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ON QUASI CLASS $Q(N)$ AND QUASI CLASS $Q^*(N)$ OPERATORS

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Abstract: Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. In this paper we introduce two new classes of operators: quasi class $Q(N)$ and quasi class $Q^*(N)$. An operator $T \in \mathcal{L}(\mathcal{H})$ is of quasi class $Q(N)$ for a fixed real number $N \geq 1$, if T satisfies

$$N\|T^2x\|^2 \leq \|T^3x\|^2 + \|Tx\|^2,$$

for all $x \in \mathcal{H}$. And an operator $T \in \mathcal{L}(\mathcal{H})$ is of quasi class $Q^*(N)$ for a fixed real number $N \geq 1$, if T satisfies

$$N\|T^*Tx\|^2 \leq \|T^3x\|^2 + \|Tx\|^2,$$

for all $x \in \mathcal{H}$. We study basic properties of these classes of operators, the structural and spectral properties, a matrix representation and also the Aluthge transformation.

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1. Introduction

In this paper let $\mathcal{L}(\mathcal{H})$ stand for the C^* algebra of all bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by ker$T$ the null space, by $T(\mathcal{H})$ the range of T. By $\sigma(T)$ we write the spectrum

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of T, the $r(T)$ is the spectral radius of operator T which is defined by $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$. The $\sigma_a(T)$ is the approximate point spectrum of operator T and it is proved that if $\lambda \in \sigma_a(T)$, then there exist the sequence (x_n), such as $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to \infty$.

The null operator and the identity on $\mathcal{H}$ will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, $T \geq O$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. If two operator $T \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ are positive operators and $TS = ST$ then TS is also positive operator.

The operator T is an isometry if $\|Tx\| = \|x\|$, for all $x \in \mathcal{H}$. The operator T is called unitary operator if $T^*T = TT^* = I$. The operator T is normaloid if $r(T) = \|T\|$ and it is quasinilpotent if $r(T) = 0$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be:

- Quasi class Q if $T^{*3}T^3 - 2T^{*2}T^2 + T^*T \geq O$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the quasi class Q if $\|T^2x\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2)$ (see [2]).

- Quasi class Q^* if $T^{*3}T^3 - 2(T^*T)^2 + T^*T \geq O$. (see [5]). It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the quasi class Q^* if $\|T^*Tx\|^2 \leq \frac{1}{2}(\|T^3x\|^2 + \|Tx\|^2)$ (see [5]).

- Class $Q(N)$ if $N\|Tx\|^2 \leq \|T^2x\|^2 + \|x\|^2$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the class $Q(N)$ if $T^{*2}T^2 - NT^*T + I \geq O$ (see [4]).

- Class $Q^*(N)$ if $N\|T^*x\|^2 \leq \|T^2x\|^2 + \|x\|^2$. It is proved that an operator $T \in \mathcal{L}(\mathcal{H})$ is of the class $Q^*(N)$ if $T^{*2}T^2 - NTT^* + I \geq O$ (see [4]).

Analyzing the good qualities of quasi class Q, quasi class Q^*, class $Q(N)$ and class $Q^*(N)$ of operators, we introduced two new classes of operators, quasi class $Q(N)$ and quasi class $Q^*(N)$ which include these classes of operators and some of their properties.

Definition 1. An operator $T \in \mathcal{L}(\mathcal{H})$ is of quasi class $Q(N)$, for a fixed real number $N \geq 1$ if T satisfies

$$N\|T^2x\|^2 \leq \|T^3x\|^2 + \|Tx\|^2,$$

for all $x \in \mathcal{H}$.

Definition 2. An operator $T \in \mathcal{L}(\mathcal{H})$ is of quasi class $Q^*(N)$, for a fixed real number $N \geq 1$ if T satisfies

$$N\|T^*Tx\|^2 \leq \|T^3x\|^2 + \|Tx\|^2,$$

for all $x \in \mathcal{H}$.

From the definitions, we see that this to new classes of operator could compared to several classes of operators

- quasi class Q = quasi class $Q(2)$;
- quasi class Q^* = quasi class $Q^*(2)$;
- class $Q(N) \subseteq$ quasi class $Q(N)$;
- class $Q^*(N) \subseteq$ quasi class $Q^*(N)$.

In this study we give basic properties of quasi class $Q(N)$ and quasi class $Q^*(N)$ operators. In Section 2, we discuss some inclusion relations, the structural and spectral properties and give some examples of this classes of operators. In Section 3, a matrix representation is obtained. And in Section 4, we give the equivalence between Aluthge transformation $*-$Aluthge transformation of quasi class $Q(N)$ and quasi class $Q^*(N)$ of operators.

