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A SUFFICIENT CONDITION FOR THE ABSOLUTE CONTINUITY OF CONJUGATION BETWEEN CIRCLE HOMEOMORPHISMS WITH BREAKS

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Abstract: Let T_1 and T_2 be circle homeomorphisms with countable many break points, that is, discontinuities in the derivative T_1 and T_2, with identical irrational rotation number ρ. Assume that the total variations of $\log DT_i$, $i = 1, 2$ are bounded. We provide a sufficient condition for the absolute continuity of conjugation between T_1 and T_2. The result extends and complements previous obtained results in [2] and [6].

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Key Words: circle homeomorphism, break point, rotation number, conjugation, absolute continuity

1. Introduction

Let $S^1 = \mathbb{R} \setminus \mathbb{Z}$ be a *unit circle*. Let $\pi : \mathbb{R} \to S^1$ denote the corresponding projection mapping that "winds" a straight line on the circle. An arbitrary homeomorphism T that preserves the orientation of the unit circle S^1 can "be

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lifted" on the straight line $\mathbb{R}$ in the form of the homeomorphism $L_T : \mathbb{R} \to \mathbb{R}$ with property $L_T(x+1) = L_T(x) + 1$ that is connected with T by relation $\pi \circ L_T = T \circ \pi$. This homeomorphism L_T is called the *lift* of the homeomorphism T and is defined up to an integer term. Poincaré proved [8] that the limit

$$\rho(T) = \lim_{i\to\infty} \frac{L_T^i(x)}{i} \mod 1,$$

exists and does not depend on the initial point $x \in \mathbb{R}$, where L_T is the lift of T. Here and below, for a given map F, F^i denotes its i-th iteration. The number $\rho(T)$ is called the *rotation number* of T and it is irrational if and only if the homeomorphism T has no periodic point. Denjoy proved [8] that if T is an orientation preserving C^1-homeomorphism with irrational rotation number ρ and $\log DT$ is of bounded variation then T is conjugate to the rigid rotation R_ρ, that is there exists an essentially unique homeomorphism h of the circle such that $T = h^{-1} \circ R_\rho \circ h$, here DT stands for the derivative of T. In this context, a natural question to ask under what condition the conjugacy is smooth? The problem of smoothness of the conjugacy of smooth diffeomorphisms has studied very well by several authors (see [10] - [12]). In this direction a remarkable result was obtained by Akhadkulov *et. al* [3]. It was shown that there exists a subset of irrational numbers of unbounded type, such that every circle diffeomorphism satisfying a certain Zygmund condition is absolutely continuously conjugate to the linear rotation provided its rotation number belongs to the above set.

A natural generalizations of diffeomorphisms are piecewise smooth homeomorphisms which are called $\mathcal{P}-$homeomorphisms.

Definition 1.1. A homeomorphism T of the circle is called $\mathcal{P}$-homeomorphism if it satisfies the following conditions:

i) T is differentiable away from countably many points $x_b \in BP(T)$, so-called break points of T, with $BP(T)$ the set of break points of T on S^1, at which left and right derivatives, denoted respectively by DT_- and DT_+, exist, and
$$\frac{DT_-(x_b)}{DT_+(x_b)} \neq 1$$
for all $x_b \in BP(T)$;

ii) there exist constants $0 < c_1 < c_2 < \infty$ with $c_1 < DT(x) < c_2$ for all $x \in S^1 \backslash BP(T)$, $c_1 < DT_-(x_b) < c_2$ and $c_1 < DT_+(x_b) < c_2$ for all $x_b \in BP(T)$;

iii) $\log DT$ has bounded variation.

