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CONSTRUCTION OF NESTED REAL IDEAL LATTICES FOR INTERFERENCE CHANNEL CODING

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Abstract: In this work we develop a new algebraic methodology which quantizes real-valued channels in order to realize interference alignment (IA) onto a real ideal lattice. Also we make use of the minimum mean square error (MMSE) criterion to estimate real-valued channels contaminated by additive Gaussian noise.

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1. Introduction

In this work we make use of rotated real lattices constructed through extension fields to develop a new methodology to perform a real-valued channel quantization in order to realize interference alignment (IA) [1] onto a real ideal lattice.

In a wireless network, a transmission from a single node is heard not only by the intended receiver, but also by all other nearby nodes. The resulting interference is usually viewed as highly undesirable and complex algorithms and protocols have been devised to avoid interference between transmitters.

Each node, indexed by $m = 1, 2, \ldots, M$, observes a noisy linear combination of the transmitted signals through the channel

$$y_m = \sum_{l=1}^{L} h_{ml} x_l + z_m, \tag{1}$$

where $h_{ml} \in \mathbb{R}$ are real-valued channel coefficients, x_l is a real lattice point whose message space presents a uniform distribution and z_m is an i.i.d. circularly symmetric real Gaussian noise. Figure 1 illustrates the corresponding channel model.

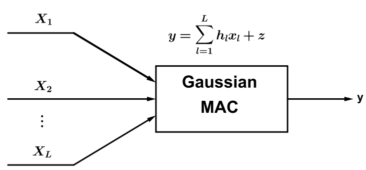


Figure 1: A Gaussian Multiple-Access Channel

Calderbank and Sloane [2] made the important observations that the signal constellation should be regarded as a finite set of points taken from an infinite lattice and the partitioning of the constellation into subsets corresponds to the partitioning of that lattice into a sublattice and its cosets. We call this general class of coded modulation schemes coset codes.

There is a great number of works based on coset codes and their applications in communications. It is not possible to discuss all of them here, but the references [3] and [4] are great indications for the interested reader.

In the literature, Trinca Watanabe et al. [5] developed a new algebraic methodology which quantizes complex-valued channels in order to realize interference alignment (IA) onto a complex ideal lattice and Andrade et al. [6] show that algebraic lattices can be associated to the rings of integers $\mathbb{Z}[\xi_{2^r} + \xi_{2^r}^{-1}]$ of the totally real number fields $K = \mathbb{Q}(\xi_{2^r} + \xi_{2^r}^{-1})$, where ξ_{2^r} denotes the 2^r -th root of unity and $r \geq 3$. These algebraic lattices are a scaled version of the $\mathbb{Z}^n$-lattices, where $n = 2^{r-2}$ and $r \geq 3$.

Therefore, in this work, we develop a new algebraic methodology to quantize real-valued channels in order to realize interference alignment (IA) [1] onto a real ideal lattice and our channel model is given by equation (1). The coding scheme only requires that each relay knows the channel coefficients from each transmitter to itself.

In this new methodology we make use of the maximal real subfield $K = \mathbb{Q}(\xi_{2^r} + \xi_{2^r}^{-1})$ of the binary cyclotomic field $\mathbb{Q}(\xi_{2^r})$, where $r \geq 3$, to provide a doubly infinite nested lattice partition chain for any dimension $n = 2^{r-2}$, where $r \geq 3$, in order to quantize real-valued channels onto these nested lattices. Such real ideal lattices are featured by their generator and Gram matrices which are developed by an algorithm [7] and, therefore, they can be described by their corresponding Construction A which furnishes us, in this case, nested lattice codes (coset codes). It is very important that the channel gain does not remove the lattice from the initial chain of nested lattices, then we show the existence of periodicity in the corresponding nested lattice partition chains.

After developing such a channel quantization, we develop a precoding to ensure onto which lattice a given real-valued channel must be quantized.

The concept of mean square error has assumed a central role in the theory and practice of estimation since the time of Gauss and Legendre. In particular, minimization of mean square error underlies numerous methods in statistical sciences. In this paper, we make use of the minimum mean square error (MMSE) to estimate real-valued channels contaminated by additive Gaussian noise.

In the following section we provide a quick preview of the concepts related to coset codes and real ideal lattices that will figure in the rest of the paper.

2. Preliminaries

Lattices have been very useful in applications in communication theory, for instance, the E_8-lattice is one of the densest lattices and in [8] we have one of the constructions of this lattice.

In this work we use real ideal lattices in order to realize interference alignment and, in this section, we present basic concepts of the lattice theory.

Definition 1. Let $v_1, v_2, \ldots, v_m$ be a set of linearly independent vectors in $\mathbb{R}^n$ such that $m \leq n$. The set of the points

$$\Lambda = \{x = \sum_{i=1}^{m} \lambda_i v_i, \text{ where } \lambda_i \in \mathbb{Z}\} \tag{2}$$

is called a *lattice* of rank m and $\{v_1, v_2, \ldots, v_m\}$ is called a basis of the lattice.

So we have that a real *lattice* Λ is simply a discrete set of vectors (points (n-tuples)) in real Euclidean n-space $\mathbb{R}^n$ that forms a group under ordinary vector addition, i.e., the sum or difference of any two vectors in Λ is in Λ. Thus Λ necessarily includes the all-zero n-tuple *0* and if λ is in Λ, then so is its additive inverse $-\lambda$.

As an example, the set $\mathbb{Z}$ of all integers is the only one-dimensional real lattice, up to scaling, and the prototype of all lattices. The set $\mathbb{Z}^n$ of all integer n-tuples is an n-dimensional real lattice, for any n, and its corresponding $\frac{n}{2}$-dimensional complex lattice is given by $\mathbb{Z}[i]^{\frac{n}{2}}$.

Lattices have only two principal structural characteristics. Algebraically, a lattice is a group; this property leads to the study of subgroups (sublattices) and partitions (coset decompositions) induced by such subgroups. Geometrically, a lattice is endowed with the properties of the space in which it is embedded, such as the Euclidean distance metric and the notion of volume in $\mathbb{R}^n$, [3].

A sublattice Λ' of Λ is a subset of the points of Λ which is itself an n-dimensional lattice. The sublattice induces a partition Λ/Λ' of Λ into $|\Lambda/\Lambda'|$ cosets of Λ', where $|\Lambda/\Lambda'|$ is the order of the partition.

The coset code $\mathcal{C}(\Lambda/\Lambda'; C)$ is the set of all sequences of signal points that lie within a sequence of cosets of Λ' that could be specified by a sequence of coded bits from C. Some lattices, including the most useful ones, can be generated as lattice codes $\mathcal{C}(\Lambda/\Lambda'; C)$, where C is a binary block code. If C is a convolutional encoder, then $\mathcal{C}(\Lambda/\Lambda'; C)$ is a trellis code, [3].

A lattice code $\mathcal{C}(\Lambda/\Lambda'; C)$, where C is a binary block code, is defined as the set of all coset leaders in Λ/Λ', i.e.,

$$\mathcal{C}(\Lambda/\Lambda'; C) = \Lambda \bmod \Lambda' = \{\lambda \bmod \Lambda' \ : \ \lambda \in \Lambda\}. \tag{3}$$

Geometrically, $\mathcal{C}(\Lambda/\Lambda'; C)$ is the intersection of the lattice Λ with the fundamental region $\mathcal{R}_{\Lambda'}$ [3], i.e.,

$$\mathcal{C}(\Lambda/\Lambda'; C) = \Lambda \cap \mathcal{R}_{\Lambda'}. \tag{4}$$

For this reason, the fundamental region $\mathcal{R}_{\Lambda'}$ is often interpreted as the *shaping region.* Note that there is a bijection between Λ/Λ' and $\mathcal{C}(\Lambda/\Lambda'; C)$; in particular,

$$|\Lambda/\Lambda'| = |\mathcal{C}(\Lambda/\Lambda'; C)|. \tag{5}$$

A lattice Λ is said to be *nested* in a lattice Λ' if $\Lambda \subseteq \Lambda'$. We refer to Λ as the coarse lattice and Λ' as the fine lattice. More generally, a sequence of lattices $\Lambda, \Lambda_1, \ldots, \Lambda_P$ is nested if $\Lambda \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_P$. Observe that nested lattices induce nested lattice codes.

In [3] an n-dimensional real lattice Λ is a *mod-2 binary lattice* if and only if it is the set of all integer n-tuples that are congruent modulo 2 to one of the codewords c in a linear binary (n, k) block code C. *Mod-2 binary lattices* are essentially isomorphic to linear binary block codes and this is "*Construction A*" of Leech and Sloane [9].