2. Basic and Spectral Properties

First, we state a proposition which gives necessary and sufficient conditions for an operator T to be of quasi class $Q(N)$.

Proposition 3. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of quasi class $Q(N)$ if and only if*

$$T^{*3}T^3 - NT^{*2}T^2 + T^*T \geq 0$$

for a fixed real number $N \geq 1$.

Proof. Since T is of quasi class $Q(N)$, for a fixed real number $N \geq 1$, then

$$N\|T^2x\|^2 \leq \|T^3x\|^2 + \|Tx\|^2,$$

for all $x \in \mathcal{H}$. Then,

$$\begin{aligned} &(T^3x|T^3x) - N(T^2x|T^2x) + (Tx|Tx) \geq 0 \\ &\Leftrightarrow (T^{*3}T^3x|x) - N(T^{*2}T^2x|x) + (T^*Tx|x) \geq 0 \\ &\Leftrightarrow ((T^{*3}T^3 - NT^{*2}T^2 + T^*T)x|x) \geq 0 \\ &\Leftrightarrow T^{*3}T^3 - NT^{*2}T^2 + T^*T \geq 0. \end{aligned}$$

□

From the definition of the class $Q(N)$ operator

$$T^{*2}T^2 - NT^*T + I \geq O,$$

and the Proposition 3 we see that every operator of the class $Q(N)$ is also an operator of the quasi class $Q(N)$. Thus, we have the following implication:

$$\text{class } Q(N) \subseteq \text{quasi class } Q(N).$$

Similarly, we state a proposition giving necessary and sufficient conditions for an operator T to be of quasi class $Q^*(N)$.

Proposition 4. *An operator $T \in \mathcal{L}(\mathcal{H})$ is of quasi class $Q^*(N)$, if and only if*

$$T^{*3}T^3 - N(T^*T)^*2 + T^*T \geq 0$$

for a fixed real number $N \geq 1$.

From the definition of the class $Q^*(N)$ operator

$$T^{*2}T^2 - NTT^* + I \geq O,$$

and the Proposition 4 we see that every operator of the class $Q^*(N)$ is also an operator of the quasi class $Q^*(N)$. Thus, we have the following implication:

$$\text{class } Q^*(N) \subseteq \text{quasi class } Q^*(N).$$

In the following we give the inclusion of these classes of operators.

Proposition 5. *Quasi class $Q^*(N) \subseteq$ quasi class $Q^*(N-1)$ for $N \geq 2$.*

Proof. Since T is of quasi class $Q^*(N)$, for a fixed real number $N \geq 2$, then

$$N\|T^*Tx\|^2 \leq \|T^3x\|^2 + \|Tx\|^2,$$

for all $x \in \mathcal{H}$. Then,

$$(N-1)\|T^*Tx\|^2 \leq N\|T^*Tx\|^2 \leq \|T^3x\|^2 + \|Tx\|^2.$$

Hence, T is an operator of quasi class $Q^*(N-1)$.

It follows that,

quasi class $Q^*(1) \supseteq$ quasi class $Q^*(2) \supseteq \ldots \supseteq$ quasi class $Q^*(N)$. $\square$

A similar inclusion can also be proved.

quasi class $Q(1) \supseteq$ quasi class $Q(2) \supseteq \ldots \supseteq$ quasi class $Q(N)$.

In the following we state a proposition which gives conditions for an operator T of the quasi class $Q(N)$ to be an operator of the class $Q(N)$.

Proposition 6. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of the quasi class $Q(N)$. If T has dense range, then T is an operator of the class $Q(N)$.*

Proof. Since T has dense range then $\overline{T(\mathcal{H})} = \mathcal{H}$. Let $y \in \mathcal{H}$. Then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{H}$ such that $T(x_n) \to y$, $n \to \infty$. Since T is an operator of the quasi class $Q(N)$, then

$$\begin{aligned}
&\big\langle (T^{*3}T^3 - NT^{*2}T^2 + T^*T)x_n, x_n \big\rangle \geq 0,\\
&\big\langle (T^*(T^{*2}T^2 - NT^*T + I)T)x_n, x_n \big\rangle \geq 0,\\
&\langle (T^{*2}T^2 - NT^*T + I)Tx_n, Tx_n \rangle \geq 0, \text{ for all } n \in N.
\end{aligned}$$

By the continuity of the inner product, we have

$$\langle (T^{*2}T^2 - 2T^*T + I)y, y \rangle \geq 0$$

Therefore T is an operator of the class $Q(N)$. □

Similarly, we state a proposition which gives conditions for an operator T of the quasi class $Q^*(N)$ to be an operator of the class $Q^*(N)$.

