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STABILITY INEQUALITIES FOR THE DELAY PSEUDO–PARABOLIC EQUATIONS

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Abstract: This paper deals with the initial-boundary value problem for linear pseudo-parabolic equation. Using the method of energy estimates the stability bounds obtained for the considered problem. Illustrative example is also presented.

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In the domain $\overline{Q} = \overline{\Omega} \times [0, T]$; $\overline{\Omega} = [0, l]$, $Q = \Omega \times (0, T]$, $\Omega = (0, l)$ we consider the following initial-boundary value problem for a pseudo-parabolic equation with delay

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$$\frac{\partial u\,(x,t)}{\partial t}-a(t)\frac{\partial^3 u\,(x,t)}{\partial t\partial x^2}=b(t)\frac{\partial^2 u\,(x,t)}{\partial x^2}+c(t)\frac{\partial^2 u\,(x,t-r)}{\partial x^2}$$

$$+\,d(t)u(x,t)+f(x,t),\quad (x,t)\in Q, \tag{1}$$

$$u(x,t)=\phi(x,t),\quad x\in\overline{\Omega},\quad -r\le t\le 0, \tag{2}$$

$$u(0,t)=u(l,t)=0,\quad t\in[0,T]\,,\quad a(t)\le\alpha\le 0, \tag{3}$$

where $r>0$ represents the delay parameter, $a\geq\alpha>0$, b, c, d, f and ϕ are given sufficiently smooth functions satisfying certain regularity conditions to be specified. The above equations are usually called Sobolev type or pseudo-parabolic equations, which appear in engineering fields, such as, for instance, flows of fluids through fissured rock, heat condition involving a thermodynamic temperature and a conductive temperature, and quasistationary processes in semiconductors (see, e.g. [1]-[6]). This existence and uniqueness result for pseudo-parabolic equations without delay can be found, e.g., in [7]-[11]. In the present study, using the method of energy estimates we have obtained the stability bounds for the problem (1)$-$(3). Illustrative example is also given.

Lemma 1. *Let $\delta(t)\geq 0$ be the continuous function such that*

$$\delta(t)\le\delta_*+\int_0^t\{c_0\delta(s)+c_1\delta(s-r)\}ds,\quad t>0,$$

$$\delta(t)=\varphi(t),\quad -r\le t\le 0,$$

with nonnegative constants δ_*, c_0, c_1 and $\varphi\in C\,[-r,0]$. Then

$$\delta(t)\le\delta_*\exp\left(c_0+c_1\int_{-r}^0\varphi(s)ds\right).$$

Proof. After replacing $s-r=\eta$ we observe that

$$\int_0^t\delta(s-r)ds=\int_{-r}^{t-r}\delta(\eta)d\eta=\begin{cases}\int_{-r}^0\varphi(t)dt, & 0\le t\le r,\\ \int_{-r}^0\varphi(t)dt+\int_0^{t-r}\delta(\eta)d\eta, & t\geq r.\end{cases}$$

Therefore we have

$$\delta(t)\le\delta_*+\left(c_0+c_1\int_{-r}^0\varphi(t)dt\right)\int_0^t\delta(s)ds,$$

which by using the Gronwall inequality completes the proof. □

Theorem 2. *For* $a, b, c, d \in C\left[0, T\right]$, $f \in C\left(\overline{Q}\right)$ *and* $\frac{\partial^k \phi}{\partial \phi^k} \in C\left(\overline{\Omega} \times [-r, 0]\right)$, $k = 0, 1$ *the solution of the delay boundary-value problem* $(1) - (3)$ *satisfies the following stability bound:*

$$\alpha\|u\|^2 + \left\|\frac{\partial u}{\partial x}\right\|^2 \leq \left[A + c_1 \int_{-r}^{0} \left(\alpha \left\|\phi\right\|^2 + \left\|\frac{\partial \phi}{\partial x}\right\|^2 dt\right)\right], \quad 0 \leq t \leq T, \tag{4}$$