Let K be a number field, i.e., an extension of finite degree of $\mathbb{Q}$. Let n be the degree of K.

Definition 2. ([10]) We call the embeddings of K the set of field homomorphisms

$$\{\sigma_i : K \to \mathbb{C},\ i = 1, 2, \ldots, n \mid \sigma_i(x) = x,\ \forall x \in \mathbb{Q}\}. \tag{6}$$

The signature (r_1, r_2) of K is defined by the number of real (r_1) and complex $(2r_2)$ embeddings such that $n = r_1 + 2r_2$. If all the embeddings of K are real (resp., complex), we say that K is *totally real* (resp., *totally complex*).

Definition 3. ([10]) Let $K = \mathbb{Q}(\theta)$ be an extension of $\mathbb{Q}$ of degree n. If the minimal polynomial of θ over $\mathbb{Q}$ has all its roots in K, we say that K is a Galois extension of $\mathbb{Q}$. The set

$$Gal(K/\mathbb{Q}) = \{\sigma : K \to K \mid \sigma(x) = x,\ \forall x \in \mathbb{Q}\} \tag{7}$$

of field automorphisms fixing $\mathbb{Q}$ is a group under the composition called the *Galois group* of K over $\mathbb{Q}$.

Note that when K is a Galois extension, the set of its embeddings coincides with its Galois group.

Definition 4. ([10]) Let K be a Galois extension of $\mathbb{Q}$. Let $x \in K$ and $Gal(K/\mathbb{Q}) = \{\sigma_i\}_{i=1}^{n}$. The trace of x over $\mathbb{Q}$ is defined as

$$Tr_{K/\mathbb{Q}}(x) = \sum_{i=1}^{n} \sigma_i(x), \tag{8}$$

while the norm of x is defined by

$$N_{K/\mathbb{Q}}(x) = \prod_{i=1}^{n} \sigma_i(x). \tag{9}$$

If the field extension is clear from the context, then we may write, respectively, $Tr(x)$ and $N(x)$.

The theory of ideal lattices gives a general framework for algebraic lattice constructions. We recall this notion in the case of totally real algebraic number fields.

Definition 5. ([10]) Let K and O_K be a totally real number field of degree n and the corresponding ring of integers of K, respectively. An ideal lattice is a lattice $\Lambda = (I, q_\alpha)$, where I is an ideal of O_K and

$$q_\alpha : I \times I \to \mathbb{Z}, \text{ where } q_\alpha(x, y) = Tr_{K/\mathbb{Q}}(\alpha x y), \ \forall x, y \in I, \tag{10}$$

where $\alpha \in K$ is totally positive (i.e., $\sigma_i(\alpha) > 0$, for all i).

If $\{w_1, w_2, \ldots, w_n\}$ is a $\mathbb{Z}$-basis of I, the generator matrix R of an ideal lattice $\sigma(I) = \Lambda = \{x = R\lambda \mid \lambda \in \mathbb{Z}^n\}$ is given by

$$R = \begin{pmatrix} \sqrt{\alpha_1}\sigma_1(w_1) & \cdots & \sqrt{\alpha_1}\sigma_1(w_n) \\ \vdots & \ddots & \vdots \\ \sqrt{\alpha_n}\sigma_n(w_1) & \cdots & \sqrt{\alpha_n}\sigma_n(w_n) \end{pmatrix}, \tag{11}$$

where $\alpha_i = \sigma_i(\alpha)$, $i = 1, \ldots, n$. One easily verifies that the Gram matrix $R^t R$ coincides with the trace form $(Tr(\alpha w_i w_j))_{i,j=1}^{n}$, where t denotes the transposition. For the $\mathbb{Z}^n$-lattice, the corresponding lattice generator matrix given in (11) becomes an orthogonal matrix ($R^{-1} = R^t$) and we talk about “rotated” $\mathbb{Z}^n$-lattices.

3. Construction of Real Nested Lattices from the Maximal Real Subfield $\mathbb{Q}(\xi_{2^r} + \xi_{2^r}^{-1})$ of $\mathbb{Q}(\xi_{2^r})$ in Order to Realize Interference Alignment

In order to realize interference alignment onto a lattice, we need to quantize the channel coefficients h_{ml}. Thus, in this section, we describe a way to find a doubly infinite nested lattice partition chain for any dimension $n = 2^{(r-2)}$, with $r \geq 3$, in order to quantize the channel coefficients. For that, we make use of the maximal real subfield $\mathbb{Q}(\xi_{2^r} + \xi_{2^r}^{-1})$ of the binary cyclotomic field $\mathbb{Q}(\xi_{2^r})$, with $r \geq 3$, $[\mathbb{Q}(\xi_{2^r}) : \mathbb{Q}] = \varphi(2^r) = 2^{(r-1)}$, where φ is the Euler function, and $[\mathbb{Q}(\xi_{2^r} + \xi_{2^r}^{-1}) : \mathbb{Q}] = 2^{(r-2)} = n$. Hence we provide a new algebraic methodology to quantize real-value channels.

Such lattices are real ideal lattices that are described by their corresponding generator and Gram matrices and provide us, in this case, nested lattice codes (nested coset codes).

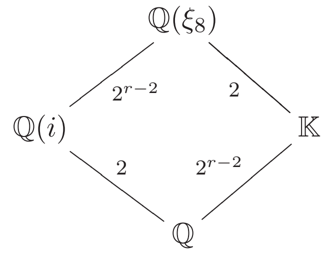
In [11] we have an example of channel quantization. For the corresponding quantization, we make use of the maximal real subfield $\mathbb{Q}(\sqrt{2})$ of the binary cyclotomic field $\mathbb{Q}(\xi_8)$. This example is related to the real dimension 2.

3.1. Quantization of real-valued channels onto a lattice

Suppose that our interference channel is real-valued, specifically $h_{ml} \in \mathbb{R}$. We suppose that all lattices used by the legitimate user and the interferers are one of a certain lattice partition chain which is extended by periodicity.

In this section, we consider n-dimensional real-valued vectors, where $n = 2^{r-2}$ and $r \geq 3$. Now we show, for a given user, how its codeword can be transformed so that we can perform the channel quantization and, for that, we make use of the maximal real subfield $\mathbb{Q}(\xi_{2^r} + \xi_{2^r}^{-1})$ of the binary cyclotomic field $\mathbb{Q}(\xi_{2^r})$, where $r \geq 3$.

In fact, consider the following Galois extensions, where $r \geq 3$ and $\mathbb{K} = \mathbb{Q}(\xi_{2^r} + \xi_{2^r}^{-1})$:



(12)

In [6], by Theorem 10, we have that $\mathbb{Z}[\xi_{2^r} + \xi_{2^r}^{-1}]$ is the ring of integers of

$\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})$ and

$$\{1, \xi_{2^r}+\xi_{2^r}^{-1}, \ldots, \xi_{2^r}^{n-1}+\xi_{2^r}^{-(n-1)}\} \tag{13}$$

is an integral basis of $\mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$.

Let $Gal(\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_n\}$ be the Galois group of $\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})$ over $\mathbb{Q}$. We find an ideal of norm equal to 2, that is, we find an element of $\mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$ with absolute algebraic norm equal to 2. In fact,

$$2 = N_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}}(1+\xi_{2^r}) = N_{\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}}(2+2cos(\pi/2^{(r-1)})). \tag{14}$$

Observe that $\xi_{2^r}+\xi_{2^r}^{-1} = 2cos(\pi/2^{(r-1)})$, then $N_{\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}}(2+\xi_{2^r}+\xi_{2^r}^{-1}) = 2$ and $2+\xi_{2^r}+\xi_{2^r}^{-1} \in \mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$. Thus $\Im = \langle 2+\xi_{2^r}+\xi_{2^r}^{-1}\rangle = (2+\xi_{2^r}+\xi_{2^r}^{-1})\mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$ is a principal ideal of $\mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$ with norm equal to 2.