The ratio $\sigma_T(x_b) := (DT_-(x_b))/(DT_+(x_b))$ is called the *jump* of T in x_b. The class of $\mathcal{P}$-homeomorphisms was introduced by Herman [9]. He studied the dynamics of piecewise linear circle homeomorphisms. The existence of conjugacy of between two $\mathcal{P}$-homeomorphisms follows directly from Denjoy's theorem. Next we consider the problem of the regularity of the conjugacy of two $\mathcal{P}$-homeomorphisms with identical irrational rotation numbers. Note that the ergodic properties of $\mathcal{P}$-homeomorphisms such as their invariant measures, their renormalizations and also their rigidity properties are rather different from those of diffeomorphisms. In this case the absolute continuity of the conjugacy depends on the size of jumps, the orbits of the break points and the rotation number. For example, in [4], it was shown that the conjugacy between two break equivalent $\mathcal{P}$-homeomorphisms with two break points and with irrational rotation numbers is singular if their jump sizes do not coincide. For the non break equivalent $\mathcal{P}$-homeomorphisms with two break points and with irrational rotation numbers of bounded type the conjugacy is still singular even their jump sizes coincide [5]. Recently, two groups of scientists have independently proved the following most general result in [1] and [7]: if the product of sizes of two $\mathcal{P}$-homeomorphisms with finite number of break points do not coincide then the conjugacy is singular.

The purpose of this work is to obtain sufficient condition for the absolute continuity of conjugating map of two $\mathcal{P}$-homeomorphisms with countable infinite number of break points. To formulate our main result let us recall some necessary notions and facts. Hereafter, we shall always assume that ρ is irrational and use its decomposition in an infinite *continued fraction* (see [13])

$$\rho = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\cfrac{\cdots}{k_n + \cfrac{1}{\cdots}}}}} := [k_1, k_2, ..., k_n, ...]. \tag{1}$$

The value of a "countable-floor" fraction is the limit of the sequence of *rational convergents* $p_n/q_n = [k_1, k_2, ..., k_n]$. The positive integers k_n, $n \geq 1$, are called *incomplete multiples* and defined uniquely for irrational ρ. The mutually prime positive integers p_n and q_n satisfy the recurrent relations $p_n = k_n p_{n-1} + p_{n-2}$ and $q_n = k_n q_{n-1} + q_{n-2}$ for $n \geq 1$, where it is convenient to define $p_{-1} = 0$, $q_{-1} = 1$ and $p_0 = 1$, $q_0 = k_1$. Given a circle homeomorphism T with irrational rotation number ρ, one may consider a *marked trajectory* (i.e. the trajectory of a marked point) $\xi_i = T^i \xi_0 \in S^1$, where $i \geq 0$, and pick out of it the sequence of the *dynamical convergents* ξ_{q_n}, $n \geq 0$, indexed by the denominators of consecutive rational convergents to ρ. We will also conventionally use $\xi_{q_{-1}} = \xi_0 - 1$. The well-understood arithmetical properties of rational convergents and the combinatorial equivalence between T and *rigid rotation* $R_\rho : \xi \to \xi + \rho \mod 1$ imply that the dynamical convergents approach the marked point, alternating their order in the following way:

$$\xi_{q_{-1}} < \xi_{q_1} < \xi_{q_3} < ... < \xi_{q_{2m+1}} < ... < \xi_0 < ... < \xi_{q_{2m}} < ... < \xi_{q_2} < \xi_{q_0}.$$

We define the n-th *fundamental interval* $\Delta^n(\xi_0)$ as the circle arc $[\xi_0, T^{q_n}(\xi_0)]$ for even n and as $[T^{q_n}(\xi_0), \xi_0]$ for odd n. For the marked trajectory, we use the notation $\Delta_0^n = \Delta^n(\xi_0)$, $\Delta_i^n = \Delta^n(\xi_i) = T^i \Delta_0^n$. It is well known, that the set $\mathbf{P}_n(\xi_0, T) = \mathbf{P}_n(T)$ of intervals with mutually disjoint interiors defined as

$$\mathbf{P}_n(T) = \left\{ \Delta_i^{n-1},\ 0 \le i < q_n;\ \Delta_j^n,\ 0 \le j < q_{n-1} \right\}$$

determines a partition of the circle for any n. The partition $\mathbf{P}_n(T)$ is called the n-th *dynamical partition* of the point ξ_0. Obviously the partition $\mathbf{P}_{n+1}(T)$ is a refinement of the partition $\mathbf{P}_n(T)$: indeed the intervals of order n are members of $\mathbf{P}_{n+1}(T)$ and each interval $\Delta_i^{n-1} \in \mathbf{P}_n(T)$ $0 \le i < q_n$, is partitioned into $k_{n+1} + 1$ intervals belonging to $\mathbf{P}_{n+1}(T)$ such that

$$\Delta_i^{n-1} = \Delta_i^{n+1} \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^n.$$

