International Journal of Applied Mathematics

Volume 32 No. 3 2019, 509-519
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v32i3.11

ON THE VOLTERRA-STIELTJES INTEGRAL EQUATION AND AXIOMATIC MEASURES OF WEAK NONCOMPACTNESS

A.A.H. Abd El-Mwla
Faculty of Science
Omar Al-Mukhtar University
"Alfataeh" Str., Derna, LIBYA

Abstract: In this note, we will use a compactness type condition in connection with the weak topology to prove the existence of weakly continuous solutions for a functional integral equation of Volterra-Stieltjes type in nonreflexive Banach spaces. A special case will be considered.

AMS Subject Classification: 35D30, 26A42, 47H10, 58C30, 28-XX
Key Words: Volterra-Stieltjes integral equation, weakly continuous, measure of weak noncompactness, Kubiaczyk's fixed point theorem

1. Introduction and preliminaries

In last years the study of the integral equations of Volterra-Stieltjes type in a Banach space has been developed extensively. However almost all of the work was done using the strong topology (see [3]-[9]), while the study of our problem involving the weak topology is lagging behind. The main tools were De Blasi measure of weak noncompactness and Kubiaczyk's fixed point theorem.

In this paper we investigate weakly continuous solutions of the integral equations of Volterra-Stieltjes type

$$x(t) = p(t) + \int_0^t f(s, x(s))\; d_s g(t,s),\;\; t \in I = [0,a], \tag{1}$$

in nonreflexive Banach spaces.

Received: February 27, 2019

The result of Szep was extended to nonreflexive Banach spaces by Boundourides [11] and Cramer-Lakshmikantham-Mitchell [12]. Recently, in [14], [15], the authors studied the existence of weak solution $x \in C[I, E]$ in a reflexive Banach space for the nonlinear Volterra-Stieltjes integral equation (1) where f is assumed to be weakly-weakly continuous.

Let E be nonreflexive Banach space with norm $\| \, . \, \|$ with its dual E^*, and we will denote by $E_\omega = (E, \omega) = (E, \sigma(E, E^*))$ the space E with its weak topology. Denote by $C[I, E_\omega]$ the Banach space of weakly continuous functions from $I = [0, a]$ to E_ω endowed with the topology of weak uniform convergence.

Now, let $r > 0$ be given and define the set

$$B_r = \{x(t) \in E, \ : \ \| x \|_0 \leq r\}. \tag{2}$$

Lemma. *Let $f : I \times B_r \to E$ be weakly-weakly continuous, then*

- *For each $t \in I$, $f(t, .)$ is weakly continuous, hence weakly sequentially continuous (see [10]),*
- *For each weakly continuous $x : I \to B_r$, $f(., x(.))$ is weakly continuous on I (see [20]),*
- *f is norm bounded, i.e., there exists an M_r such that*

$$\| f(t, x) \| \leq M_r \tag{3}$$

for all $(t, x) \in I \times B_r$ (see [19]).

Assume that a nonnegative function which is continuous function $h : [0, \infty) \to [0, \infty)$ satisfies the following conditions:

(A_1) $h(0) = 0$.

(A_2) $z(t) \equiv 0$ is the unique continuous solution of the integral inequality

$$u(t) \leq \int_0^t h(u(s)) \, d_s g(t, s), \ t \in I \tag{4}$$

satisfying the condition

$$u(0) = 0. \tag{5}$$

Further on, denote by m_E the family of nonempty and bounded subsets of E. In this paper by a measure of weak noncompactness we will understand a function $\beta : m_E \to [0, \infty)$ such that $(A, \ B \in m_E)$:

(a_1) $\beta(A) = 0 \Leftrightarrow A$ is relatively weakly compact in E,

(a_2) $\beta(A) = \beta(\overline{conv} A)$,

(a_3) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$,

(a_4) $\beta(A \cup \{x\}) = \beta(A),\ x \in E$,

(a_5) $\beta(\lambda A) = |\lambda|.\beta(A),\ \lambda \in R$,

(a_6) $\beta(A + B) \leq \beta(A) + \beta(B)$,

(a_7) $\beta(A \cup B) = \max\ (\ \beta(A), \beta(B)\ )$.

It is necessary to remark that if β has these properties, then the following theorem is true.

Theorem 1. ([2, 13])
Let $V \subset \mathcal{C}_\omega$ be a family of strongly equicontinuous functions. Then the function $t \mapsto v(t) = \beta(V(t))$ is continuous and $\beta(V(I)) = \sup\{\beta(V(t)) : t \in I\}$.

Now we have the following Kubiaczyk's fixed point theorem that will be needed in this paper (see [18]).