In this work, we suppose that the columns of a matrix generate the $\mathbb{Z}^n$-lattice. Thus, by [6], page 7, we can conclude that the generator matrix of the rotated $\mathbb{Z}^n$-lattice is given by

$$M_0 = \frac{1}{\sqrt{2^{r-1}}} A M^t T \quad \text{and} \quad M_0^t M_0 = I, \tag{15}$$

where t denotes the transposition, I is the $n \times n$ identity matrix, M is given by

$$\begin{pmatrix} \sigma_1(1) & \sigma_2(1) & \cdots & \sigma_n(1) \\ \sigma_1(\xi_{2^r}+\xi_{2^r}^{-1}) & \sigma_2(\xi_{2^r}+\xi_{2^r}^{-1}) & \cdots & \sigma_n(\xi_{2^r}+\xi_{2^r}^{-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1(\xi_{2^r}^{n-1}+\xi_{2^r}^{-(n-1)}) & \sigma_2(\xi_{2^r}^{n-1}+\xi_{2^r}^{-(n-1)}) & \cdots & \sigma_n(\xi_{2^r}^{n-1}+\xi_{2^r}^{-(n-1)}) \end{pmatrix}, \tag{16}$$

$$A = diag\left(\sqrt{\sigma_i(2-\xi_{2^r}+\xi_{2^r}^{-1})}\right)_{i=1}^{n}, \tag{17}$$

and

$$T = \begin{pmatrix} -1 & -1 & \cdots & -1 & -1 \\ -1 & -1 & \cdots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{18}$$

At the receiver, we suppose that we apply M_0 to the received vector (1) to obtain

$$\bar{y}_m = M_0 y_m = \sum_{l=1}^{L} h_{ml} M_0 x_l + M_0 z_m. \tag{19}$$

As z_m is an i.i.d. circularly symmetric complex Gaussian noise and M_0 is an orthogonal matrix, then the noise in (19) is also i.i.d. circularly symmetric complex Gaussian.

Now we observe the vectors of the form $h_{ml}M_0x_l$. Hence we can rewrite it as

$$\begin{pmatrix} h_{ml} & 0 & \cdots & 0 \\ 0 & h_{ml} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{ml} \end{pmatrix} \cdot M_0 \cdot x_l = H_{ml} \cdot M_0 \cdot x_l. \tag{20}$$

We have that $\Im^k$, where $k \in \mathbb{Z}$, is an ideal of $\mathbb{Z}[\xi_{2^r} + \xi_{2^r}^{-1}]$ generated by $(2 + \xi_{2^r} + \xi_{2^r}^{-1})^k$, that is, $\Im^k = (2 + \xi_{2^r} + \xi_{2^r}^{-1})^k \mathbb{Z}[\xi_{2^r} + \xi_{2^r}^{-1}]$.

Now if $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[\xi_{2^r} + \xi_{2^r}^{-1}]$, then we can see that

$$\{(2 + \xi_{2^r} + \xi_{2^r}^{-1})^k\gamma_1, (2 + \xi_{2^r} + \xi_{2^r}^{-1})^k\gamma_2, \ldots, (2 + \xi_{2^r} + \xi_{2^r}^{-1})^k\gamma_n\} \tag{21}$$

is a $\mathbb{Z}$-basis of $\Im^k$, since the set of invertible fractional ideals form an abelian group related to the product of ideals. Thus a generator matrix of the real algebraic lattice $\sigma(\Im^k)$ [10] is given by

$$\begin{aligned} M'_k &= A \cdot \begin{pmatrix} (2+\theta)^k & 0 & \cdots & 0 \\ 0 & \sigma_2((2+\theta)^k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n((2+\theta)^k) \end{pmatrix} \cdot M^tT \\ &= AA'_kM^tT = A'_kAM^tT, \end{aligned} \tag{22}$$

where $\theta = \xi_{2^r} + \xi_{2^r}^{-1}$, $A'_k = diag(\sigma_i((2 + \xi_{2^r} + \xi_{2^r}^{-1})^k))_{i=1}^n$ and $AA'_k = A'_kA$, since A and A'_k are diagonal matrices.

Since M_0 and AM^tT generate the same lattice, the $\mathbb{Z}^n$-lattice, and by comparing the equations (20) and (22), then the conclusion is that the matrix H_{ml} can be approximated by

$$A'_k = \begin{pmatrix} (2+\theta)^k & 0 & \cdots & 0 \\ 0 & \sigma_2((2+\theta)^k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n((2+\theta)^k) \end{pmatrix}. \tag{23}$$

Thus the diagonal matrix H_{ml} is quantized by the diagonal matrix A'_k whose elements are components of the canonical embedding of the power (positive or negative) of an element of $\mathbb{Z}[\xi_{2^r} + \xi_{2^r}^{-1}]$ with absolute algebraic norm equal to 2.

Now, by using the concept of equivalent lattices, observe that

$$A'_kM_0 = \begin{pmatrix} (2+\theta)^k & 0 & \cdots & 0 \\ 0 & \sigma_2((2+\theta)^k) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n((2+\theta)^k) \end{pmatrix} \cdot M_0$$

$$= M_0 M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})^k}, \tag{24}$$

where $M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})^k}$ is an $n \times n$ matrix whose entries belong to $\mathbb{Z}$; this means that if $(2+\xi_{2^r}+\xi_{2^r}^{-1})^k$ generates the ideal $(2+\xi_{2^r}+\xi_{2^r}^{-1})^k \mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$, then the matrix $M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})^k}$ is a generator matrix of the lattice that is the canonical embedding of the ideal $\Im^k$ whose position compared to the $\mathbb{Z}^n$-lattice is equal to k.

Since for $k = 1$ we have

$$\begin{pmatrix} 2+\xi_{2^r}+\xi_{2^r}^{-1} & 0 & \cdots & 0 \\ 0 & \sigma_2(2+\xi_{2^r}+\xi_{2^r}^{-1}) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n(2+\xi_{2^r}+\xi_{2^r}^{-1}) \end{pmatrix} \cdot M_0$$

$$= M_0 M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})}, \tag{25}$$

then we can see, by induction, that $A'_k M_0 = M_0 (M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})})^k$, for $k \geq 1$; that is, $M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})^k} = (M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})})^k$, for $k \geq 1$.

Now in the following section we present a method that describes for any dimension $n = 2^{r-2}$, with $r \geq 3$, a doubly infinite nested lattice partition chain in order to quantize real-valued channels onto a lattice, that is, in order to realize interference alignment onto a lattice and, for that, we make use of an algorithm.

3.2. Construction of real nested ideal lattices from the channel quantization

In [11] we have that the lattice partition chain related to $r = 3$ $(n = 2)$ is given by

$$\cdots \supset (2\Lambda_1)^* \supset \Lambda_2^* \supset \Lambda_1^* \supset \Lambda_0 = \mathbb{Z}^2 \supset$$

$$\supset \Lambda_1 = 2\mathbb{Z}^2 + C_1 \supset \Lambda_2 = 2\mathbb{Z}^2 \supset 2\Lambda_1 \supset \cdots \tag{26}$$

where $*$ denotes the dual of a lattice and C_1 is the binary block code generated by the vector (0 1).

Here we describe a way to find a doubly infinite nested lattice partition chain for any dimension $n = 2^{r-2}$, where $r \geq 3$.

Consider the following equation:

$$A'_k(AM^tT) = (AM^tT) M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})^k}. \tag{27}$$

The following algorithm, Algorithm 1, calculates in an equivalent way equation (27). Such an algorithm furnishes us the generator matrix $M_k = M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})^k}$ of the lattice related to k and the corresponding Gram matrix G_k of such a lattice.

In this algorithm, for each r and k, we find the generator and Gram matrices of a lattice which is isomorphic to the canonical embedding of the ideal $\Im^k$, where we know that $\Im^k = (2+\xi_{2^r}+\xi_{2^r}^{-1})^k\mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$ is an ideal of $\mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$ generated by $(2+\xi_{2^r}+\xi_{2^r}^{-1})^k$. The position of this lattice compared to the $\mathbb{Z}^n$-lattice in the nested lattice partition chain related to r is exactly k.

Algorithm 1 Algorithm for calculating equation (27)

$n = 2^{r-2}$

$A = diag\left(\sqrt{\sigma_i(2-\xi_{2^r}+\xi_{2^r}^{-1})}\right)_{i=1}^{n}$

$A'_k = diag\left(\sigma_i((2+\xi_{2^r}+\xi_{2^r}^{-1})^k)\right)_{i=1}^{n}$

compute M_0 from equation (15)

compute $P = M_0 * A$ and P^{-1}

calculate integer entrances of $M_k = P * A'_k * P^{-1}$ and M_k^T

compute G_k which is the integer entrances of the LLL reduction of $M_k * M_k^T$

From Algorithm 1, for $k = n$, we have that the corresponding Gram matrix G_n is given by

$$G_n = \begin{pmatrix} 2^n & 0 & \cdots & 0 \\ 0 & 2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2^n \end{pmatrix}. \tag{28}$$

Then M_n is a generator matrix of the lattice Λ_n and by observing the corresponding Gram matrix G_n, we can conclude that the algebraic lattice Λ_n is the lattice $2\mathbb{Z}^n$, that is, $\Lambda_n = 2\mathbb{Z}^n$.

