

ON THE FAITHFULNESS OF JONES-WENZL
REPRESENTATION OF THE BRAID GROUP B_4

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Abstract: We consider the Jones-Wenzl representation $\rho_4^{(k,r)}$ of the braid group B_4 . For each pair of integers (k,r) with $r \geq k + 1$, we exclude a family of words in the generators from belonging to the kernel of $\rho_4^{(k,r)}$. Then, we prove that $\rho_4^{(k,r)}$ is not faithful for any even integer r .

AMS Subject Classification: 20F36

Key Words: braid group, Jones-Wenzl representation, faithfulness

1. Introduction

The linear representation of the braid group on n strings, namely the Burau representation, has been shown that it is not faithful for $n \geq 6$ [6]. Since then, irreducible representations of the braid group have been constructed to attack the question of the faithfulness of the braid group B_n or its normal subgroup P_n . An attempt was done on the Burau representation of the braid group on four strings [1].

In [4], H. Haidar and M. Abdulrahim constructed a representation of P_3 , the pure braid group on three strands. They studied its irreducibility and unitarity as a tool to study its faithfulness in the future.

We consider Jones-Wenzl representation of the braid group B_n . This new family of representations ρ of the braid groups were discovered by V. Jones in 1983. While many sought to bend Jones' theory toward classical topological

objectives, M. Freedman, M. Larsen, and Z. Wang have found that the relation between the Jones polynomial and physics allows potentially realistic models of quantum computation to be created. For more details (see [3]).

A feature of Jones representation is the two-eigen value property. That is, the image of each braid generator has only two distinct eigenvalues $\{-1, q\}$. This property was used in 2002, in [2], in the identification of the closed images of the irreducible components of Jones representation. Also, the closed images of the special unitarity groups $SU(N)$ were identified completely for the general case.

Theorem 1. *Fix an integer $r \geq 5$, $r \neq 6, 10$, $n \geq 3$ or $r=10$, $n \geq 5$. Let*

$$\rho_n^{(2,r)} = \bigoplus_{\lambda \in \Lambda_n^{(2,r)}} B_n \rightarrow \prod_{\lambda \in \Lambda_n^{(2,r)}} U(\lambda)$$

be the unitary Jones representation of the n -strand braid group B_n . Then the closed image $\rho_n^{(2,r)}(B_n)$ contains $\prod_{\lambda \in \Lambda_n^{(2,r)}} SU(\lambda)$.

The original motivation for studying Jones representation is for quantum computation. The special case $r = 5$ has already been used to show that the $SU(2)$ Witten-Chern-Simons modular functor at the fifth root of unity is universal for quantum computation. Combining that paper with the above result, it was concluded that the $SU(2)$ Witten-Chern-Simons modular functor at an r^{th} root of unity is universal for quantum computation if $r \neq 3, 4, 6$. For more details (see [2]).

The unitary Jones-Wenzl representation $\rho_n^{(k,r)}$ of the braid group B_n was defined for a pair of integers (k, r) with $r \geq k + 1$.

In our work, we consider Jones representation $\rho_n^{(k,r)}$ for $n = 4$. We determine the explicit form of the matrices of the generators of B_4 for $k = 3$ and $r = 8$. We then approach the question of the faithfulness of the unitary Jones representation of B_4 . Our first result is to exclude a family of words in the generators from belonging to the kernel of the considered representation $\rho_4^{(k,r)}$, (Theorem 2). Also, we prove that $\rho_4^{(k,r)}$ is not faithful for any even integer r , (Theorem 3).

2. Preliminaries

Definition 1. ([5]) The braid group on n strings, B_n , is the abstract group with presentation:

$$B_n = \left\{ \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, \dots, n - 2 \right\}.$$

Definition 2. Let $GL(V)$ be the general linear group of the vector space V over the field K . A representation ρ of a group G on V is said to be faithful if the group homomorphism $\rho : G \rightarrow GL(V)$ is injective.

Definition 3. ([2]) The Hecke algebra $H_n(q)$ of type A is the (finite dimensional) complex algebra generated by e_1, \dots, e_{n-1} such that

1. $e_i^2 = e_i$,
2. $e_i e_{i+1} e_i - [2]^{-2} e_i = e_{i+1} e_i e_{i+1} - [2]^{-2} e_{i+1}$,
3. $e_i e_j = e_j e_i$ if $|i - j| \geq 2$.

Here $[2] = q^{1/2} + q^{-1/2} = 2 \cos \frac{\pi}{r}$ ($r \in \mathbb{Z}$).

Definition 4. Let π be a representation of the Hecke algebra $H_n(q)$ generated by $e_i, i = 1, 2, \dots, n - 1$, on a Hilbert space. The representation π is called a \mathbb{C}^* representation if each $\pi(e_i)$ is self-adjoint.

3. Jones-Wenzl Representation

Hecke algebra representations of the braid groups in the root of unity case are indexed by two parameters: a compact Lie group and an integer $l \geq 1$, called the level of the theory. The cases of Jones and Wenzl representations correspond to the special unitary groups $SU(k), k \geq 2$. For each pair of integers (k, r) with $r \geq k + 1$, there is a unitary representation of the braid groups with level $l = r - k$.