Theorem 2. *Let Q be a closed convex and equicontinuous subset of a metrizable locally convex vector space E and let F be a weakly sequentially continuous mapping of Q into itself. If for some $x \in Q$ the implication*

$$\overline{V} \ = \ \overline{conv}(F(V) \cup \{x\}) \Rightarrow V \text{ is relatively weakly compact} \tag{6}$$

holds, for every subset V of Q, then F has a fixed point.

2. Existence of solutions

Let us denote by $J = [0, T]$ where $T = \min\{a, \frac{r}{M_r}\}$, $\mathcal{C}_\omega = C[J, E_\omega]$. We begin this section by using Theorem 2 to establish the existence solutions for the Volterra-Stieltjes integral equation (1) will be sought in $\mathcal{C}_\omega$. By a weak solution to (1) we mean a weakly continuous function x which satisfies the integral equation (1). This is equivalent to finding $\mathcal{C}_\omega$ with

$$\phi(x(t)) = \phi(p(t) + \int_0^t f(s, x(s))\ d_s g(t, s)),\ t \in I. \tag{7}$$

To facilitate our discussion, let us first state the following assumptions:

(i) $p : I \to E$ is weakly continuous function.

(ii) $f : I \times B_r \to E$ is weakly-weakly continuous.

(iii) $g : [0, T] \times [0, T] \to R$ such that

(iv) The functions $t \to g(t, t)$ and $t \to g(t, 0)$ are continuous on J.

(v) For all $t_1, t_2 \in J$ such that $t_1 < t_2$ the function $s \to g(t_2, s) - g(t_1, s)$ is nondecreasing on J.

(vi) $g(0, s) = 0$ for any $s \in J$.

Remark. Observe that Assumptions (v) and (vi) imply that the function $s \to g(t, s)$ is nondecreasing on the interval J, for any fixed $t \in J$ (Remark 1 in [8]). Indeed, putting $t_2 = t,\ t_1 = 0$ in (v) and keeping in mind (vi), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in J$, the function $s \to g(t, s)$ is of bounded variation on J.

Theorem 3. *Under the assumptions (i) − (vi), if*

$$\beta(f(J \times X)) \leq h(\beta(X)) \tag{8}$$

for each $X \subset B_r,\ J \subset I$, then there exists at least one weak solution $x(.) \in \mathcal{C}_\omega$.

Proof. We define the integral operator $F : \mathcal{C}_\omega \to \mathcal{C}_\omega$ associated to the integral equation (1) by

$$Fx(t) = p(t) + \int_0^t f(s, x(s))\, d_s g(t, s),\ t \in J. \tag{9}$$

According to the last lemma for every weakly continuous function x, $f(., x(.))$, is weakly continuous on J, means that $\phi(f(., x(.)))$
is continuous, for every $\phi \in E^*$, g is of bounded variation. Hence $f(., x(.))$ is weakly Riemann-Stieltjes integrable on J with respect to $s \to g(t, s)$. Thus F makes sense.

Now, define the closed, convex, bounded, equicontinuous subset of $Q \subset \mathcal{C}_\omega$ by

$$Q = \{x \in \mathcal{C}_\omega,\ \parallel x \parallel_0 \leq r \text{ and } x \text{ is norm continuous}\}.$$

For notational purposes $\parallel x \parallel_0 = \sup_{t \in J} \parallel x(t) \parallel$. We claim that $F : Q \to Q$ is weakly sequentially continuous and FQ is weakly relatively compact. Once the claim is established, then Theorem 2 with $\mathcal{C}_\omega$ guarantees a fixed point of F, and hence (1) has a solution in $\mathcal{C}_\omega$. First we show that FQ is an equicontinuous set.

Let $t_1,\ t_2 \in J,\ t_2 > t_1,\ x \in Q$, without loss of generality, assume that $Fx(t_2) - Fx(t_1) \neq 0$:

$$\| Fx(t_2) - Fx(t_1) \|$$

$$\begin{aligned}
\leq \ & | \phi(p(t_2) - p(t_1)) | \ + \ | \int_0^{t_2} \phi(f(s,x(s)))\ d_s g(t_2,s) \\
- \ & \int_0^{t_1} \phi(f(s,x(s)))\ d_s g(t_1,s) \,| \\
\leq \ & \| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s,x(s)))\ d_s g(t_2,s) \\
+ \ & \int_{t_1}^{t_2} \phi(f(s,x(s)))\ d_s g(t_2,s) - \int_0^{t_1} \phi(f(s,x(s)))\ d_s g(t_1,s) \,| \\
\leq \ & \| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s,x(s)))\ d_s [g(t_2,s) - g(t_1,s)] \,| \\
+ \ & | \int_{t_1}^{t_2} \phi(f(s,x(s)))\ d_s g(t_2,s) \,| \leq \| p(t_2) - p(t_1) \| \\
+ \ & \int_0^{t_1} | \phi(f(s,x(s))) |\ d_s [\bigvee_{z=0}^{s} (g(t_2,z) - g(t_1,z))] \\
+ \ & \int_{t_1}^{t_2} | \phi(f(s,x(s))) |\ \ d_s [\bigvee_{z=0}^{s} g(t_2,z)] \\
\leq \ & \| p(t_2) - p(t_1) \| + M [\int_0^{t_1} d_s [g(t_2,s) - g(t_1,s)] \\
+ \ & \int_{t_1}^{t_2} d_s g(t_2,s)] \\
\leq \ & \| p(t_2) - p(t_1) \| + M [g(t_2,t_1) - g(t_1,t_1) - [g(t_2,0) - g(t_1,0)] \\
+ \ & [g(t_2,t_2) - g(t_2,t_1)]].
\end{aligned}$$