This means that if $(2+\xi_{2^r}+\xi_{2^r}^{-1})^n$ generates the ideal $\Im^n$, then the matrix M_n is a generator matrix of the lattice $\Lambda_n = 2\mathbb{Z}^n$ whose position in the nested lattice partition chain compared to the $\mathbb{Z}^n$-lattice is equal to $k = n$.

Let $M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})}$ represent the generator matrix of the lattice related to the position $k = 1$ calculated by using equation (25). Hence the following theorem gives us the extension by periodicity of the nested lattice partition chain for the positive positions, that is, $k \geq 1$.

Theorem 6. *For $k = n\beta + j$, where $\beta \in \mathbb{N}$ and $0 \leq j \leq n-1$, we have* *that* $M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})^{(n\beta+j)}} = (M_{(2+\xi_{2^r}+\xi_{2^r}^{-1})})^{k=(n\beta+j)}$ *is a generator matrix of the lattice* $2^{\beta}\Lambda_j$ *seen as a* $\mathbb{Z}$*-lattice, where* Λ_j *is the lattice featured by Algorithm 1 related to the position* k *compared to the* $\mathbb{Z}^n$*-lattice.*

Proof. See Appendix. □

Thus, by Theorem 6, we can conclude that the periodicity of the nested lattice partition chain for the positive positions is equal to $k = n$ because $\sigma(\Im^n) = 2\mathbb{Z}^n$, that is, $\sigma(\Im^n)$ is a scaled version of the $\mathbb{Z}^n$-lattice.

Therefore, we have the interference alignment onto a lattice for $k \geq 0$. Now the following theorem furnishes us the extension by periodicity of the nested lattice partition chain for the negative positions, that is, $k \leq -1$.

Theorem 7. *For all* $k \in \mathbb{N}^*$*, we have* $\sigma(\Im^{-k}) = \sigma(\Im^k)^*$*, where* $\sigma(\Im^k)^*$ *indicates the dual lattice of* $\sigma(\Im^k)$*.*

Proof. Let $\overrightarrow{x}$ and $\overrightarrow{y}$ be arbitrary elements of $\sigma(\Im^k)$ and $\sigma(\Im^{-k})$, respectively, where $k \in \mathbb{N}^*$. Then we have

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = Tr_{\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}}(x \cdot y), \tag{29}$$

where $x \in \Im^k$ and $y \in \Im^{-k}$. So $x = (2+\xi_{2^r}+\xi_{2^r}^{-1})^k x_0$, where $x_0 \in \mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$, and $y = (2+\xi_{2^r}+\xi_{2^r}^{-1})^{-k} y_0$, where $y_0 \in \mathbb{Z}[\xi_{2^r}+\xi_{2^r}^{-1}]$.

It is easy to see that

$$\begin{aligned} Tr_{\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}}(x \cdot y) &= \sum_{i=1}^{n} \sigma_i(x \cdot y) = \sum_{i=1}^{n} \sigma_i(x)\sigma_i(y) \\ &= \sum_{i=1}^{n} \sigma_i(x_0)\sigma_i(y_0) = Tr_{\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}}(x_0 \cdot y_0). \end{aligned} \tag{30}$$

We have that $Tr_{\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}}(x_0 \cdot y_0) \in \mathbb{Z}$, then

$$\langle \overrightarrow{x}, \overrightarrow{y} \rangle = Tr_{\mathbb{Q}(\xi_{2^r}+\xi_{2^r}^{-1})/\mathbb{Q}}(x \cdot y) \in \mathbb{Z}. \tag{31}$$

Thus, $\sigma(\Im^{-k}) \subset \sigma(\Im^k)^*$, for all $k \in \mathbb{N}^*$. We also have that

$$Vol[\sigma(\Im^k)^*] = \frac{1}{Vol[\sigma(\Im^k)]} = Vol[\sigma(\Im^{-k})]. \tag{32}$$

So the index $|\sigma(\Im^k)^*/\sigma(\Im^{-k})|$ is equal to 1 and, then, $\sigma(\Im^{-k}) = \sigma(\Im^k)^*$. □

By using Theorems 6 and 7 we can conclude that we have n $(k = 0, 1, 2, \ldots, n-1)$ different lattices in the doubly infinite nested lattice partition chain.

Hence, in this section, we have constructed a doubly infinite nested lattice partition chain related to any dimension $n = 2^{r-2}$, where $r \geq 3$, in order to realize interference alignment onto a lattice. Thus, for the real case, we have a generalization to obtain a doubly infinite nested lattice partition chain in order to quantize the channel coefficients in order to realize interference alignment onto a lattice.

The corresponding doubly infinite nested lattice partition chain is given as it follows:

$$\cdots \supset (2\mathbb{Z}^n)^* \supset (\Lambda_{n-1})^* \supset \cdots \supset (\Lambda_1)^* \supset$$

$$\supset \Lambda_0 = \mathbb{Z}^2 \supset \Lambda_1 \supset \cdots \supset \Lambda_{n-1} \supset 2\mathbb{Z}^n \supset \cdots \tag{33}$$

Besides, consequently, we have constructed nested lattice codes (nested coset codes) with $2\mathbb{Z}^n$ being the corresponding sublattice. The following algorithm, Algorithm 2, calculates the construction A of the corresponding lattices Λ_k, where $k = 0, 1, 2, \ldots, n-1$, and, then, we obtain the respective nested lattice codes.

Algorithm 2 Algorithm for calculating the construction A of the lattices Λ_k, where $k = 0, 1, 2, \ldots, n-1$

1: $k = 0, 1, \ldots, n-1$
2: v_i, where $i = 1, 2, \ldots, n$, is the i-th column of the matrix M_k which is a basis of the lattice Λ_k
3: compute the entrances of each v_i modulo 2, where $i = 1, 2, \ldots, n$
4: c_i is the binary vector coming from the vector v_i modulo 2, where $i = 1, 2, \ldots, n$
5: M_{C_k} is the matrix in which its columns are formed by the binary vectors c_i, where $i = 1, 2, \ldots, n$
6: M_{C_k} is a generator matrix of the linear binary block code C_k
7: lattice code $\frac{\Lambda_k}{2\mathbb{Z}^n} \simeq C_k$ related to k
8: $\Lambda_k = 2\mathbb{Z}^n + C_k$ is the construction A of the lattice Λ_k, where $k = 0, 1, 2, \ldots, n-1$
9: $C_{n-1} \subset C_{n-2} \subset \cdots \subset C_1 \subset C_0$ are nested lattice codes, where C_0 is the universal linear binary block code

4. Precoder

In Section 3 we show that complex-valued channels can be quantized onto a lattice. Therefore, precoding is essential to ensure onto which lattice a given complex-valued channel coefficient must be quantized. Hence, in this section, we provide the details of such a precoding which is related to the dimension $n = 2^{r-2}$, where $r \geq 3$.

Observe that $\xi_{2^r}^n = i \in \mathbb{Z}[i]$, where $n = 2^{r-2}$. A generator of an ideal of a ring of integers multiplied by a unit of this ring of integers also generates such an ideal. Thus we must analyse all the possible generators and, for each case, utilize a precoding for that the respective channel approximations be aligned onto one of the n different lattices related to the doubly infinite nested lattice partition chain constructed in Section 3.2. As generators, note that $(1 + \xi_{2^r})^n = (1 + i) \in \mathbb{Z}[i]$.

In Section 3.2 we have n different lattices related to the doubly infinite nested lattice partition chain, the other lattices are equivalent to one of these n different lattices. Observe that these n different lattices are the lattices related to the positions $0, 1, 2, 3, \ldots, n-1$ of the doubly infinite nested lattice partition chain.

Remember that the position of the lattices in the doubly infinite nested lattice partition chain is related to the power of the principal ideal $(1+\xi_{2^r})\mathbb{Z}[\xi_{2^r}] = \Im$, that is, let $(1 + \xi_{2^r})^k$ and by computing k modulo n, we have that $k \in \{0, 1, 2, 3, \ldots, n-1\}$ and the ideal $(1 + \xi_{2^r})^k \mathbb{Z}[\xi_{2^r}] = \Im^k$ furnishes us, by using the Galois embedding, the lattice related to the position k of the doubly infinite nested lattice partition chain.

We have that all the possible generators are $(\xi_{2^r})^{k'}(1 + \xi_{2^r})^k \lambda$ [12], where $\lambda \in \mathbb{Z}[i]$ and $k, k' \in \mathbb{Z}$. Then we have to analyse the product $(\xi_{2^r})^{k'}(1 + \xi_{2^r})^k$, since $\lambda \neq 1$ removes the element $(\xi_{2^r})^{k'}(1 + \xi_{2^r})^k$ from the origin. Therefore, all the possible generators of the ideals are the elements $(\xi_{2^r})^{k'}(1+\xi_{2^r})^k$, where $k, k' \in \mathbb{Z}$.

