}_{j,k,l}(t_1,t_2,t_3;[a,b]\times[c,d]\times[e,f];\vec{h}).
\end{aligned}
$$

Theorem 8. *(Dual Functional Property of the* $\vec{h}$*-blossom)*
Let $P(t_1,t_2,t_3)$ *be an* $\vec{h}$*-Bézier volume of degree* m *in* t_1, n *in* t_2, *and* p *in* t_3, *over a solid* $[a,b]\times[c,d]\times[e,f]$ *with* $\vec{h}$*-blossom* $p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h})$. *Then the* $\vec{h}$*-Bézier control points of* $P(t_1,t_2,t_3)$ *are given by*

$$P_{j,k,l} = p(a-jh_1,\dots,a-(m-1)h_1,b,b-h_1,\dots,b-(j-1)h_1;$$

$$c-kh_2,\dots,c-(n-1)h_2,d,d-h_2,\dots,d-(k-1)h_2;$$
$$e-lh_3,\dots,e-(p-1)h_3,f,f-h_3,\dots,f-(l-1)h_3;\vec{h}),$$

where $j=0,\dots,m$, $k=0,\dots,n$, *and* $l=0,\dots,p$.

Theorem 9. *Let* $P(t_1,t_2,t_3)$ *be an* $\vec{h}$*-Bézier volume of degree* m *in* t_1, n *in* t_2, *and* p *in* t_3, *over* $[a,b]\times[c,d]\times[e,f]$, *with control points* $\{P_{j,k,l}\}$, $j=0,\dots,m$, $k=0,\dots,n$, *and* $l=0,\dots,p$. *Let* $P^{\alpha,\beta,\gamma}_{j,k,l}$, $\alpha=0,\dots,m$, $\beta=0,\dots,n$, $\gamma=0,\dots,p$ *and* $j=0,\dots,m-\alpha$, $k=0,\dots,n-\beta$, *and* $l=0,\dots,p-\gamma$, *be the nodes in the* $\vec{h}$*-evaluation algorithm for* $P(t_1,t_2,t_3)$ *for the identity permutations. Then*

$$P^{\alpha,\beta,\gamma}_{j,k,l}(t_1,t_2,t_3)=\sum_{r=0}^{\alpha}\sum_{q=0}^{\beta}\sum_{s=0}^{\gamma}P_{j+r,k+q,l+s}$$
$$\times B^{\alpha,\beta,\gamma}_{r,q,s}\left(t_1+jh_1,t_2+kh_2;t_3+lh_3;[a,b]\times[c,d]\times[e,f];\vec{h}\right).$$

4. A Subdivision Algorithm for $\vec{h}$-Bézier Volumes

We present an analogue of the de Casteljau subdivision algorithm and we illustrate it on examples using *Mathematica*.

Theorem 10. *Let* $\{P_{j,k,l}\}$, $j=0,\dots,m$, $k=0,\dots,n$, *and* $l=0,\dots,p$, *be the control points of an* $\vec{h}$*-Bézier volume* $P(t_1,t_2,t_3)$ *of degree* m *in* t_1, n *in* t_2, *and* p *in* t_3, *over a solid* $\mathcal{S}:=[a,b]\times[c,d]\times[e,f]$. *Let* $p(u_1,\dots,u_m;v_1,\dots,v_n;w_1,\dots,w_p;\vec{h})$ *be the* $\vec{h}$*-blossom of* $P(t_1,t_2,t_3)$ *and let* $x\in(a,b)$, $y\in(c,d)$, *and* $z\in(e,f)$ *be fixed.*

• *A control polygon for the volume* $P(t_1,t_2,t_3)$ *over* $[a,x]\times[c,y]\times[e,z]$ *is generated by selecting* $\sigma_1(j)=j$, $j=1,\dots,m$, $\sigma_2(k)=k$, $k=1,\dots,n$, *and* $\sigma_3(l)=l$, $l=1,\dots,p$, *in Theorem 6. Then*

$$P(t_1,t_2,t_3)=\sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p}P^{LLL}_{j,k,l}B^{m,n,p}_{j,k,l}(t_1,t_2,t_3;[a,x]\times[c,y]\times[e,z];\vec{h}),$$

where

$$P^{LLL}_{j,k,l}=p(a-jh_1,\dots,a-(m-1)h_1,x,x-h_1,\dots,x-(j-1)h_1;$$

$$c-kh_2,\dots,c-(n-1)h_2,y,y-h_2,\dots,y-(k-1)h_2;$$
$$e-lh_3,\dots,e-(p-1)h_3,z,z-h_3,\dots,z-(l-1)h_3;\vec{h}),$$