Let T_1 and T_2 be $\mathcal{P}$-homeomorphisms with identical irrational rotation number ρ. We consider dynamical partitions $\mathbf{P}_n(\xi, T_1) = \mathbf{P}_n(T_1)$ and $\mathbf{P}_n(h(\xi), T_2) = \mathbf{P}_n(T_2)$ appropriate to the homeomorphisms T_1 and T_2. Denote by $\widehat{\Delta}^n$ intervals of partition of $\mathbf{P}_n(T_2)$. Since the function h is a conjugation function between T_1 and T_2 we have $h(\Delta^n) = \widehat{\Delta}^n$ for any $\Delta^n \in \mathbf{P}_n(T_1)$. Denote by $|A|$ the Lebesgue measure of the corresponding set of $A \subset S^1$.

Our main result is the following theorems.

Theorem 1.2. *Let T_1 and T_2 be $\mathcal{P}$-homeomorphisms with the identical irrational rotation numbers with continued fraction expansion $\rho_1 = \rho_2 = [k_1, k_2, ..., k_n, ...]$. If there exists a sequence (τ_n) such that $\sum_{n=1}^{\infty} (k_n \tau_n)^2 < \infty$ and*

$$\left| \frac{|\Delta_1|}{|\Delta_2|} - \frac{|\widehat{\Delta}_1|}{|\widehat{\Delta}_2|} \right| \le \tau_n \tag{2}$$

for each pair of adjacent intervals Δ_1, $\Delta_2 \in \mathbf{P}_n(T_1)$, for all $n > 1$. Then the conjugation h between T_1 and T_2 is an absolutely continuous function.

Remark

- Note that our main theorems extend the main result of [6]. In [6], there was obtained a sufficient condition for the absolute continuity of conjugation of two $\mathcal{P}$-homeomorphisms with finite number of break points and with the same bounded type irrational rotation numbers. In our main results, the number of break points of $\mathcal{P}$-homeomorphisms is countable infinite and the rotation numbers are any irrational that is not of bounded type.

- Our result complements the main result of [2]. Indeed, in [2], there was obtained a necessary condition for absolute continuity of the invariant measure of $\mathcal{P}$-homeomorphisms with countable infinite number of break points. It is well known that the invariant measure is a conjugacy between a circle homeomorphism and linear rotation.

- We will use the consideration of the theory of martingales in the proofs of main theorems and the proofs follow closely that of [6]. The idea of using the theory of martingales was established in [11].

2. Martingales and Martingale Convergence Theorem

Our objective in this section is to develop the fundamentals of the theory of martingales, and prepare for the main results and applications that will be presented in the subsequent sections.

Definition 2.1. Let $(X, \mathfrak{F})$ be a measurable space. A sequence $(\mathfrak{F}_m)$ of σ-algebras on X is said to be a **filtration** in $\mathfrak{F}$, if

$$\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \subseteq ... \subseteq \mathfrak{F}.$$

Statement 1. *The sequence of algebras generated by dynamical partitions, which is also denoted by* $(\boldsymbol{P}_m)$ *(by abuse of notation) is a filtration in* $\mathcal{B}$*, where* $\mathcal{B}$ *is a Borel* σ*-algebra on* S^1*.*

Definition 2.2. Let $(\mathcal{R}_m)$ be a sequence of random variables on a measurable space $(X, \mathfrak{F})$ and $(\mathfrak{F}_m)$ a filtration in $\mathfrak{F}$. We say that $(\mathcal{R}_m)$ is **adapted to** $(\mathfrak{F}_m)$ if, for each positive integer m, $\mathcal{R}_m$ is $\mathfrak{F}_m$-measurable.