We also have that k and k', for the dimension $n = 2^{r-2}$ $(r \geq 3)$, each of them has n possibilities of values, since $\xi_{2^r}^n = i \in \mathbb{Z}[i]$ and $k \in \{0, 1, 2, 3, \ldots, n-1\}$. So, by analysing the element $(\xi_{2^r})^{k'}(1+\xi_{2^r})^k$, we have a total of n^2 possibilities of values for it.

Now as it is not possible to discuss all the cases for k and k' in order to precode the complex-valued channel coefficients h_{ml}, then we explain the process to realize the precoding in each case, i.e., for each case, we ensure that the complex-valued channel coefficient belongs to a corresponding lattice (one of the n different lattices). For that, we observe the form of the generator in each case.

For the case $k \equiv 0$ modulo n and $k' \equiv 0$ modulo n, we have no precoding because h_{ml} is approximated by an element that belongs in $\mathbb{Z}[i]$.

For the other cases we fix a particular one, then h_{ml} is approximated by $(\xi_{2^r})^{k'}(1+\xi_{2^r})^k$, that is,

$$h_{ml} \to (\xi_{2^r})^{k'}(1+\xi_{2^r})^k, \tag{34}$$

for some fixed k and k'.

Thereby, for each i such that $1 \leq i \leq n$, the element $(\xi_{2^r})^{k'}(1+\xi_{2^r})^k$ must be multiplied by a constant ζ_i such that $(\xi_{2^r})^{k'}(1+\xi_{2^r})^k \cdot \zeta_i = \sigma_i((\xi_{2^r})^{k'}(1+\xi_{2^r})^k)$ (for $i = 1$, we have $\zeta_i = 1$). We need this kind of multiplication to ensure the precoding which is given as it follows:

$$\begin{pmatrix} h_{ml} & 0 & 0 & 0 & \cdots & 0 \\ 0 & h_{ml}\cdot\zeta_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & h_{ml}\cdot\zeta_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h_{ml}\cdot\zeta_n \end{pmatrix} \to$$

$$\to \begin{pmatrix} \sigma_1((\xi_{2^r})^{k'}(\mu)^k) & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2((\xi_{2^r})^{k'}(\mu)^k) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n((\xi_{2^r})^{k'}(\mu)^k) \end{pmatrix} \sim M'_k, \tag{35}$$

where $\mu = 1+\xi_{2^r}$.

Consequently, we ensure onto which lattice a given complex-valued channel coefficient must be quantized.

Now we need to argue how we can find, given an arbitrary $h_{ml} \in \mathbb{C}$, the appropriate k and k', that is, given an arbitrary $h_{ml} \in \mathbb{C}$, we find k and k' such that $h_{ml} \to (\xi_{2^r})^{k'}(1+\xi_{2^r})^k$. Hence, after finding the appropriate integers k and k', we compute them modulo n and then we use one of the n^2 possible cases in order to realize the complex-valued channel quantization for dimension n.

So let $h_{ml} \in \mathbb{C}$. From the new algebraic methodology described in Section 3 in order to realize interference alignment onto a lattice, it is natural the approximation $\parallel h_{ml} \parallel \to \parallel 1+\xi_{2^r} \parallel^k$, where $k \in \mathbb{Z}$. Consequently, to find the appropriate k, we have $\frac{log\|h_{ml}\|}{log\|1+\xi_{2^r}\|} \to k \in \mathbb{Z}$, that is, we choose k as being the closest integer value to the value $\frac{log\|h_{ml}\|}{log\|1+\xi_{2^r}\|}$.

Now, after finding k, finally we can find k' by using the argument function. In fact, we have that $h_{ml} \to (\xi_{2^r})^{k'}(1+\xi_{2^r})^k$ (note that we already know k), then, to find k', we have $\frac{arg(h_{ml})-n arg(1+\xi_{2^r})}{\pi/2^{r-1}} \to k' \in \mathbb{Z}$, that is, we choose k' as being the closest integer value to the value $\frac{arg(h_{ml})-n arg(1+\xi_{2^r})}{\pi/2^{r-1}}$.

Then, by knowing k and k', we can realize for dimension n the corresponding complex-valued channel quantization described in Section 3.1 by using the process of precoding for dimension n described in this section.

5. Minimum Mean Square Error Criterion for the Complex-Valued Channel Quantization

In Section 3.1 we introduce a new algebraic methodology to quantize complex-valued channel coefficients. The purpose of this section is to minimize the mean square error related to the quantization of this work, consequently, it provides us the best estimation for such a quantization.

In Section 3.1, for fixed m and l, we have that the matrix H_{ml} is quantized by M'_k, where Section 3.2 guarantees that $k \in \{0, 1, \ldots, n-1\}$.

In this section we have $l = 1, 2, \ldots, L$, then for the sake of simplicity we denote M'_k by M'_{k_l} and $M_{(1+\xi_{2^r})^k}$ by $M_{(1+\xi_{2^r})^{k_l}}$, where $k_l \in \{0, 1, \ldots, n-1\}$.

The following theorem furnishes us the computation of the corresponding mean square error.

Theorem 8. *The $n \times n$ matrix $B = \frac{1}{\eta}\sum_{l=1}^{L} h_{ml}(M_0 M_{(1+\xi_{2^r})^{k_l}} M_0^H)$ minimizes the mean square error $E[\overrightarrow{v}_m^H \overrightarrow{v}_m]$, where*

$$\eta = (\|h\|^2 + \frac{1}{\rho}), \qquad h = (h_{m1}, h_{m2}, \ldots, h_{mL}),$$

ρ is the signal-to-noise ratio (SNR),

$$\overrightarrow{v}_m = \sum_{l=1}^{L} \left(h_{ml} \left(M_0^H B M_0 \right) - M_{(1+\xi_{2^r})^{k_l}} \right) \overrightarrow{v_l} + M_0^H B \overrightarrow{z}_m, \tag{36}$$

$\overrightarrow{v}_l \in \mathbb{Z}[i]^n$ and $M'_{k_l} M_0 = M_0 M_{(1+\xi_{2^r})^{k_l}}$, for $l = 1, \ldots, L$, with $M_{(1+\xi_{2^r})^{k_l}} \in \mathbb{M}_n(\mathbb{Z}[i])$ and H denotes the transpose conjugate of a matrix, where $\mathbb{M}_n(\mathbb{Z}[i])$ denotes the set of the $n \times n$ matrices with integer complex entries. The equality $M'_{k_l} M_0 = M_0 M_{(1+\xi_{2^r})^{k_l}}$ means that the matrices M'_{k_l} and $M_{(1+\xi_{2^r})^{k_l}}$ generate the same lattice. In addition, the mean square error is given by

$$P_s \frac{1}{\eta}(\eta \sum_{l=1}^{L} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}(((1+\xi_{2^r})^{k_l})^2)$$

$$- \sum_{l,j=1}^{L} h_{ml} h_{mj} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}((1+\xi_{2^r})^{k_l}(1+\xi_{2^r})^{k_j})), \tag{37}$$

where P_s is the signal power.

Proof. See Appendix. □

Equation (37) is an expression of the mean square error and, by minimizing such an equation, the minimum solution of the mean square error is obtained.

Thereby, for finding the corresponding minimum solution, we have to minimize the following expression:

$$\eta \sum_{l=1}^{L} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}(((1+\xi_{2^r})^{k_l})^2)$$

$$- \sum_{l,j=1}^{L} h_{ml} h_{mj} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}((1+\xi_{2^r})^{k_l}(1+\xi_{2^r})^{k_j}). \tag{38}$$

Equation (38) is a quadratic form whose variables are $a_{l0}, a_{l1}, \ldots$ $\ldots, a_{l(n-1)} \in \mathbb{Z}[i]$, where $l = 1, \ldots, L$, and

$$(1+\xi_{2^r})^{k_l} = a_{l0} + a_{l1}\xi_{2^r} + a_{l2}\xi_{2^r}^2 + \cdots + a_{l(n-1)}\xi_{2^r}^{n-1}. \tag{39}$$

We can associate the quadratic form (38) to the following functional

$$F(a) = \eta \sum_{l=1}^{L} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}(((1+\xi_{2^r})^{k_l})^2)$$

$$- \sum_{l,j=1}^{L} h_{ml} h_{mj} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}((1+\xi_{2^r})^{k_l}(1+\xi_{2^r})^{k_j}) = a^t Q a, \tag{40}$$

where $a = (a_{10}, a_{11}, \ldots, a_{1(n-1)}, \ldots, a_{L0}, a_{L1}, \ldots, a_{L(n-1)}) \in \mathbb{Z}[i]^{Ln}$ and Q is the corresponding $Ln \times Ln$ symmetric matrix.