$j=0,\dots,m$, $k=0,\dots,n$, and $l=0,\dots,p$. Moreover,

$$P^{LLL}_{j,k,l}=\sum_{\alpha=0}^{j}\sum_{\beta=0}^{k}\sum_{\gamma=0}^{l}P_{\alpha,\beta,\gamma}B^{j,k,l}_{\alpha,\beta,\gamma}(x,y,z;\mathcal{S};\vec{h}).$$

• *A control polygon for the volume $P(t_1,t_2,t_3)$ over $[a,x]\times[y,d]\times[e,z]$ is generated by selecting $\sigma_1(j)=j$, $j=1,\dots,m$, $\sigma_2(k)=n+1-k$, $k=1,\dots,n$, and $\sigma_3(l)=l$, $l=1,\dots,p$ in Theorem 6. Then*

$$P(t_1,t_2,t_3)=\sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p}P^{LRL}_{j,k,l}B^{m,n,p}_{j,k,l}(t_1,t_2,t_3;[a,x]\times[y,d]\times[e,z];\vec{h}),$$

where

$$P^{LRL}_{j,k,l}=p(a-jh_1,\dots,a-(m-1)h_1,x,x-h_1,\dots,x-(j-1)h_1;$$
$$y-kh_2,\dots,y-(n-1)h_2,d,d-h_2,\dots,d-(k-1)h_2;$$
$$e-lh_3,\dots,e-(p-1)h_3,z,z-h_3,\dots,z-(l-1)h_3;\vec{h}),$$

$j=0,\dots,m$, $k=0,\dots,n$, and $l=0,\dots,p$. Moreover,

$$P^{LRL}_{j,k,l}=\sum_{\alpha=0}^{j}\sum_{\beta=k}^{n}\sum_{\gamma=0}^{l}P_{\alpha,\beta,\gamma}B^{j,n-k,l}_{\alpha,\beta-k,\gamma}(x,y,z;\mathcal{S};\vec{h}).$$

• *A control polygon for the volume $P(t_1,t_2,t_3)$ over $[x,b]\times[c,y]\times[e,z]$ is generated by selecting $\sigma_1(j)=m+1-j$, $j=1,\dots,m$, $\sigma_2(k)=k$, $k=1,\dots,n$, and $\sigma_3(l)=l$, $l=1,\dots,p$ in Theorem 6. Then*

$$P(t_1,t_2,t_3)=\sum_{j=0}^{m}\sum_{k=0}^{n}\sum_{l=0}^{p}P^{RLL}_{j,k,l}B^{m,n,p}_{j,k,l}(t_1,t_2,t_3;[x,b]\times[c,y]\times[e,z];\vec{h}),$$

where

$$P^{RLL}_{j,k,l}=p(x-jh_1,\dots,x-(m-1)h_1,b,b-h_1,\dots,b-(j-1)h_1;$$
$$c-kh_2,\dots,c-(n-1)h_2,y,y-h_2\dots,y-(k-1)h_2;$$
$$e-lh_3,\dots,e-(p-1)h_3,z,z-h_3\dots,z-(l-1)h_3;\vec{h}),$$

$j = 0, \dots, m$, $k = 0, \dots, n$, *and* $l = 0, \dots, p$. *Moreover,*

$$P^{RLL}_{j,k,l} = \sum_{\alpha=j}^{m} \sum_{\beta=0}^{k} \sum_{\gamma=0}^{l} P_{\alpha,\beta,\gamma} B^{m-j,k,l}_{\alpha-j,\beta,\gamma}(x, y, z; \mathcal{S}; \vec{h}).$$