Hence,

$$\| Fx(t_2) - Fx(t_1) \|$$

$$\begin{aligned}
\leq \ & \| p(t_2) - p(t_1) \| + M \{ | g(t_2,t_2) - g(t_1,t_1) | \\
+ \ & | g(t_2,0) - g(t_1,0) | \}.
\end{aligned}$$

Thus $Fx \in \mathcal{C}_\omega$ and the operator F is well defined.

Let $t \in J$. There exists $\phi \in E^*$ such that $\| \phi \|= 1$, and $\| Fx(t) \|= \phi(Fx)(t)$. Then, using the assumptions $(i)-(vi)$, we have

$$\| Fx(t) \|= \phi(Fx(t))$$

$$\begin{aligned}
&\leq \ | \phi(p(t)) | + | \phi(\int_0^t f(s,x(s))\ d_s g(t,s)) | \\
&\leq \ \| p \|_0 + \int_0^t | \phi(f(s,x(s))) |\ \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \ \| p \|_0 + M_r \int_0^t \ d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \ \| p \|_0 + M_r \int_0^t \ d_s g(t,s) \\
&\leq \ \| p \|_0 + M_r[g(t,t) - g(t,0)] \\
&\leq \ \| p \|_0 + M_r[| g(t,t) | + | g(t,0) |] \\
&\leq \ \| p \|_0 + M_r[\sup_{t\in I} | g(t,t) | + \sup_{t\in I} | g(t,0) |] \\
&\leq \ \| p \|_0 + M_r[k_1 + k_2] \leq r,
\end{aligned}$$

where $k_1 = \sup_{t\in J} | g(t,t) |;\ k_2 = \sup_{t\in J} | g(t,0) |$. Then

$$\| Fx \|_0 = \sup_{t\in J} \| Fx(t) \| \leq r. \tag{10}$$

From the above equation (10) hence, $Fx \in Q$ and $FQ \subset Q$ which prove that $F : Q \to Q$. Also $F : Q \to Q$ is weakly sequentially continuous. To see this, let $\{x_n(t)\}$ be sequence in Q weakly convergent to $x(t)$ in E, since Q is closed we have $x \in Q$. Fix $t \in J$, since f satisfies (ii), Thus $f(t, x_n(t))$ converges weakly to $f(t, x(t))$. By the Lebesgue dominated convergence theorem (see assumption (ii)) ([16]), we have for each $\phi \in E^*,\ s \in J$,

$$\phi(\int_0^t f(s,x_n(s))\ d_s g(t,s)) = \int_0^t \phi(f(s,x_n(s)))\ d_s g(t,s)$$

$$\to \int_0^t \phi(f(s,x(s)))\ d_s g(t,s),\ \forall \phi \in E^*,\ t \in J,$$

hence

$$| \phi(Fx_n(t) - Fx(t)) | \leq \int_0^t | \phi(f(s,x_n(s)) - f(s,x(s))) |\ \ d_s g(t,s)$$

$$\leq \epsilon. \tag{11}$$

Thus F restricted to Q is a weakly sequentially continuous.

Finally put

$$V(t) := \{x(t) : x(.) \in V\}$$

$$(FV)(t) := \{(Fx)(t) : x(.) \in V\}.$$

Suppose that $V \subset Q$ such that $\overline{V} \subset \overline{conv}(F(V) \cup \{0\})$. We will show that V is weakly relatively compact. Since Q is bounded and equicontinuous it follows that V is also bounded and equicontinuous. Put $v(t) = \beta(V(t))$ for $t \in J$. Obviously

$$V(t) \subset \overline{conv}(F(V)(t) \bigcup \{0\}),\ t \in J. \tag{12}$$

Using the properties of β, we have

$$v(t) \leq \beta(F(V)(t) \bigcup \{0\}) = \beta(F(V)(t)),\ t \in J.$$

As $V \subset Q$ is equicontinuous, by Theorem 1 the function $t \mapsto v(t)$ is continuous on J. Fix $t \in J$, divide the interval $[0, t]$ into n parts

$$0 = t_0 < t_1 < ... < t_n = t,\ t_i - t_{i-1} < \delta,\ i = 1, 2, 3, ..., n.$$