Proposition 7. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of the quasi class $Q^*(N)$. If T has dense range, then T is an operator of the class $Q^*(N)$.*

Now we will prove some properties of quasi class $Q(N)$ and quasi class $Q^*(N)$ operators.

Proposition 8. *Let T be an operator of quasi class $Q(N)$.*
a) If T double commutes with an isometric operator S, then TS is an operator of the quasi class $Q(N)$.
b) If S is unitarily equivalent to operator T, then S is an operator of the quasi class $Q(N)$.

Proof. a) Let $B = TS, TS = ST, S^*T = TS^*$ and $S^*S = I$.

$$B^{*3}B^3 - NB^{*2}B^2 + B^*B$$

$$\begin{aligned}&= (TS)^{*3}(TS)^3 - N(TS)^{*2}(TS)^2 + (TS)^*(TS)\\ &= T^{*3}T^3 - NT^{*2}T^2 + T^*T \geq 0,\end{aligned}$$

so TS is an operator of quasi class $Q(N)$.

b) Since operator S is unitarily equivalent to operator T, then there exists an unitary operator U such that $S = U^*TU$. Since T is an operator of class $Q(N)$, then

$$T^{*3}T^3 - NT^{*2}T^2 + T^*T \geq 0.$$

Hence,

$$\begin{aligned}&S^{*3}S^3 - NS^{*2}S^2 + S^*S\\ &= (U^*TU)^{*3}(U^*TU)^3 - N(U^*TU)^{*2}(U^*TU)^2 + (U^*TU)^*(U^*TU)\\ &= U^*(T^{*3}T^3 - NT^{*2}T^2 + T^*T)U \geq 0\end{aligned}$$

so S is an operator of quasi class $Q(N)$.

□

Proposition 9. *Let T be an operator of quasi class $Q^*(N)$. If S is unitarily equivalent to operator T, then S is an operator of the quasi class $Q^*(N)$.*

Proposition 10. *Let $T \in L(H)$. If $\|T\| \leq \frac{1}{\sqrt{N}}$, then T is an operator of quasi class $Q(N)$.*

Proof. From $\|T\| \leq \frac{1}{\sqrt{N}}$, we have $\|T\|^2 \leq \frac{1}{N}$. Then,

$$\begin{aligned}&\|Tx\|^2 \leq \frac{1}{N}\|x\|^2, \forall x \in H\\ &\langle Tx, Tx\rangle - \frac{1}{N}\langle x, x\rangle \leq 0, \forall x \in H\\ &\langle (I - NT^*T)x, x\rangle \geq 0, \forall x \in H\\ &I - NT^*T \geq 0,\\ &T^{*2}T^2 - NT^*T + I \geq 0,\\ &T^*(T^{*2}T^2 - NT^*T + I)T \geq 0,\\ &T^{*3}T^3 - NT^{*2}T^2 + T^*T \geq 0,\end{aligned}$$

so T is an operator of quasi class $Q(N)$.

□

Proposition 11. *Let $T \in L(H)$. If $\|T^*\| \leq \dfrac{1}{\sqrt{N}}$, then T is an operator of quasi class $Q^*(N)$.*

Proposition 12. *Let M be a closed T invariant subset of $\mathcal{H}$. Then, the restriction $T_{|M}$ of a quasi class $Q(N)$ operator T to M is of quasi class $Q(N)$.*

Proof. Let be $u \in M$. Then,

$$\begin{aligned}
&\|(T\ |_M)^2 u\|^2 \\
&\qquad = \|T^2 u\|^2 \leq \frac{1}{N}\left(\|T^3 u\|^2 + \|Tu\|^2\right) \\
&\qquad = \frac{1}{N}\left(\|(T\ |_M)^3 u\|^2 + \|Tu\|^2\right).
\end{aligned}$$

This implies that $T\ |_M$ is an operator of quasi class $Q(N)$. □

Proposition 13. *Let M be a closed T invariant subset of $\mathcal{H}$. Then, the restriction $T_{|M}$ of a quasi class $Q^*(N)$ operator T to M is of quasi class $Q^*(N)$.*