• *A control polygon for the volume* $P(t_1, t_2, t_3)$ *over* $[x, b] \times [y, d] \times [e, z]$ *is generated by selecting* $\sigma_1(j) = m + 1 - j$, $j = 1, \dots, m$, $\sigma_2(k) = n + 1 - k$, $k = 1, \dots, n$, *and* $\sigma_3(l) = l$, $l = 1, \dots, p$, *in Theorem 6. Then*

$$P(t_1, t_2, t_3) = \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} P^{RRL}_{j,k,l} B^{m,n,p}_{j,k,l}(t_1, t_2, t_3; [x, b] \times [y, d] \times [e, z]; \vec{h}),$$

where

$$\begin{aligned} P^{RRL}_{j,k,l} =& p(x - jh_1, \dots, x - (n-1)h_1, b, b - h_1, \dots, b - (j-1)h_1;\\ & y - kh_2, \dots, y - (n-1)h_2, d, d - h_2, \dots, d - (k-1)h_2;\\ & e - lh_3, \dots, e - (p-1)h_3, z, z - h_3 \dots, z - (l-1)h_3; \vec{h}), \end{aligned}$$

$j = 0, \dots, m$, $k = 0, \dots, n$, *and* $l = 0, \dots, p$. *Moreover,*

$$P^{RRL}_{j,k,l} = \sum_{\alpha=j}^{m} \sum_{\beta=k}^{n} \sum_{\gamma=0}^{l} P_{\alpha,\beta,\gamma} B^{m-j,n-k,l}_{\alpha-j,\beta-k,\gamma}(x, y, z; \mathcal{S}; \vec{h}).$$

• *A control polygon for the volume* $P(t_1, t_2, t_3)$ *over* $[a, x] \times [c, y] \times [z, f]$ *is generated by selecting* $\sigma_1(j) = j$, $j = 1, \dots, m$, $\sigma_2(k) = k$, $k = 1, \dots, n$, *and* $\sigma_3(l) = p + 1 - l$, $l = 1, \dots, p$, *in Theorem 6. Then*

$$P(t_1, t_2, t_3) = \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} P^{LLR}_{j,k,l} B^{m,n,p}_{j,k,l}(t_1, t_2, t_3; [a, x] \times [c, y] \times [z, f]; \vec{h}),$$

where

$$\begin{aligned} P^{LLR}_{j,k,l} =& p(a - jh_1, \dots, a - (m-1)h_1, x, x - h_1, \dots, x - (j-1)h_1;\\ & c - kh_2, \dots, c - (n-1)h_2, y, y - h_2, \dots, y - (k-1)h_2;\\ & z - lh_3, \dots, z - (p-1)h_3, f, f - h_3, \dots, f - (l-1)h_3; \vec{h}), \end{aligned}$$

$j = 0, \dots, m$, $k = 0, \dots, n$, *and* $l = 0, \dots, p$. *Moreover,*

$$P^{LLR}_{j,k,l} = \sum_{\alpha=0}^{j} \sum_{\beta=0}^{k} \sum_{\gamma=l}^{p} P_{\alpha,\beta,\gamma} B^{j,k,p-l}_{\alpha,\beta,\gamma-l}(x, y, z; \mathcal{S}; \vec{h}).$$

• *A control polygon for the volume* $P(t_1, t_2, t_3)$ *over* $[a, x] \times [y, d] \times [z, f]$ *is generated by selecting* $\sigma_1(j) = j$, $j = 1, \ldots, m$, $\sigma_2(k) = n + 1 - k$, $k = 1, \ldots, n$, *and* $\sigma_3(l) = p + 1 - l$, $l = 1, \ldots, p$ *in Theorem 6. Then*

$$P(t_1, t_2, t_3) = \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} P^{LRR}_{j,k,l} B^{m,n,p}_{j,k,l}(t_1, t_2, t_3; [a, x] \times [y, d] \times [z, f]; \vec{h}),$$

where

$$\begin{aligned} P^{LRR}_{j,k,l} =& p(a - jh_1, \ldots, a - (m-1)h_1, x, x - h_1, \ldots, x - (j-1)h_1; \\ & y - kh_2, \ldots, y - (n-1)h_2, d, d - h_2, \ldots, d - (k-1)h_2; \\ & z - lh_3, \ldots, z - (p-1)h_3, f, f - h_3, \ldots, f - (l-1)h_3; \vec{h}), \end{aligned}$$