Put $T_i = [t_{i-1}, t_i]$. In view of Theorem 1 it follows that for each $i \in [1, 2, ..., n]$ there exists $\tau_i \in T_i$ such that

$$\beta(V(T_i)) = v(\tau_i), \quad i = 1, ..., n.$$

By the mean value theorem we have

$$\begin{aligned}
Fx(t) &= p(t) + \int_0^t f(s, x(s))\ d_s g(t, s) \\
&= p(t) + \Sigma_{i=1}^n \int_{T_i} f(s, x(s))\ d_s g(t, s) \\
&\in p(t) + \Sigma_{i=1}^n \mu(g(t, T_i))\ \overline{Conv}\{f(s, x(s)) :\ s \in T_i\} \\
&\in p(t) + \Sigma_{i=1}^n [g(t, t_i) - g(t, t_{i-1})]\ \overline{Conv} f(T_i \times V(T_i)) \\
&\subset p(t) + \Sigma_{i=1}^n [g(t, t_i) - g(t, t_{i-1})]\ \overline{Conv} f(J \times V(T_i)).
\end{aligned}$$

Hence

$$FV(t) \subset p(t) + \Sigma_{i=1}^n [g(t, t_i) - g(t, t_{i-1})]\ \overline{Conv} f(J \times V(T_i)),$$

by using the properties of the measure of weak noncompactness β we obtain

$$\begin{aligned}\beta(FV(t)) &\leq \Sigma_{i=1}^{n}[g(t,t_i)-g(t,t_{i-1})]\ \beta(f(J\times V(T_i)))\\ &\leq \Sigma_{i=1}^{n}[g(t,t_i)-g(t,t_{i-1})]\ h(\beta(V(T_i)))\\ &= \Sigma_{i=1}^{n}[g(t,t_i)-g(t,t_{i-1})]\ h(v(\tau_i)).\end{aligned}$$

Letting $n\to\infty$ we deduce that

$$\beta(FV(t)) \leq \int_0^t h(v(s))\ d_s g(t,s),\ t\in J.$$

Finally,

$$v(t) \leq \int_0^t h(v(s))\ d_s g(t,s),\ t\in J$$

the condition (A_2) implies that the integral inequality above has only trivial solution, i.e. $\beta(FV(t))=0,\ t\in J$. Thus, $V(t)$ is weakly relatively compact in E. Consequently, Ascoli's theorem proves that V is relatively compact in $\mathcal{C}_\omega$.

Since all conditions of Theorem 2 are satisfied, then the operator F has at least one fixed point $x\in Q$ and the nonlinear Stieltjes integral equation (1) has at least one weak solution $x\in\mathcal{C}_\omega$. □

Now, we show that the Volterra integral equation

$$x(t)=x_0+\int_0^t f(s,x(s))\ ds,\ t\in I \tag{13}$$

can be considered as a special case of the Volterra-Stieltjes integral equation (1), where the integral is in the sense of weakly Riemann. By consider that the functions $g(t,s)=s,\ p(t)=x_0$.

Finally, we can formulate the following existence result concerning the Volterra integral equation (13).

Corollary 4. *Under the assumption* (ii), *if*

$$\beta(f(J\times X))\leq h(\beta(X))$$

for each $X\subset B_r,\ J\subset I$, *then the Volterra integral equation (13) has at least one weak solution* $x(.)\in\mathcal{C}_\omega$.

3. Cauchy problem

As an application for the existence of weak solutions for Cauchy proplem

$$\frac{dx}{dt} = f(t, x(t)),\ x(0) = x_0,\ t \in (0, a] \tag{14}$$

in Banach spaces.

Since, a Riemann-Pettis integrable function is sometime called a weakly Riemann integrable function. Also every weakly continuous function from I into E is Riemann pettis integrable on I (see [1]). It is easy to see that every Riemann-Pettis integrable function is Pettis integrable (see [17]).

A function $x(.) \in \mathcal{C}_\omega$ is a weak solution of the Volterra integral equation (13) if and only if x is a solution of the Cauchy problem (14). By the equivalent between them.

Corollary 5. [13] *Under the assumption* (ii), *if*

$$\beta(f(J \times X)) \leq h(\beta(X))$$

for each $X \subset B_r,\ J \subset I$, *then the problem (14) has at least one weak solution* $x(.) \in \mathcal{C}_\omega$ *in the nonreflexive Banach space* E.

Corollary 6. [19] *If* E *is a reflexive Banach space, and* f *is weakly-weakly continuous, then there exists at least one weak solution* $x(.) \in \mathcal{C}_\omega$ *of (14) on* J.

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