where

$$(g, h) = \int_0^l g(x)h(x)dx, \quad \|g\|^2 = \int_0^l g^2(x)dx,$$

$$A = 2\alpha \left\|\phi\right\|^2 + \left\|\frac{\partial \phi}{\partial x}\right\|^2 + 4\alpha T \int_0^T \left\|f\right\|^2 ds,$$

$$c_0 = T \max\left(4\overline{c}^2, 2\overline{b}^2\right), \qquad c_1 = 4\alpha^{-2}\overline{d}^2 T, \qquad \overline{g} = \max_{[0,T]} |g(t)|.$$

Proof. Consider the identity

$$\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) - a(t)\left(\frac{\partial^3 u}{\partial t \partial x^2}, \frac{\partial u}{\partial t}\right) = b(t)\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t}\right)$$

$$+ c(t)\left(\frac{\partial^2 u(\cdot, t-r)}{\partial x^2}, \frac{\partial u}{\partial t}\right) + d(t)\left(u, \frac{\partial u}{\partial t}\right) + \left(f(t), \frac{\partial u}{\partial t}\right). \tag{5}$$

Next we will use the following relations

$$\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) = \left\|\frac{\partial u}{\partial t}\right\|^2,$$

$$\left(\frac{\partial^3 u}{\partial t \partial x^2}, \frac{\partial u}{\partial t}\right) = \left(\frac{\partial^2 u}{\partial t \partial x}, \frac{\partial^2 u}{\partial t \partial x}\right) = \left\|\frac{\partial^2 u}{\partial t \partial x}\right\|^2,$$

$$\left|b(t)\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t}\right)\right| = \left|b(t)\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t \partial x}\right)\right| \leq \mu_1 \left\|\frac{\partial^2 u}{\partial t \partial x}\right\|^2 + \frac{\bar{b}^2(t)}{4\mu_1}\left\|\frac{\partial u}{\partial x}\right\|^2,$$

$$\left|c(t)\left(\frac{\partial^2 u(\cdot, t-r)}{\partial x^2}, \frac{\partial u}{\partial t}\right)\right| = \left|c(t)\left(\frac{\partial u(\cdot, t-r)}{\partial x}, \frac{\partial^2 u}{\partial t \partial x}\right)\right|$$

$$\leq \mu_2 \left\|\frac{\partial^2 u}{\partial t \partial x}\right\|^2 + \frac{\bar{c}^2(t)}{4\mu_2}\left\|\frac{\partial u(\cdot, t-r)}{\partial x}\right\|^2,$$

$$\left|d(t)\left(u, \frac{\partial u}{\partial t}\right)\right| \leq \mu_3 \left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{\bar{d}^2(t)}{4\mu_3}\left\|u\right\|^2,$$

$$\left|\left(f(t), \frac{\partial u}{\partial t}\right)\right| \le \mu_4 \left\|\frac{\partial u}{\partial t}\right\|^2 + \frac{1}{4\mu_4}\|f\|^2 .$$

Then from (5) we have

$$(1-\mu_3-\mu_4)\left\|\frac{\partial u}{\partial t}\right\|^2 + (a(t)-\mu_1-\mu_2)\left\|\frac{\partial^2 u}{\partial t \partial x}\right\|^2 \le \frac{\bar{b}^2}{4\mu_1}\left\|\frac{\partial u}{\partial x}\right\|^2$$

$$+\frac{\bar{c}^2}{4\mu_2}\left\|\frac{\partial u(\cdot, t-r)}{\partial x}\right\|^2 + \frac{\bar{d}^2}{4\mu_3}\|u\|^2 + \frac{1}{4\mu_4}\|f\|^2 .$$