Since Q is a complex symmetric square matrix, we apply the Takagi decomposition of the matrix $Q = VDV^t$, where D is a real nonnegative diagonal matrix and V is unitary.

The goal is to find $a \in (\mathbb{Z}[i]^{Ln} - \{0\})$ such that a is the vector which minimizes $F(a)$. Hence

$$\min_{a \in (\mathbb{Z}[i]^{Ln} - \{0\})} F(a) = \min_{a \in (\mathbb{Z}[i]^{Ln} - \{0\})} a^t Q a$$

$$= \min_{a\in(\mathbb{Z}[i]^{Ln}-\{0\})} a^t V D V^t a = \min_{b\in\Lambda'} b^t D b, \tag{41}$$

where $b = V^t a$ and Λ' is the corresponding lattice.

Thereby, given complex-valued channels h_{ml}, where $l = 1, 2, \ldots, L$, we find $a \in \mathbb{Z}[i]^{Ln}$ which gives us the best estimation for the respective equations in (39), therefore, we obtain the best estimation for the corresponding quantizations $M'_{k_l} \sim (M_{(1+\xi_{2^r})})^{k_l}$. Notice that by applying the stipulated value for the complex-valued channels h_{ml}, where $l = 1, 2, \ldots, L$, we have the value of η by conditioning a value for ρ and, through Section 4, we can find the value of the corresponding powers k_l, where $l = 1, 2, \ldots, L$.

As we perform the complex-valued channel quantization described in Section 3.1, the corresponding codewords x_l, where $l = 1, 2, \ldots, L$, are transformed in lattice points which belong to one of the n lattices constructed in Section 3.2. By using the minimum mean square error criterion, the corresponding estimation for $h_{ml}x_l$ is a point of the lattice related to the power k_l which is associated to a coset of this lattice with $(1+i)\mathbb{Z}[i]^n$ being the corresponding sublattice and, consequently, we have an efficient decoder for such a complex-valued channel quantization and the corresponding achievable computation rate at each node is maximized.

5.1. Minimum mean square error criterion for the two-complex dimensional quantization

In [17], for the two-complex dimensional case and $L = 2$, we have the corresponding complex-valued channel quantization and the construction of complex nested ideal lattices from such a channel quantization.

By (40) the functional related to such a minimization is given by

$$F(a) = \eta \sum_{l=1}^{2} Tr_{\mathbb{Q}(\xi_8)/\mathbb{Q}(i)}(((1+\xi_8)^{k_l})^2)$$

$$- \sum_{l,j=1}^{2} h_{ml} h_{mj} Tr_{\mathbb{Q}(\xi_8)/\mathbb{Q}(i)}((1+\xi_8)^{k_l}(1+\xi_8)^{k_j}) = a^t Q a, \tag{42}$$

where $a \in \mathbb{Z}[i]^4$, $\eta = (\|h\|^2 + \frac{1}{\rho})$, $h = (h_{m1}, h_{m2})$, ρ is the signal-to-noise ratio (SNR) and Q is the corresponding 4×4 symmetric complex matrix.

Since Q is a complex symmetric square matrix, we apply the Takagi decomposition of the matrix $Q = VDV^t$, where D is a real nonnegative diagonal matrix and V is unitary.

The goal is to find $a \in (\mathbb{Z}[i]^4 - \{0\})$ such that a is the vector which minimizes $F(a)$. Hence

$$\min_{a\in(\mathbb{Z}[i]^4-\{0\})} F(a) = \min_{a\in(\mathbb{Z}[i]^4-\{0\})} a^tQa$$
$$= \min_{a\in(\mathbb{Z}[i]^4-\{0\})} a^tVDV^ta = \min_{b\in\Lambda'} b^tDb, \tag{43}$$

where $b = V^ta$ and Λ' is the corresponding lattice.

Following the theoretical construction developed in Section 5, for the two-complex dimensional case and $L = 2$, the input elements h_{m1}, h_{m2} of the functional (42) are uniformly distributed random numbers. Also we randomly generate the values of the SNR ρ for each computational experiment i, where $i = 1, 2, \dots, 10$. Thereby we compute the values of η in the second column of Table 1. Therefrom the functional (42) and its respective quadratic form is obtained. The minimum of the equation (43) corresponds to a $b \in \Lambda'$ such that b is the closest lattice point to the origin.

For each computational experiment i, we find the vector $a \in (\mathbb{Z}[i]^4 - \{0\})$ which gives us the best estimation for the respective equations in (39). In (39) we have

$$\begin{cases} (1+\xi_8)^{k_1} = a_{10} + a_{11}\xi_8 & \text{(a)} \\ (1+\xi_8)^{k_2} = a_{20} + a_{21}\xi_8 & \text{(b)} \end{cases}, \tag{44}$$

where (a) and (b) correspond, respectively, to the best estimation of the quantizations M'_{k_1} and M'_{k_2}. Each k_l, where $l = 1, 2$, is computed by taking the closest integer of the following value

$$\frac{\log \|h_{ml}\|}{\log \|1+\xi_8\|} \tag{45}$$

and, after that, we compute such an integer value mod 2 to obtain k_l. In Figure 2 each lattice is represented by either $\Lambda_0 = \mathbb{Z}[i]^2$ (blue dots) or $\Lambda_1 = D_4$ (red crosses).

The corresponding estimations for $h_{m1}x_1$ and $h_{m2}x_2$ are represented in Table 1 by the vectors $P1_i$ and $P2_i$, respectively, for the computational experiments $i = 1, \dots, 10$. These estimations are points of the lattices related to the powers k_1 and k_2, respectively, and are associated to a coset of such lattices with $(1+i)\mathbb{Z}[i]^2$ being the corresponding sublattice. Consequently, we have an efficient decoder for such a two-complex dimensional channel quantization and the corresponding achievable computation rate at each node is maximized.

In Figure 2, for the sake of illustration, 5 computational experiments from Table 1 ($i = 4, 6, 7, 8, 10$) are used for the representation of the corresponding estimations (for the other ones such a representation is analogous). The points $P1_i$ (continous) and $P2_i$ (dashed) are estimations for $h_{m1}x_1$ and $h_{m2}x_2$, respectively.

From the computational experiments, we observe that we obtain a two-dimensional hyperplane by taking the values of h_{m1} and h_{m2} such that $||h_{m1}|| = ||h_{m2}|| = 1$. We can generate such a two-dimensional hyperplane through the projection of the last complex coordinate.

i-th	η	k_1	k_2	$P1_i$ (Cont.)	$P2_i$ (Dashed)	Color
1	3.4781	1	1	(0, 22+26i)	(0, 7+35i)	-
2	4.5525	0	0	(0, -72-72i)	(0, -90-90i)	-
3	336.0012	0	0	(0, -18-98i)	(0, -18-162i)	-
4	2.5855	0	1	(0, -4-2i)	(0, -2-4i)	Blue
5	300.7523	1	0	(0, -12-16i)	(0, -3-13i)	-
6	3.6555	1	0	(0, 2-18i)	(0, 3-22i)	Magenta
7	2.5852	1	0	(0, 20+4i)	(0, 6+8i)	Red
8	3.1153	0	0	(0, -12-16i)	(0, -3-13i)	Black
9	2.7803	0	0	(0, -50-66i)	(0, -25-70i)	-
10	2.9633	1	0	(0, -4i)	(0, 2-3i)	Green

Table 1: Data from the Computational Experiments

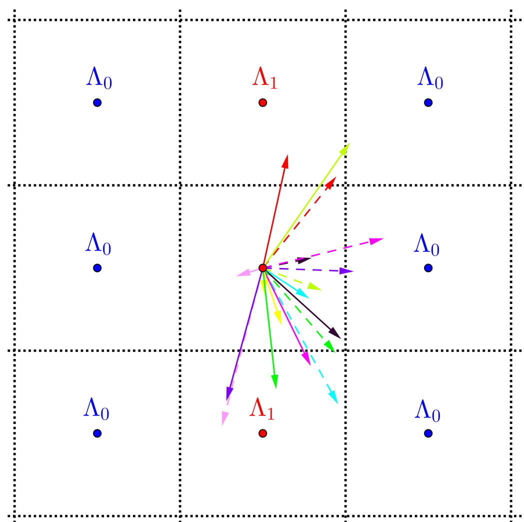


Figure 2: Representation of the Vectors $P1_i$ and $P2_i$ from Table I. Note that $\Lambda_1 \subseteq \Lambda_0$.