Theorem 14. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator and S be an operator such that S commutes with operator T^*T. Then, S is of quasi class $Q(N)$ if and only if TST^{-1} is of quasi class $Q(N)$.*

Proof. Let S be an operator of quasi class $Q(N)$. Then,

$$S^{*3}S^3 - NS^{*2}S^2 + S^*S \geq 0.$$

From this we have that

$$T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^* \geq 0.$$

Now, we prove that operator TT^* commutes with operator $T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*$. Since operator S commutes with operator T^*T, operator S^* also commutes with operator T^*T. From this we have:

$$\begin{aligned}
&T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*[TT^*] \\
=&T[S^{*3}S^3 - NS^{*2}S^2 + S^*S][T^*T]T^* \\
=&T[T^*T][S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^* \\
=&[TT^*]T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*.
\end{aligned}$$

Thus, operator TT^* commutes with operator $T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*$. Then, operator $[TT^*]^{-1}$ also commutes with operator

$$T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*.$$

Since the operators $[TT^*]^{-1}$ and $T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*$ are positive, then:

$$T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*[TT^*]^{-1} \geq 0.$$

Since operator S commutes with operator T^*T, we get,

$$(TST^{-1})^*(TST^{-1}) = T^{*-1}S^*T^*TST^{-1} = TS^*ST^{-1} \tag{1}$$

$$(TST^{-1})^{*2}(TST^{-1})^2 = TS^{*2}S^2T^{-1} \tag{2}$$

$$(TST^{-1})^{*3}(TST^{-1})^3 = TS^{*3}S^3T^{-1} \tag{3}$$

To prove that TNT^{-1} is an operator of quasi class $Q(N)$, we substitute equations 1, 2 and 3 in the above expression:

$$(TST^{-1})^{*3}(TST^{-1})^3 - N(TST^{-1})^{*2}(TST^{-1})^2 + (TST^{-1})^*(TST^{-1})$$

and we obtain:

$$\begin{aligned}
&(TST^{-1})^{*3}(TST^{-1})^3 - N(TST^{-1})^{*2}(TST^{-1})^2 + (TST^{-1})^*(TST^{-1}) \\
=&TS^{*3}S^3T^{-1} - NTS^{*2}S^2T^{-1} + TS^*ST^{-1} \\
=&T^{*-1}T^*[T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^{-1}]TT^{-1} \\
=&T^{*-1}T^*T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^{-1}TT^{-1} \\
=&T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^{-1}.
\end{aligned}$$

Now we prove that the last expression is positive. Since:

$$T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*[TT^*]^{-1} \geq 0$$

we have

$$\begin{aligned}
&T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*[TT^*]^{-1} \geq 0 \\
&T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^*T^{*-1}T^{-1} \geq 0 \\
&T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^{-1} \geq 0.
\end{aligned}$$

Hence, TST^{-1} is an operator of quasi class $Q(N)$.

Conversely, let TST^{-1} be an operator of quasi class $Q(N)$ then:

$$(TST^{-1})^{*3}(TST^{-1})^3 - N(TST^{-1})^{*2}(TST^{-1})^2 + (TST^{-1})^*(TST^{-1}) \geq 0.$$

Similarly, after substituting equations 1 , 2 and 3 we have:

$$T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^{-1} \geq 0$$
$$T^*T[S^{*3}S^3 - NS^{*2}S^2 + S^*S]T^{-1}T \geq 0$$
$$[T^*T][S^{*3}S^3 - NS^{*2}S^2 + S^*S] \geq 0.$$

Since operator $[T^*T]$ commutes with operator S and hence, with operator $[T^*T][S^{*3}S^3 - NS^{*2}S^2 + S^*S]$. Therefore, operator $[T^*T]^{-1}$ commutes with operator $[T^*T][S^{*3}S^3 - NS^{*2}S^2 + S^*S]$. Since operators $[T^*T]^{-1}$ and $[T^*T][S^{*2}S^2 - NS^*S + I]$ are positive we have

$$[T^*T]^{-1}[T^*T][S^{*3}S^3 - NS^{*2}S^2 + S^*S \geq 0.$$

Therefore,

$$S^{*3}S^3 - NS^{*2}S^2 + S^*S \geq 0.$$

Hence, S is an operator of quasi class $Q(N)$. □

Theorem 15. *Let $T \in \mathcal{L}(\mathcal{H})$ be an invertible operator and S be an operator such that S commutes with operator T^*T. Then, S is of quasi class $Q^*(N)$ if and only if TST^{-1} is of quasi class $Q^*(N)$.*

The next two proposition give necessary and sufficient conditions for a weighted shift operator T with decreasing weighted sequence (α_n) to be an operator of these classes of operators.