Choosing

$$\mu_1 = \mu_2 = \frac{\alpha}{4}, \quad \mu_3 = \mu_4 = \frac{1}{4},$$

we get

$$\left\|\frac{\partial u}{\partial t}\right\|^2 + \alpha\left\|\frac{\partial^2 u}{\partial t \partial x}\right\|^2 \le \frac{\bar{b}^2}{\alpha}\left\|\frac{\partial u}{\partial x}\right\|^2 + 2\bar{d}^2\|u\|^2 + \frac{2\bar{c}^2}{\alpha}\left\|\frac{\partial u(\cdot, t-r)}{\partial x}\right\|^2$$

$$+2\|f\|^2 . \tag{6}$$

After integrating (6) on $(0,t)$ and using the inequality

$$g^2(t) \le 2g^2(0) + 2T\int_0^t |g'(s)|^2 ds,$$

we obtain

$$\int_0^t \left\|\frac{\partial u}{\partial t}\right\|^2 ds \ge \frac{1}{2T}\|u\|^2 - \frac{1}{T}\|\phi\|^2 ,$$

$$\int_0^t \left\|\frac{\partial^2 u}{\partial t \partial x}\right\|^2 ds \ge \frac{1}{2T}\left\|\frac{\partial u}{\partial x}\right\|^2 - \frac{1}{T}\left\|\frac{\partial \phi}{\partial x}\right\|^2 .$$

Therefore the inequality (6) reduces to

$$\alpha\|u\|^2 + \left\|\frac{\partial u}{\partial x}\right\|^2 \le 4\alpha T\bar{d}^2\int_0^t \|u\|^2 ds + 2T\bar{b}^2\int_0^t \left\|\frac{\partial u}{\partial x}\right\|^2 ds$$

$$+4T\bar{c}^2\int_0^t \left\|\frac{\partial u(\cdot, t-r)}{\partial x}\right\|^2 ds + A.$$

Denote

$$\delta(t) = \alpha\|u\|^2 + \left\|\frac{\partial u}{\partial x}\right\|^2 ,$$

then

$$\delta(t) \leq A + c_0 \int_0^t \delta(s)ds + c_1 \int_0^t \delta(s-r)ds.$$

From here by fLemma1 we have

$$\delta(t) \leq \left[A + c_1 \int_{-r}^0 \left(\alpha \left\| \phi \right\|^2 + \left\| \frac{\partial \phi}{\partial x} \right\|^2 \right) dt \right] \exp\left(c_0 T + c_1 \exp(c_0 T) t \right)$$

which immediately leads to (4). □

Example

$$\frac{\partial u}{\partial t} - (1+t)^2 \frac{\partial^3 u}{\partial t \partial x^2} = e^{-t} \frac{\partial^2 u}{\partial x^2} + \sqrt{2+t^2} \frac{\partial^2 u(x,t-1)}{\partial x^2} + tu + t \sin \pi x,$$

$$0 < t \leq 1, \quad 0 < x < 1,$$

$$u(x,t) = te^{-t}, \quad 0 < x < 1, \quad -1 \leq t \leq 0.$$

Using the inequality (4) with

$$\alpha = 1, \quad \bar{b} = 1, \quad \bar{c} = \sqrt{3}, \quad \bar{d} = 1,$$

$$\int_{-r}^0 \left\| \phi \right\|^2 dt = \int_{-r}^0 \frac{t}{2} \left(1 - e^{-2x} \right) dt = \frac{1 - e^- 2x}{6},$$

$$\int_{-r}^0 \left\| \frac{\partial \phi}{\partial x} \right\|^2 dt = \frac{1 - e^- 2x}{6},$$

gives us the following stability estimate for the solution of our particular problem:

$$v(t) \geq 0,$$

$$\left\| u \right\|^2 + \left\| \frac{\partial u}{\partial x} \right\|^2 \leq \left[2 + 12 \left(\frac{1 - e^- 2x}{6} \right) \right] \exp(4T + 12 \exp(4T) t).$$

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