$j = 0, \ldots, m$, $k = 0, \ldots, n$, *and* $l = 0, \ldots, p$. *Moreover,*

$$P^{LRR}_{j,k,l} = \sum_{\alpha=0}^{j} \sum_{\beta=k}^{n} \sum_{\gamma=l}^{p} P_{\alpha,\beta,\gamma} B^{j,n-k,p-l}_{\alpha,\beta-k,\gamma-l}(x, y, z; \mathcal{S}; \vec{h}).$$

• *A control polygon for the volume* $P(t_1, t_2, t_3)$ *over* $[x, b] \times [c, y] \times [z, f]$ *is generated by selecting* $\sigma_1(j) = m + 1 - j$, $j = 1, \ldots, m$, $\sigma_2(k) = k$, $k = 1, \ldots, n$, *and* $\sigma_3(l) = p + 1 - l$, $l = 1, \ldots, p$ *in Theorem 6. Then*

$$P(t_1, t_2, t_3) = \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} P^{RLR}_{j,k,l} B^{m,n,p}_{j,k,l}(t_1, t_2, t_3; [x, b] \times [c, y] \times [z, f]; \vec{h}),$$

where

$$\begin{aligned} P^{RLR}_{j,k,l} =& p(x - jh_1, \ldots, x - (m-1)h_1, b, b - h_1, \ldots, b - (j-1)h_1; \\ & c - kh_2, \ldots, c - (n-1)h_2, y, y - h_2 \ldots, y - (k-1)h_2; \\ & z - lh_3, \ldots, z - (p-1)h_3, f, f - h_3, \ldots, f - (l-1)h_3; \vec{h}), \end{aligned}$$

$j = 0, \ldots, m$, $k = 0, \ldots, n$, *and* $l = 0, \ldots, p$. *Moreover,*

$$P^{RLR}_{j,k,l} = \sum_{\alpha=j}^{m} \sum_{\beta=0}^{k} \sum_{\gamma=l}^{p} P_{\alpha,\beta,\gamma} B^{m-j,k,p-l}_{\alpha-j,\beta,\gamma-l}(x, y, z; \mathcal{S}; \vec{h}).$$

• *A control polygon for the volume* $P(t_1, t_2, t_3)$ *over* $[x, b] \times [y, d] \times [z, f]$ *is generated by selecting* $\sigma_1(j) = m + 1 - j$, $j = 1, \ldots, m$, $\sigma_2(k) = n + 1 - k$, $k = 1, \ldots, n$, *and* $\sigma_3(l) = p + 1 - l$, $l = 1, \ldots, p$, *in Theorem 6. Then*

$$P(t_1, t_2, t_3) = \sum_{j=0}^{m} \sum_{k=0}^{n} \sum_{l=0}^{p} P^{RRR}_{j,k,l} B^{m,n,p}_{j,k,l}(t_1, t_2, t_3; [x, b] \times [y, d] \times [z, f]; \vec{h}),$$

where

$$\begin{aligned}P^{RRR}_{j,k,l} =& p(x-jh_1,\dots,x-(n-1)h_1,b,b-h_1,\dots,b-(j-1)h_1;\\ & y-kh_2,\dots,y-(n-1)h_2,d,d-h_2,\dots,d-(k-1)h_2;\\ & z-lh_3,\dots,z-(p-1)h_3,f,f-h_3,\dots,f-(l-1)h_3;\vec{h}),\end{aligned}$$

$j=0,\dots,m$, $k=0,\dots,n$, *and* $l=0,\dots,p$. *Moreover,*

$$P^{RRR}_{j,k,l} = \sum_{\alpha=j}^{m}\sum_{\beta=k}^{n}\sum_{\gamma=l}^{p} P_{\alpha,\beta,\gamma} B^{m-j,n-k,p-l}_{\alpha-j,\beta-k,\gamma-l}(x,y,z;\mathcal{S};\vec{h}).$$

Example 11. We consider an $\vec{h}$-Bézier volume linear in t_1, quadratic in t_2, and linear in t_3, with control points given in Example 3 and with $h_2 = 0.29$, implying $P_1(t_1,t_2,t_3) = t_1 - 0.224806t_2 - 0.775194t_2^2$, $P_2(t_1,t_2,t_3) = t_2$, and $P_3(t_1,t_2,t_3) = t_3$. In Figure 2 we plot this $\vec{h}$-Bézier volume, and in Figure 3 we plot the control points from the first three iterations of the midpoint subdivision algorithm.