Denote by $E(\mathcal{R}|\mathfrak{F})$ conditional expectation of random variables $\mathcal{R}$ with respect to partition $\mathfrak{F}$.

Definition 2.3. Let $(\mathcal{R}_m)$ be a sequence of random variables on a probability space $(X, \mathfrak{F}, \mathbf{P})$ and $(\mathfrak{F}_m)$ a filtration in $\mathfrak{F}$. The sequence $(\mathcal{R}_m)$ is said to be a **martingale** with respect to $(\mathfrak{F}_m)$ if, for every positive integer m,

(i) $(\mathcal{R}_m)$ is integrable;

(ii) $(\mathcal{R}_m)$ is adapted to $(\mathfrak{F}_m)$;

(iii) $E(\mathcal{R}_{m+1}|\mathfrak{F}_m) = \mathcal{R}_m$.

Lemma 2.4. (see [14]) Let $(\mathcal{R}_m)$ be a sequence of random variables on a probability space $(X, \mathfrak{F}, \mathbf{P})$. If $\sup\limits_m E(|\mathcal{R}_m|^p) < \infty$ for some $p > 1$ and $(\mathcal{R}_m)$ is a martingale, then there exists an integrable $\mathcal{R} \in L_1(X, \mathfrak{F})$ such that

$$\lim_{m\to\infty} \mathcal{R}_m = \mathcal{R} \text{ (a.e } \mathbf{P}) \text{ and } \mathcal{R}_m \to \mathcal{R} \text{ in } L_1 - \text{norm.}$$

Suppose f is a homeomorphism (not necessary to be $\mathcal{P}$-homeomorphism) of the circle S^1. Using the homeomorphism f and sequence of dynamical partitions $(\mathbf{P}_m)$ we define the sequence of random variables on the circle which is generating a martingales. For any $m \geq 1$ we set

$$\mathcal{R}_m(x) = \frac{|f(\Delta^m)|}{|\Delta^m|}, \text{ if } x \in \Delta^m, \ \Delta^m \in \mathbf{P}_m.$$

Lemma 2.5. The sequence $(\mathcal{R}_m)$ of random variables is a martingale with respect to $(\mathbf{P}_m)$.

Proof. To prove the martingale, it suffices to check $E(\mathcal{R}_{m+1}|\mathbf{P}_m) = \mathcal{R}_m$, for any $m \geq 1$, because the sequence of random variables $(\mathcal{R}_m)$ is sequence of step functions, so the sequence of step functions is integrable and adapted to $(\mathbf{P}_m)$. Denote by χ_I indicator function of interval I. Using definition of conditional expectation of random variables $(\mathcal{R}_m)$ with respect to partition $(\mathbf{P}_m)$ we get

$$E(\mathcal{R}_{m+1}|\mathbf{P}_m) = \sum_{i=0}^{q_{m-1}-1} E(\mathcal{R}_{m+1}|\Delta_i^m)\chi_{\Delta_i^m} + \sum_{i=0}^{q_m-1} E(\mathcal{R}_{m+1}|\Delta_i^{m-1})\chi_{\Delta_i^{m-1}}. \quad (3)$$

Now, we calculate each sum of (3) separately. Note, that each interval of $\mathbf{P}_m$ order m are members of $\mathbf{P}_{m+1}$ and each interval $\Delta_i^{m-1} \in \mathbf{P}_m$, $0 \leq i < q_m$, is partitioned into $k_{m+1}+1$ intervals belonging to $\mathbf{P}_{m+1}$ such that

$$\Delta_i^{m-1} = \Delta_i^{m+1} \cup \bigcup_{s=0}^{k_{m+1}-1} \Delta^m_{i+q_{m-1}+sq_m}.$$