6. Conclusion

This work presents a new algebraic methodology to quantize complex-valued channels in order to realize interference alignment (IA) [1] onto a complex ideal lattice. Such a methodology makes use of the binary cyclotomic field $\mathbb{Q}(\xi_{2^r})$, where $r \geq 3$, to provide a doubly infinite nested lattice partition chain for any dimension $n = 2^{r-2}$, where $r \geq 3$, in order to quantize complex-valued channels onto these nested lattices.

We prove the existence of periodicity in the corresponding nested lattice partition chains to guarantee that the channel gain does not remove the lattice from the initial chain of nested complex ideal lattices.

Precoding is essential to ensure onto which lattice a given complex-valued channel must be quantized. Therefore Section 4 provides us such a precoder.

In this work we minimize the mean square error related to the corresponding quantization to providing us the best estimation for such a quantization. Consequently, we obtain an efficient decoder. In Section 5.1 we exemplify this new algebraic methodology through the two-complex dimensional channel quantization and show all the corresponding computational experiments.

The proposed algebraic methodology is original and can be approached to applications such as compute-and-forward [15] and homomorphic encryption schemes.

7. Appendix 1: Extension by periodicity of the nested lattice partition chain for the positive positions, that is, $k \geq 0$

In Section 3.2, we have the lattices Λ_j, where $0 \leq j \leq n-1$ and Λ_j is the lattice related to the position j. Also we have that $M_{(1+\xi_{2^r})^j} = (M_{(1+\xi_{2^r})})^j$ is a generator matrix of the lattice Λ_j and we know that the matrices $(M_{(1+\xi_{2^r})})^n$ and $(1+i)I_{n\times n}$ are equivalent matrices, where $I_{n\times n}$ is the $n\times n$ identity matrix.

Then, for $k = n$, the matrix $(M_{(1+\xi_{2^r})})^n$ generates the lattice $(1+i)\mathbb{Z}[i]^n$; for $k = n+j$, we have $M_{(1+\xi_{2^r})^{(n+j)}} = (M_{(1+\xi_{2^r})})^{(n+j)} = ((M_{(1+\xi_{2^r})})^n)(M_{(1+\xi_{2^r})})^j = (1+i)(M_{(1+\xi_{2^r})})^j$ as being a generator matrix of the lattice $(1+i)\Lambda_j$ and, for $k = 2n+j$, we have $M_{(1+\xi_{2^r})^{(2n+j)}} = (M_{(1+\xi_{2^r})})^{(2n+j)} = ((M_{(1+\xi_{2^r})})^{2n})(M_{(1+\xi_{2^r})})^j = (1+i)^2(M_{(1+\xi_{2^r})})^j = 2(M_{(1+\xi_{2^r})})^j$ as being a generator matrix of the lattice $2\Lambda_j$, since the matrices $(M_{(1+\xi_{2^r})})^n$ and $(1+i)I_{n\times n}$ are equivalent.

Then we suppose, by hypothesis of induction, that $(M_{(1+\xi_{2^r})})^{n\beta+j}$, where $\beta \in \mathbb{N}$ and $0 \leq j \leq n-1$, is a generator matrix of the lattice $(1+i)^\beta\Lambda_j$.

We show, for $k = n(\beta+1)+j$, that the lattice $(1+i)^{\beta+1}\Lambda_j$ has a generator matrix as being the matrix $(M_{(1+\xi_{2^r})})^{(n(\beta+1)+j)}$. In fact, $(M_{(1+\xi_{2^r})})^{n(\beta+1)+j} = ((M_{(1+\xi_{2^r})})^n)((M_{(1+\xi_{2^r})})^{(n\beta+j)})$, by using the hypothesis of induction and the fact that $(1+i)I_{n\times n}$ and $(M_{(1+\xi_{2^r})})^n$ are equivalent matrices, we have $(M_{(1+\xi_{2^r})})^{n(\beta+1)+j}$ as a generator matrix of the lattice $(1+i)^{\beta+1}\Lambda_j$.

Hence, we show, for $k = n\beta + j$, where $\beta \in \mathbb{N}$ and $0 \le j \le n-1$, that the matrix $(M_{(1+\xi_{2^r})})^{n\beta+j}$ is a generator matrix of the lattice $(1+i)^{\beta}\Lambda_j$.

Therefore, if β is even, we have $\beta = 2\epsilon$, where $\epsilon \in \mathbb{N}$, and $(1+i)^{\beta}\Lambda_j = 2^{\beta/2}\Lambda_j$, for $\beta \neq 0$; for $\beta = 0$, we have the lattice Λ_j. Now if β is odd, we have $\beta = 2\epsilon+1$, where $\epsilon \in \mathbb{N}$, and $(1+i)^{\beta}\Lambda_j = 2^{(\beta-1)/2}(1+i)\Lambda_j$.

8. Appendix 2: Providing an expression for the corresponding mean square error

From equation (1), we have

$$B\overrightarrow{y}_m = \sum_{l=1}^{L} B(h_{ml}I)\overrightarrow{x}_l + B\overrightarrow{z}_m$$

$$= \sum_{l=1}^{L} M'_{k_l}\overrightarrow{x}_l + \sum_{l=1}^{L}(B(h_{ml}I) - M'_{k_l})\overrightarrow{x}_l + B\overrightarrow{z}_m,$$

where $M'_{k_l}\overrightarrow{x}_l = M'_{k_l}(M_0\overrightarrow{v}_l) = M_0(M_{(1+\xi_{2^r})^{k_l}}\overrightarrow{v}_l)$, with $M_{(1+\xi_{2^r})^{k_l}} \in \mathbb{M}_n(\mathbb{Z}[i])$. Hence

$$\sum_{l=1}^{L} M'_{k_l}\overrightarrow{x}_l = \sum_{l=1}^{L} M_0(M_{(1+\xi_{2^r})^{k_l}}\overrightarrow{v}_l) = M_0\sum_{l=1}^{L}(M_{(1+\xi_{2^r})^{k_l}}\overrightarrow{v}_l).$$

We also have that

$$\sum_{l=1}^{L}(B(h_{ml}I) - M'_{k_l})\overrightarrow{x}_l = \sum_{l=1}^{L}((M_0M_0^H)h_{ml}B - M'_{k_l})\overrightarrow{x}_l$$

$$= \sum_{l=1}^{L}((M_0M_0^H)h_{ml}B)\overrightarrow{x}_l - \sum_{l=1}^{L} M'_{k_l}\overrightarrow{x}_l$$

$$= \sum_{l=1}^{L} (M_0 M_0^H) h_{ml} B (M_0 M_0^H) \overrightarrow{x}_l - \sum_{l=1}^{L} M_0 (M_{(1+\xi_{2^r})^{k_l}} \overrightarrow{v}_l)$$

$$= \sum_{l=1}^{L} (M_0 M_0^H) h_{ml} B M_0 \overrightarrow{v}_l - \sum_{l=1}^{L} M_0 (M_{(1+\xi_{2^r})^{k_l}} \overrightarrow{v}_l)$$

$$= M_0 \left(\sum_{l=1}^{L} (M_0^H h_{ml} B M_0) \overrightarrow{v}_l \right) - M_0 \sum_{l=1}^{L} M_{(1+\xi_{2^r})^{k_l}} \overrightarrow{v}_l$$

$$= M_0 \sum_{l=1}^{L} (h_{ml} (M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}}) \overrightarrow{v}_l,$$

and $B\overrightarrow{z}_m = M_0(M_0^H B \overrightarrow{z}_m)$.