Proposition 16. *A weighted shift operator T with decreasing weighted sequence (α_n) is an operator of quasi class $Q(N)$ if and only if*

$$|\alpha_{n+1}|^2|\alpha_{n+2}|^2 - N|\alpha_{n+1}|^2 + 1 \geq 0$$

for every n.

Proof. Since T is a weighted shift, its adjoint T^* is also a weighted shift and defined by $T(e_n) = |\alpha_n|e_{n+1}$ we have:

$$T^*(e_n) = |\alpha_{n-1}|e_{n-1},$$

$$\begin{aligned}&(T^*T)(e_n) = |\alpha_n|^2 e_n,\\&(T^{*2}T^2)(e_n) = |\alpha_n|^2|\alpha_{n+1}|^2 e_n,\\&(T^{*3}T^3)(e_n) = |\alpha_n|^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2 e_n.\end{aligned}$$

Now, since T is an operator of quasi class $Q(N)$ then,

$$\begin{aligned}&T^{*3}T^3 - NT^{*2}T^2 + T^*T \geq 0\\ \Leftrightarrow&|\alpha_n|^2|\alpha_{n+1}|^2|\alpha_{n+2}|^2 - N|\alpha_n|^2|\alpha_{n+1}|^2 + |\alpha_n|^2 \geq 0.\\ \Leftrightarrow&|\alpha_{n+1}|^2|\alpha_{n+2}|^2 - N|\alpha_{n+1}|^2 + 1 \geq 0.\end{aligned}$$

□

Example 1. A weighted shift operator T with decreasing weighted sequence $\alpha_n = \dfrac{1}{\sqrt{N-1}}$, is an operator of quasi class $Q(N)$ for every fixed real number $N \geq 2$ (it is clear from Proposition 16).

Proposition 17. *A weighted shift operator T with decreasing weighted sequence (α_n) is an operator of quasi class $Q^*(N)$ if and only if*

$$|\alpha_{n+1}|^2|\alpha_{n+2}|^2 - N|\alpha_n|^2 + 1 \geq 0$$

for every n.

Example 2. A weighted shift operator T with decreasing weighted sequence $\alpha_n = \dfrac{1}{\sqrt{N}}$, is an operator of quasi class $Q^*(N)$ for every fixed real number $N \geq 1$ (it is clear from Proposition 17).

In the following we give one example where it is shown that there exists an operator from the quasi class $Q^*(3)$, which is not a quasi class $Q(3)$ operator. This shows that the classes of operators quasi class $Q(N)$ and quasi class $Q^*(N)$ are independent.

Example 3. Let T be a weighted shift operator with decreasing weighted sequence as follows:

$$\alpha_n = \begin{cases} 0, & n \leq 0\\ 1, & n = 1\\ \dfrac{1}{2}, & n = 2\\ 4, & n \geq 3 \end{cases}.$$

After some calculations from Proposition 17, it follows that T is an operator of quasi class $Q^*(3)$ for every n.

But, from Proposition 16, it follows that T is not an operator of quasi class $Q(3)$. For example for $n = 0$ we have

$$\alpha_1^2 \cdot \alpha_2^2 - 3 \cdot \alpha_1^2 + 1 = 1 \cdot \frac{1}{4} - 3 \cdot 1 + 1 < 0.$$

In following we give the inclusion of approximate point spectrum of these two classes of operators.

Theorem 18. *Let $T \in L(H)$ be a regular quasi class $Q(N)$ operator. Then the approximate point spectrum of operator T lies in the disc*

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{N}}{\|T^{-2}\| \cdot \sqrt{\|T^2\|^2 + 1}} \leq |\lambda| \leq \|T\|\}.$$

Proof. Let T be a regular quasi class $Q(N)$ operator. For every unit vector x in Hilbert space $\mathcal{H}$, we have:

$$\begin{aligned}
&\|x\|^2 = \|T^{-2} \cdot T^2 x\|^2 \\
\leq &\|T^{-2}\|^2 \cdot \|T^2 x\|^2 \\
\leq &\|T^{-2}\|^2 \cdot \frac{1}{N} \cdot (\|T^3 x\|^2 + \|Tx\|^2) \\
\leq &\frac{1}{N} \cdot \|T^{-2}\|^2 \cdot (\|T^2\|^2 \cdot \|Tx\|^2 + \|Tx\|^2).
\end{aligned}$$