Using this we get:

$$E(\mathcal{R}_{m+1}|\Delta_i^m) = \frac{1}{|\Delta_i^m|}\int_{\Delta_i^m} \mathcal{R}_{m+1}(x)\ell(dx) = \frac{1}{|\Delta_i^m|}\int_{\Delta_i^m} \mathcal{R}_m(x)\ell(dx) \qquad (4)$$

$$E(\mathcal{R}_{m+1}|\Delta_i^{m-1}) = \frac{1}{|\Delta_i^{m-1}|}\int_{\Delta_i^{m-1}} \mathcal{R}_{m+1}(x)\ell(dx)$$

$$= \frac{1}{|\Delta_i^{m-1}|}\Big[\int_{\Delta_i^{m+1}} \mathcal{R}_{m+1}(x)\ell(dx)\Big] + \qquad (5)$$

$$\frac{1}{|\Delta_i^{m-1}|}\Big[\sum_{s=0}^{k_{m+1}-1}\int_{\Delta^m_{i+q_{m-1}+sq_m}} \mathcal{R}_{m+1}(x)\ell(dx)\Big] = \frac{1}{|\Delta_i^{m-1}|}\int_{\Delta_i^{m-1}} \mathcal{R}_m(x)\ell(dx).$$

Finally, summing (3), (4) and (5) we get

$$E(\mathcal{R}_{m+1}|\mathbf{P}_m) = \sum_{i=0}^{q_{m-1}-1} \mathcal{R}_m(x)\chi_{\Delta_i^m} + \sum_{i=0}^{q_m-1} \mathcal{R}_m(x)\chi_{\Delta_i^{m-1}} = \mathcal{R}_m.$$

□

The following lemmas will be used in the proof of main result.

Lemma 2.6. Given $a, b, c, d > 0$ the following inequalities hold

$$\min\left\{\frac{a}{b}, \frac{c}{d}\right\} \le \frac{a+c}{b+d} \le \max\left\{\frac{a}{b}, \frac{c}{d}\right\}.$$

Proof. Consider points $A = (a, b)$, $B = (c, d)$ and $C = (a + c, b + d)$ on the plan xOy. The slope of the ray OC lies between slops of rays OA and OB. □

3. Proof of Main Theorems

3.1. Necessary lemmas

Let h be the conjugation homeomorphism between T_1 and T_2, i.e. $h \circ T_1 = T_2 \circ h$. Without loss of generality we assume $h(0) = 0$. Consider the dynamical partition $\mathbf{P}_m(T_1)$. Define sequence of random variables $(\mathcal{R}_m)$ on the S^1, by this formula:

$$\mathcal{R}_m(x) = \frac{|h(\Delta^m)|}{|\Delta^m|}, \text{ if } x \in \Delta^m, \ \Delta^m \in \mathbf{P}_m(T_1).$$

Denote by $\Theta_m(x) = \mathcal{R}_m(x) - \mathcal{R}_{m-1}(x)$, $m \geq 1$ and $\mathcal{R}_0(x) :\equiv 0$, $x \in S^1$.

Lemma 3.1. Let T_1 and T_2 satisfy the conditions of Theorem 1.2. We have

$$|\Theta_m(x)| \leq k_m \tau_m, \ x \in S^1$$

for all $m \geq 1$, where the sequence (τ_m) is defined in Theorem 1.2.

Proof. For a given $x \in \Delta^m$ we denote $\mathcal{R}_m(\Delta^m) := \mathcal{R}_m(x)$. Utilizing Lemma 2.6 and the inequality (2) we get

$$\mathcal{R}_m(\Delta^{m-1}) - \mathcal{R}_m(\Delta^m(x)) \leq \max_{0 \leq s \leq k_m} \mathcal{R}_m(\Delta_s^m) - \mathcal{R}_m(\Delta^m(x)) \leq k_m \tau_m, \quad (6)$$