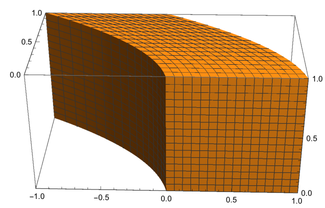

Figure 2: The $\vec{h}$-Bézier volume from Example 11.

Example 12. Consider an $\vec{h}$-Bézier volume, linear in t_1, quadratic in t_2 and linear in t_3, with control points

$P_{0,0,0}(0,0,0)$	$P_{0,1,0}(0,1/2,0)$	$P_{0,2,0}(-1,1,0)$
$P_{0,0,1}(0,-1,1)$	$P_{0,1,1}(0,-1/2,1)$	$P_{0,2,1}(-1,0,1)$
$P_{1,0,0}(1,0,0)$	$P_{1,1,0}(1,1/2,0)$	$P_{1,2,0}(0,1,0)$
$P_{1,0,1}(1,-1,1)$	$P_{1,1,1}(1,-1/2,1)$	$P_{1,2,1}(0,0,1)$.

Then $P(t_1,t_2,t_3)$ depends only on h_2 and we let $h_2 = 0.25$ which gives $P_1(t_1,t_2,t_3) = t_1 - 0.2t_2 - 0.8t_2^2$, $P_2(t_1,t_2,t_3) = t_2 - t_3$, and $P_3(t_1,t_2,t_3) = t_3$. We plot this $\vec{h}$-Bézier volume in Figure 4 and in Figure 5 we plot the control points from the first three iterations of the midpoint subdivision algorithm.

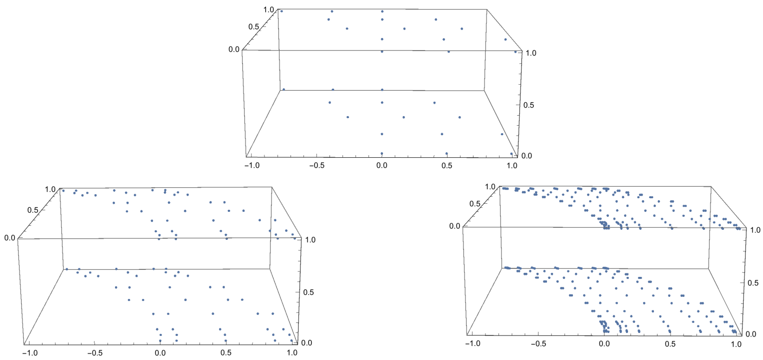

Figure 3: The control points from the first, second and the third iterations of the midpoint subdivision algorithm for Example 11.

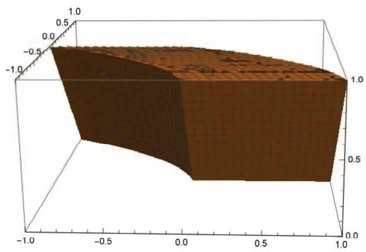

Figure 4: The $\vec{h}$-Bézier volume from Example 12.

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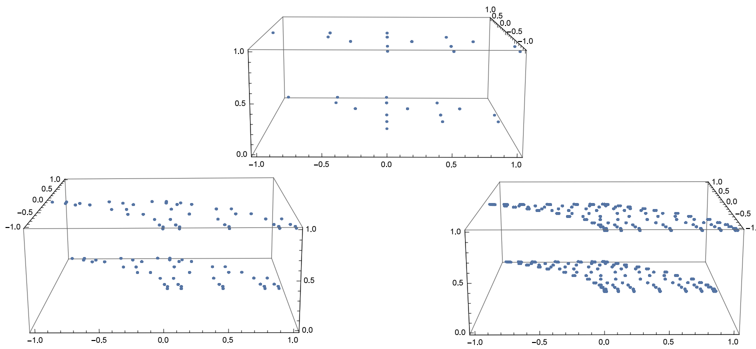

Figure 5: The control points from the first, second and the third iterations of the midpoint subdivision algorithm for Example 12.

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