So,

$$N \leq \|Tx\|^2 \cdot \|T^{-2}\|^2 \cdot (\|T^2\|^2 + 1),$$

where we have

$$\|Tx\| \geq \frac{\sqrt{N}}{\|T^{-2}\| \cdot \sqrt{\|T^2\|^2 + 1}}.$$

Now, assume that $\lambda \in \sigma_a(T)$, then there exists a sequence (x_n), such as $\|x_n\| = 1$ and $\|(T - \lambda I)x_n\| \to 0, n \to \infty$.

From the last inequation we have:

$$\|Tx_n - \lambda x_n\| \geq \|Tx_n\| - |\lambda| \cdot \|x_n\| \geq \frac{\sqrt{N}}{\|T^{-2}\| \cdot \sqrt{\|T^2\|^2 + 1}} - |\lambda|.$$

Now, when $n \to \infty$ we have

$$|\lambda| \geq \frac{\sqrt{N}}{\|T^{-2}\| \cdot \sqrt{\|T^2\|^2 + 1}}.$$

So, we have

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{N}}{\|T^{-2}\| \cdot \sqrt{\|T^2\|^2 + 1}} \leq |\lambda| \leq \|T\|\}.$$

Therefore the proof is completed. □

Theorem 19. *Let $T \in L(H)$ be a regular quasi class $Q^*(N)$ operator. Then the approximate point spectrum of operator T lies in the disc*

$$\sigma_a(T) \subseteq \{\lambda \in C : \frac{\sqrt{N}}{\|(T^*T)^{-1}\| \cdot \sqrt{\|T^2\|^2 + 1}} \leq |\lambda| \leq \|T\|\}.$$

3. Matrix Representation

In this section we give some results for the matrix representation of these two classes of operators.

Proposition 20. *Let $T \in \mathcal{L}(\mathcal{H})$ be the operator defined as*

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}.$$

If A is operator of class $Q(N)$, then T is an operator of quasi class $Q(N)$.

Proof. A simple calculation shows that:

$$T^* = \begin{pmatrix} A^* & 0 \\ B^* & 0 \end{pmatrix},$$

$$T^{*2} = \begin{pmatrix} A^{*2} & 0 \\ B^*A^* & 0 \end{pmatrix},$$

$$T^2 = \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix},$$

$$T^{*3} = \begin{pmatrix} A^{*3} & 0 \\ B^*A^{*2} & 0 \end{pmatrix},$$
$$T^3 = \begin{pmatrix} A^3 & A^2B \\ 0 & 0 \end{pmatrix},$$
$$T^{*3}T^3 = \begin{pmatrix} A^{*3}A^3 & A^{*3}A^2B \\ B^*A^{*2}A^3 & B^*A^{*2}A^2B \end{pmatrix}.$$
$$T^{*3}T^3 - NT^{*2}T^2 + T^*T$$
$$= \begin{pmatrix} A^*(A^{*2}A^2 - NA^*A + I)A & A^*(A^{*2}A^2 - NA^*A + I)B \\ B^*(A^{*2}A^2 - NA^*A + I)A & B^*(A^{*2}A^2 - NA^*A + I)B \end{pmatrix}$$

Let $u = x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Then,

$$\begin{aligned}
&\langle (T^{*3}T^3 - NT^{*2}T^2 + T^*Tu, u\rangle \\
&= \langle A^*(A^{*2}A^2 - NA^*A + I)Ax, x\rangle + \langle A^*(A^{*2}A^2 - NA^*A + I)By, x\rangle \\
&+ \langle {}^*(A^{*2}A^2 - NA^*A + I)Ax, y\rangle + \langle B^*(A^{*2}A^2 - NA^*A + I)By, y\rangle \\
&= \langle (A^{*2}A^2 - NA^*A + I)Ax, Ax\rangle + \langle (A^{*2}A^2 - NA^*A + I)By, Ax\rangle \\
&+ \langle (A^{*2}A^2 - NA^*A + I)Ax, By\rangle + \langle (A^{*2}A^2 - NA^*A + I)By, By\rangle \\
&= \langle (A^{*2}A^2 - NA^*A + I)(Ax + By), (Ax + By)\rangle \geq 0
\end{aligned}$$

because A is operator of class $Q(N)$ then, $A^{*2}A^2 - 2A^*A + I \geq O$, so this proves the result. □