and

$$\mathcal{R}_m(\Delta^{m-1}) - \mathcal{R}_m(\Delta^m(x)) \geq \mathcal{R}_m(\Delta^m(x)) - \min_{0 \leq s \leq k_m} \mathcal{R}_m(\Delta_s^m) \geq -k_m \tau_m \quad (7)$$

where $\Delta^m(s)$, $0 \leq s \leq k_m$ are sub-intervals of Δ^{m-1} and $\Delta^m(x)$ is the sub-interval of Δ^{m-1} which containing the point x. Combining the inequalities (6) and (7) we get

$$|\Theta_m(x)| \leq k_m \tau_m, \ x \in S^1.$$

□

Lemma 3.2. Let T_1 and T_2 satisfy the conditions of Theorem 1.2. We have

$$\sup_m E(\mathcal{R}_m^2) < \infty.$$

Proof. First we show that $\Theta_m(x)$ and $\mathcal{R}_{m-1}(x)$ is orthogonal i.e.

$$\int_{S^1} \Theta_m(x)\mathcal{R}_{m-1}(x)dx = 0. \tag{8}$$

Indeed, by the definitions of Θ_m and $\mathcal{R}_{m-1}$ we have

$$\int_{S^1} \Theta_m(x)\mathcal{R}_{m-1}(x)dx = \int_{S^1} \mathcal{R}_m(x)\mathcal{R}_{m-1}(x)dx - \int_{S^1} \mathcal{R}^2_{m-1}(x)dx. \tag{9}$$

Now we evaluate the first integral of the right hand site of (9). Since $\mathcal{R}_{m-1}$ is a constant on the intervals of the partition $\mathbf{P}_{m-1}(T_1)$ we have

$$\int_{S^1} \mathcal{R}_m(x)\mathcal{R}_{m-1}(x)dx = \sum_{\Delta\in\mathbf{P}_{m-1}(T_1)} \mathcal{R}_{m-1}(\Delta)\int_{\Delta} \mathcal{R}_m(x)dx$$

$$= \sum_{\Delta\in\mathbf{P}_{m-1}(T_1)} \mathcal{R}_{m-1}(\Delta)\Bigg[\sum_{\substack{I\subset\Delta\\ I\in\mathbf{P}_m(T_1)}} \int_I \mathcal{R}_m(x)dx\Bigg]$$

$$= \sum_{\Delta\in\mathbf{P}_{m-1}(T_1)} \mathcal{R}_{m-1}(\Delta)\Bigg[\sum_{\substack{I\subset\Delta\\ I\in\mathbf{P}_m(T_1)}} |I|\mathcal{R}_m(I)\Bigg] = \sum_{\Delta\in\mathbf{P}_{m-1}(T_1)} |\Delta|\mathcal{R}^2_{m-1}(\Delta)$$

$$= \sum_{\Delta\in\mathbf{P}_{m-1}(T_1)} \int_{\Delta} \mathcal{R}^2_{m-1}(x)dx = \int_{S^1} \mathcal{R}^2_{m-1}(x)dx.$$

From this and (9) it follow (8). Since $\Theta_m(x)$ and $\mathcal{R}_{m-1}(x)$ is orthogonal we get

$$\|\mathcal{R}_m\|^2_{L_2} = \langle \Theta_m(x) - \mathcal{R}_{m-1}\,, \Theta_m(x) - \mathcal{R}_{m-1}\rangle = \|\Theta_m\|^2_{L_2} + \|\mathcal{R}_{m-1}\|^2_{L_2}.$$