Then we conclude that

$$\overrightarrow{y}'_m = M_0^H B \overrightarrow{y}_m = \sum_{l=1}^{L} M_{(1+\xi_{2^r})^{k_l}} \overrightarrow{v}_l$$

$$+ \sum_{l=1}^{L} (h_{ml} (M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}}) \overrightarrow{v}_l + M_0^H B \overrightarrow{z}_m,$$

where $\overrightarrow{v}_m = \sum_{l=1}^{L} (h_{ml} (M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}}) \overrightarrow{v}_l + M_0^H B \overrightarrow{z}_m$ is the noise term ($\overrightarrow{v}_m$ is an $n \times 1$ column vector). Thus the mean square error is given by

$$E[\overrightarrow{v}_m^H \overrightarrow{v}_m] = Tr(E[\overrightarrow{v}_m^H \overrightarrow{v}_m])$$

$$= E[Tr(\overrightarrow{v}_m^H \overrightarrow{v}_m)] = E[Tr(\overrightarrow{v}_m \overrightarrow{v}_m^H)] = Tr(E[\overrightarrow{v}_m \overrightarrow{v}_m^H]) \text{ and}$$

$$Tr(E[\overrightarrow{v}_m \overrightarrow{v}_m^H])$$

$$= Tr(E[(\sum_{l=1}^{L} (h_{ml} (M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}}) \overrightarrow{v}_l + M_0^H B \overrightarrow{z}_m)$$

$$\times (\sum_{l=1}^{L}(h_{ml}(M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}})\overrightarrow{v}_l + M_0^H B \overrightarrow{z}_m)^H]).$$

Since the variables $\overrightarrow{v}_l$ and $\overrightarrow{z}_m$ are uncorrelated, for $l = 1, \ldots, L$, we have

$$E[\overrightarrow{v}_m \overrightarrow{v}_m^H] = \sum_{l=1}^{L}(h_{ml}(M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}}) \cdot E[\overrightarrow{v}_l \overrightarrow{v}_l^H]$$

$$\times (h_{ml}(M_0^H B^H M_0) - M_{(1+\xi_{2^r})^{k_l}}^H)$$

$$+ M_0^H B E[\overrightarrow{z}_m \overrightarrow{z}_m^H] B^H M_0.$$

Hence

$$E[\overrightarrow{v}_m^H \overrightarrow{v}_m] = Tr(\sum_{l=1}^{L}(h_{ml}(M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}}) \cdot E[\overrightarrow{v}_l \overrightarrow{v}_l^H]$$

$$\times t(h_{ml}(M_0^H B^H M_0) - M_{(1+\xi_{2^r})^{k_l}}^H)$$

$$+ M_0^H B E[\overrightarrow{z}_m \overrightarrow{z}_m^H] B^H M_0).$$

Let $E[\overrightarrow{v}_l \overrightarrow{v}_l^H] = P_s$, for all $l = 1, \ldots, L$, and $E[\overrightarrow{z}_m \overrightarrow{z}_m^H] = \sigma_N^2$, where P_s is the signal power, σ_N^2 is the noise variance and $\rho = \frac{P_s}{\sigma_N^2}$ is the signal-to-noise ratio (SNR). Then

$$E[\overrightarrow{v}_m^H \overrightarrow{v}_m] = P_s Tr(\sum_{l=1}^{L}(h_{ml}(M_0^H B M_0) - M_{(1+\xi_{2^r})^{k_l}})$$

$$\times (h_{ml}(M_0^H B^H M_0) - M_{(1+\xi_{2^r})^{k_l}}^H) + \frac{1}{\rho} M_0^H B B^H M_0)$$

$$= P_s Tr(\sum_{l=1}^{L} h_{ml}^2 (M_0^H B B^H M_0)$$

$$-\sum_{l=1}^{L} h_{ml}[(M_0^H B M_0) M_{(1+\xi_{2^r})^{k_l}}^H + M_{(1+\xi_{2^r})^{k_l}} (M_0^H B^H M_0)]$$

$$+\sum_{l=1}^{L} M_{(1+\xi_{2^r})^{k_l}} M_{(1+\xi_{2^r})^{k_l}}^H + \frac{1}{\rho} M_0^H B B^H M_0).$$

Thereby we have

$$E[\overrightarrow{\upsilon}_m^H \overrightarrow{\upsilon}_m] = P_s Tr((\|h\|^2 + \frac{1}{\rho}) M_0^H B B^H M_0$$

$$-\sum_{l=1}^{L} h_{ml} (M_0^H B M_0) M_{(1+\xi_{2^r})^{k_l}}^H - \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}} (M_0^H B^H M_0)$$

$$+\sum_{l=1}^{L} M_{(1+\xi_{2^r})^{k_l}} M_{(1+\xi_{2^r})^{k_l}}^H),$$

where $h = (h_{m1}, h_{m2}, \ldots, h_{mL})$.

Let $F = M_0^H B M_0$ ($F^H = M_0^H B^H M_0$) and $\eta = (\|h\|^2 + \frac{1}{\rho})$. Then,

$$E[\overrightarrow{\upsilon}_m^H \overrightarrow{\upsilon}_m] = P_s \eta Tr(F \cdot F^H - \frac{1}{\eta} F \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}}^H$$

$$-\frac{1}{\eta} F^H \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}} + \frac{1}{\eta} \sum_{l=1}^{L} M_{(1+\xi_{2^r})^{k_l}} M_{(1+\xi_{2^r})^{k_l}}^H)$$

$$= P_s \eta Tr((F - \frac{1}{\eta} \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})(F - \frac{1}{\eta} \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})^H$$

$$+\frac{1}{\eta} \sum_{l=1}^{L} M_{(1+\xi_{2^r})^{k_l}} M_{(1+\xi_{2^r})^{k_l}}^H$$

$$-\frac{1}{\eta^2} (\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})(\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})^H).$$

Observe that $F = \frac{1}{\eta}\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}}$ minimizes $E[\overrightarrow{\upsilon}_m^H \overrightarrow{\upsilon}_m]$. Since $F = M_0^H B M_0$, it follows that

$$BM_0 = \frac{1}{\eta} M_0 \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}}$$

$$\Leftrightarrow B = \frac{1}{\eta} M_0 (\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}}) M_0^H$$

$$= \frac{1}{\eta} \sum_{l=1}^{L} h_{ml} (M_0 M_{(1+\xi_{2^r})^{k_l}} M_0^H).$$

Hence $B = \frac{1}{\eta}\sum_{l=1}^{L} h_{ml}(M_0 M_{(1+\xi_{2^r})^{k_l}} M_0^H)$ minimizes $E[\overrightarrow{\upsilon}_m^H \overrightarrow{\upsilon}_m]$ and the mean square error is given by

$$P_s Tr(\sum_{l=1}^{L} M_{(1+\xi_{2^r})^{k_l}} M^H_{(1+\xi_{2^r})^{k_l}}$$

$$-\frac{1}{\eta}(\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})(\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})^H)$$

$$= P_s(\sum_{l=1}^{L} Tr(M_{(1+\xi_{2^r})^{k_l}} M^H_{(1+\xi_{2^r})^{k_l}}) - \frac{1}{\eta} \parallel \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}} \parallel_F^2)$$

$$= P_s(\sum_{l=1}^{L} \parallel M_{(1+\xi_{2^r})^{k_l}} \parallel_F^2 - \frac{1}{\eta} \parallel \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}} \parallel_F^2),$$

with $\parallel A \parallel_F = \sqrt{Tr(AA^t)}$, where A is an $m \times n$ complex matrix. The norm $\parallel \cdot \parallel_F$ is called *Frobenius norm.*

Since

$$\parallel M_{(1+\xi_{2^r})^{k_l}} \parallel_F^2 = Tr(M_{(1+\xi_{2^r})^{k_l}} M^H_{(1+\xi_{2^r})^{k_l}})$$

$$= Tr(M_{(1+\xi_{2^r})^{k_l}} M^H_{(1+\xi_{2^r})^{k_l}} M_0^H M_0)$$

$$= Tr(M_0 M_{(1+\xi_{2^r})^{k_l}} M^H_{(1+\xi_{2^r})^{k_l}} M_0^H)$$

$$= \sum_{i=1}^{n} \sigma_i((1+\xi_{2^r})^{k_l})^2 = \sum_{i=1}^{n} \sigma_i(((1+\xi_{2^r})^{k_l})^2)$$

$$= Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}(((1+\xi_{2^r})^{k_l})^2) \text{ and}$$

$$\| \sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}} \|_F^2 = Tr((\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})(\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})^H)$$

$$= Tr(M_0(\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})(\sum_{l=1}^{L} h_{ml} M_{(1+\xi_{2^r})^{k_l}})^H M_0^H)$$

$$= Tr((\sum_{l=1}^{L} h_{ml} M_0 M_{(1+\xi_{2^r})^{k_l}})(\sum_{l=1}^{L} h_{ml} M^H_{(1+\xi_{2^r})^{k_l}} M_0^H))$$

$$= \sum_{l,j=1}^{L} h_{ml} h_{mj} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}((1+\xi_{2^r})^{k_l}(1+\xi_{2^r})^{k_j}),$$

the mean square error is given by

$$P_s(\sum_{l=1}^{L} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}(((1+\xi_{2^r})^{k_l})^2)$$

$$-\frac{1}{\eta}\sum_{l,j=1}^{L} h_{ml} h_{mj} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}((1+\xi_{2^r})^{k_l}(1+\xi_{2^r})^{k_j}))$$

$$= P_s \frac{1}{\eta}(\eta \sum_{l=1}^{L} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}(((1+\xi_{2^r})^{k_l})^2)$$

$$- \sum_{l,j=1}^{L} h_{ml} h_{mj} Tr_{\mathbb{Q}(\xi_{2^r})/\mathbb{Q}(i)}((1+\xi_{2^r})^{k_l}(1+\xi_{2^r})^{k_j})).$$

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