Proposition 21. *Let $T \in \mathcal{L}(\mathcal{H})$ be a quasi class $Q(N)$ operator, the range of T not to be dense, and*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on} \;\; \mathcal{H} = \overline{T(\mathcal{H})} \oplus \ker T^*.$$

Then, A is an operator of the class $Q(N)$ on $\overline{T(\mathcal{H})}$, $C = O$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. Suppose that $T \in \mathcal{L}(\mathcal{H})$ is an operator of quasi class $Q(N)$. Since that T does not have dense range, we can represent T as the upper triangular matrix:

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{on} \;\; \mathcal{H} = \overline{T(\mathcal{H})} \oplus \ker T^*.$$

Since T is an operator of quasi class $Q(N)$, we have

$$T^{*3}T^3 - NT^{*2}T^2 + T^*T \geq 0$$

$$\Rightarrow T^*(T^{*2}T^2 - NT^*T + I)T \geq 0.$$

Therefore, after some calculation similar as in Proposition 20 we get:

$$\langle (T^{*2}T^2 - NT^*T + I)x, x\rangle = \langle (A^{*2}A^2 - NA^*A + I)x, x\rangle \geq 0,$$

for all $x \in \overline{T(\mathcal{H})}$.

Hence

$$A^{*2}A^2 - NA^*A + I \geq 0.$$

This shows that A is an operator of the class $Q(N)$ on $\overline{T(\mathcal{H})}$.

Let P be the orthogonal projection of H onto $\overline{T(\mathcal{H})}$. For any

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{H} = \overline{T(\mathcal{H})} \oplus \ker T^*.$$

Then

$$\langle Cx_2, x_2\rangle = \langle T(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^*(I-P)x\rangle = 0.$$

Thus $T^* = 0$.

Since $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$ by [3, Corollary 7]. Since $\sigma(A) \cap \sigma(C)$ has no interior points, then $\sigma(T) = \sigma(A) \cup \sigma(C) = \sigma(A) \cup \{0\}$ and $C^k = 0$.

□

Proposition 22. *Let $T \in \mathcal{L}(\mathcal{H})$ be a quasi class $Q^*(N)$ operator, the range of T not to be dense, and*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{T(\mathcal{H})} \oplus \ker T^*.$$

Then, A is an operator of the class $Q^*(N)$ on $\overline{T(\mathcal{H})}$, $C = O$ and $\sigma(T) = \sigma(A) \cup \{0\}$.

4. Aluthge Transformation

Aluthge in [1] defined a transformation $\tilde{T}$ of operator T by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where $T = U|T|$ is the polar decomposition of operator T. $\tilde{T}$ is called Aluthge transformation.

Yamazaki in [6] defined the $^*-$Aluthge transformation of operator T. The $*-$Aluthge transformation is defined by $\tilde{T}^{(*)} \stackrel{def}{=} (\tilde{T}^*)^* = |T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}$.

It is proved that:

$U^*|T^*|^{\frac{1}{2}} = |T|^{\frac{1}{2}}U^*, U^*|T^*| = |T|U^*, U|T|^{\frac{1}{2}} = |T^*|^{\frac{1}{2}}U, U|T| = |T^*|U.$

Now we will give the equivalence between Aluthge transformation and $*-$Aluthge transformation of quasi class $Q(N)$ and quasi class $Q^*(N)$ of operators.

Theorem 23. *Let $T \in L(H)$. Then, $\tilde{T}$ is an operator of quasi class $Q(N)$ if and only if $\tilde{T}^{(*)}$ is an operator of quasi class $Q(N)$.*

Proof. Assume that $\tilde{T}$ is an operator of quasi class $Q(N)$, then

$$\tilde{T}^{*3}\tilde{T}^3 - N\tilde{T}^{*2}\tilde{T}^2 + \tilde{T}^*\tilde{T} \geq 0.$$

We need to prove that $\tilde{T}^{(*)}$ is an operator of quasi class $Q(N)$.