By Lemma 3.1 and assumptions of Theorem 1.2 it imply

$$\|\mathcal{R}_m\|^2_{L_2} \le \sum_{n=1}^{m} k_n\tau_n.$$

Hence

$$\sup_m E(\mathcal{R}^2_m) \le \sqrt{\sum_{n=1}^{\infty} k_n\tau_n}.$$

□

3.2. Proof of Theorem 1.2.

Proof. By Lemma 2.5 the sequence $(\mathcal{R}_m)$ of random variables is a martingale with respect to $(\mathbf{P}_m)$. From Lemmas 2.4 and 3.2 it follow that the sequence of random variables $(\mathcal{R}_m)$ converges to some function $\mathcal{R}$ in L_1 norm. We show that $\mathcal{R} = Dh$ almost everywhere on S^1. Denote by α_m and β_m the end-points of interval Δ^m of dynamical partition $\mathbf{P}_m(T_1)$. By definition of $\mathcal{R}_m$ we get

$$|h(x) - \int_0^x \mathcal{R}_m(x)dx| \leq |h(x) - h(\alpha_m)| + \frac{|h(\Delta^m)|}{|\Delta^m|}|x - \alpha_m| \leq 2|h(\Delta^m)|.$$

Using the last inequality we obtain

$$|h(x) - \int_0^x \mathcal{R}(x)dx| \leq |h(x) - \int_0^x \mathcal{R}_m(x)dx| + \int_0^x |\mathcal{R}(x) - \mathcal{R}_m(x)|dx \leq$$

$$2|h(\Delta^m)| + \|\mathcal{R}_m - \mathcal{R}\|_{L_1}.$$

Taking the limit when $m \to \infty$ we get

$$h(x) = \int_0^x \mathcal{R}(x)dx.$$

Since $\mathcal{R} \in L_1(S^1, d\ell)$ we can conclude that h is an absolutely continuous function and $Dh(x) = \mathcal{R}(x)$ almost everywhere on S^1. Thus Theorem 1.2 is completely proved. □

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References

[1] A. Adouani and H. Marzougui, Non-rigidity for circle homeomorphisms with several break points, *Ergod. Th. & Dynam. Sys.*, (2017); https://doi.org/10.1017/etds.2017.121.

[2] H. Akhadkulov, M.S. Noorani, A. Saaban and S. Akhatkulov, A necessary condition for the absolute continuity of invariant measure of circle maps with countably infinite number of break points, *Far East J. Math.*, **101**, No 3 (2017), 675-688.

[3] H. Akhadkulov, A. Dzhalilov and K. Khanin, Notes on a theorem of Katznelson and Orstein, *Dis. Con. Dyn. Sys.*, **37**, No 9 (2017), 4587-4609.

[4] H. Akhadkulov, A. Dzhalilov and D. Mayer, On conjugations of circle homeomorphisms with two break points, *Ergod. Th. & Dynam. Sys.*, **34**, No 3 (2014), 725-741.

[5] Habibulla A.D. Akhadkulov and M.S. Md Noorani, On conjugacies between piecewise-smooth circle maps, *Nonlinear Analysis: Theory, Methods & Applications*, **99** (2014), 1-15.

[6] H. Akhadkulov and M.S. Noorani, On absolute continuity of conjugations between circle maps with break points, *Abst. Appl. Anal.*, **2012** (2012).

[7] A. Dzhalilov, D. Mayer and U. Safarov, On the conjugation of piecewise smooth circle homeomorphisms with a finite number of break points, *Nonlinearity*, **28**, No 7 (2015), 2441-2460.

[8] I. Cornfeld, S. Fomin and Ya. Sinai, *Ergodic Theory*, Springer Verlag, Berlin (1982).

[9] M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Inst. Hautes Etudes Sci. Publ. Math.*, **49** (1979), 5-234.

[10] Y. Katznelson and D. Ornstein, The differentiability of the conjugation of certain diffeomorphisms of the circle, *Ergod. Theor. Dyn. Syst.*, **9** (1989), 643-680.

[11] Y. Katznelson and D. Ornstein, The absolute continuite of the conjugation of certain diffeomorphisms of the circle, *Ergod. Theor. Dyn. Syst.*, **9** (1989), 681-690.

[12] K. Khanin and Ya. Sinai, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Russ. Math. Surv.*, **44** (1989), 69-99.

[13] A. Khinchin, *Continued Fractions*, University of Chicago Press (1964).

[14] A. Shiryayev, *Probability*, New York (1984).