$$\begin{aligned}
&\tilde{T}^{(*)*3}\tilde{T}^{(*)3} - N\tilde{T}^{(*)*2}\tilde{T}^{(*)2} + \tilde{T}^{(*)*}\tilde{T}^{(*)}\\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*3}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^3\\
&- N|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*2}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^2\\
&+ (|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
=&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&- N(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}}\\
&+ (|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
=&(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&- N(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&+ (|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
=&UU^*(|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*|U^*|T^*U|T^*|U|T^*|U|T^*|^{\frac{1}{2}}\\
&- N|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}}\\
&+ |T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}})UU^*\\
=&U(U^*|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*|U^*|T^*U|T^*|U|T^*|U|T^*|^{\frac{1}{2}}U
\end{aligned}$$

$$
\begin{aligned}
&- NU^*|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*U|T^*|U|T^*|^{\frac{1}{2}}U \\
&+ U^*|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}U)U^* \\
=&U(|T|^{\frac{1}{2}}U^*|T|U^*|T|U^*|T|U|T|U|T|U|T|^{\frac{1}{2}} \\
&- N|T|^{\frac{1}{2}}U^*|T|U^*|T|U|T|U|T|^{\frac{1}{2}} \\
&+ |T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}})U^*
\end{aligned}
$$

$$
\begin{aligned}
=&U(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&- N|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&+ (|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}))U^* \\
=&U(\tilde{T}^{*3}\tilde{T}^3 - N\tilde{T}^{*2}\tilde{T}^2 + \tilde{T}^*\tilde{T})U^* \geq 0.
\end{aligned}
$$

Therefore,

$$\tilde{T}^{(*)*3}\tilde{T}^{(*)3} - N\tilde{T}^{(*)*2}\tilde{T}^{(*)2} + \tilde{T}^{(*)*}\tilde{T}^{(*)} \geq 0.$$

Hence, $\tilde{T}^{(*)}$ is an operator of quasi class $Q(N)$.

Conversely, assume that $\tilde{T}^{(*)}$ is an operator of quasi class $Q(N)$, then

$$\tilde{T}^{(*)*3}\tilde{T}^{(*)3} - N\tilde{T}^{(*)*2}\tilde{T}^{(*)2} + \tilde{T}^{(*)*}\tilde{T}^{(*)} \geq 0.$$

We need to prove that $\tilde{T}$ is an operator of quasi class $Q(N)$.

Consider

$$
\begin{aligned}
&\tilde{T}^{*3}\tilde{T}^3 - N\tilde{T}^{*2}\tilde{T}^2 + \tilde{T}^*\tilde{T} \\
=&(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&- N(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}) \\
&+ (|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})
\end{aligned}
$$

$$=U^*U[(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})$$

$$
\begin{aligned}
&(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})\\
&-N|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})\\
&+(|T|^{\frac{1}{2}}U^*|T|^{\frac{1}{2}})(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})]U^*U\\
=&U^*[U|T|^{\frac{1}{2}}U^*|T|U^*|T|U^*|T|U|T|U|T|U|T|^{\frac{1}{2}}U^*\\
&-NU|T|^{\frac{1}{2}}U^*|T|U^*|T|U|T|U|T|^{\frac{1}{2}}U^*\\
&+U|T|^{\frac{1}{2}}U^*|T|U|T|^{\frac{1}{2}}U^*]\\
=&U^*[|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*|U^*|T^*|U|T^*|U|T^*|U|T^*|^{\frac{1}{2}}\\
&-N|T^*|^{\frac{1}{2}}U^*|T^*|U^*|T^*|U|T^*|U|T^*|^{\frac{1}{2}}\\
&+|T^*|^{\frac{1}{2}}U^*|T^*|U|T^*|^{\frac{1}{2}}]U\\
=&U^*[(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})\\
&(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&-N(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})\\
&+(|T^*|^{\frac{1}{2}}U^*|T^*|^{\frac{1}{2}})(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})]U\\
=&U^*[(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*3}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^3\\
&-N(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^{*2}(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^2\\
&+(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})^*(|T^*|^{\frac{1}{2}}U|T^*|^{\frac{1}{2}})]U\\
=&U^*[\tilde{T}^{(*)*3}\tilde{T}^{(*)3}-N\tilde{T}^{(*)*2}\tilde{T}^{(*)2}+\tilde{T}^{(*)*}\tilde{T}^{(*)}]U\geq 0.
\end{aligned}
$$

Therefore,

$$\tilde{T}^{*3}\tilde{T}^3-N\tilde{T}^{*2}\tilde{T}^2+\tilde{T}^*\tilde{T}\geq 0.$$

Hence, $\tilde{T}$ is an operator of class $Q(N)$. □

Theorem 24. *Let $T\in L(H)$. Then $\tilde{T}$ is an operator of quasi class $Q^*(N)$ if and only if $\tilde{T}^{(*)}$ is an operator of quasi class $Q^*(N)$.*

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