To describe the Jones-Wenzl representation, we let $q = e^{\pm \frac{2\pi i}{r}}$ and $[m]$ be the quantum integer $\frac{q^{m/2} - q^{-m/2}}{q^{1/2} - q^{-1/2}}$.

Lemma 1. Each \mathbb{C}^* representation of the Hecke algebra $H_n(q)$ gives rise to a unitary representation of the braid group B_n by the formula:

$$\rho(\sigma_i) = q - (1 + q)\pi(e_i).$$

Jones-Wenzl \mathbb{C}^* representation of $H_n(q)$ are reducible. Their irreducible constituents are indexed by Young diagrams. A Young diagram with n boxes is a diagram of a partition of n such that $\lambda = [\lambda_1, \dots, \lambda_k]$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$. $\sum_{i=1}^k \lambda_i = n$. Let λ be a Young diagram with n boxes. A standard tableau of shape λ is an assignment of integers $\{1, 2, \dots, n\}$ into the boxes so that the entries of each row and column are increasing.

Definition 5. ([2]) Suppose t is a standard tableau with n boxes, and m_1 and m_2 are two entries in t . Suppose m_i appears in row r_i and column c_i of t .

1. Set $d_{t,m_1,m_2} = (c_1 - c_2) - (r_1 - r_2)$.
2. Set $\alpha_{t,i} = \frac{[d_{t,i,i+1}+1]}{[2][d_{t,i,i+1}]}$ if $[d_{t,i,i+1}] \neq 0$ and $\beta_{t,i} = \sqrt{\alpha_{t,i}(1 - \alpha_{t,i})}$.
3. A Young diagram $\lambda = [\lambda_1, \dots, \lambda_k]$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$ is (k,r) -admissible if $\lambda_1 - \lambda_k \leq r - k$.
4. Suppose t is a standard tableau of shape λ with n boxes, let $t^{(i)}$ ($1 \leq i \leq n$) be the standard tableaux obtained from t by deleting boxes with entries $n, n-1, \dots, n-i+1$. A standard tableau t is (k,r) -admissible if the shape of each tableau $t^{(i)}$ is a (k,r) -admissible Young diagram.

The irreducible sectors of the Jones-Wenzl representations of the Hecke algebras $H_n(q)$ are indexed by the pair (k,r) and a (k,r) -admissible Young diagram λ with n boxes. Note that λ is allowed to have empty rows.

To construct a \mathbb{C}^* representation $\pi_\lambda^{(k,r)}$ of the Hecke algebra $H_n(q)$, the (k,r) -admissible standard tableaux t of shape λ are considered. By interchanging the entries i and $i+1$ in t , the tableau $s_i(t)$ is obtained.

If $s_i(t)$ is also (k,r) -admissible standard tableau, then

$$\pi_\lambda^{(k,r)}(e_i)(v_t) = \alpha_{t,i}v_t + \beta_{t,i}v_{s_i(t)},$$

where $\{v_t\}$ is the basis of the complex vector space $V_\lambda^{(k,r)}$.

If $s_i(t)$ is not (k,r) -admissible standard tableau, set $\beta_{t,i} = 0$. In this case, $\alpha_{t,i}$ is either 0 or 1.

Consequently, $\pi_\lambda^{(k,r)}(e_i)$ is a matrix consisting of only 2×2 blocks

$$\begin{pmatrix} \alpha_{t,i} & \beta_{t,i} \\ \beta_{t,i} & 1 - \alpha_{t,i} \end{pmatrix}$$

and 1×1 blocks 0 or 1.

The restriction of $\pi_\lambda^{(k,r)}$ to B_n is denoted by $\rho_\lambda^{(k,r)}$. Direct calculations show that $\rho_\lambda^{(k,r)}(\sigma_i)$ is a matrix consisting of only 2×2 blocks

$$\rho_\lambda(\sigma_i) = \begin{pmatrix} q - (1 + q)\alpha_{t,i} & -\beta_{t,i} \\ -\beta_{t,i} & q - (1 + q)(1 - \alpha_{t,i}) \end{pmatrix}$$

and 1×1 blocks q or -1 .

Definition 6. ([2]) Given a pair of integers (k, r) with $r \geq k + 1$. Let $\Lambda_n^{(k,r)}$ be the set of all (k, r) -admissible Young diagrams with n boxes. The Jones-Wenzl representation of the braid group B_n is:

$$\rho_n^{(k,r)} = \bigoplus_{\lambda \in \Lambda_n^{(k,r)}} \rho_\lambda^{(k,r)} : B_n \rightarrow \prod_{\lambda \in \Lambda_n^{(k,r)}} U(\lambda).$$

Here we write $U(\lambda)$ for the unitary group of the Hilbert space $V_\lambda^{(k,r)}$ with the orthonormal basis $\{v_t\}$.

4. Jones-Wenzl Representation for $k=3$ and $r=8$

Using the explicit description of Jones-Wenzl representation in Section 3, we write explicitly the 9×9 matrices of the generators of B_4 for $k = 3$ and $r = 8$. The representation $\rho_4^{(3,8)}$ is defined as follows:

$$\rho_4^{(3,8)}(\sigma_1) = \left(\begin{array}{ccc|ccc|c} -1 & 0 & 0 & & & & \\ 0 & q & 0 & & 0 & & 0 \\ 0 & 0 & q & & & & 0 \\ \hline & & & -1 & 0 & 0 & \\ 0 & & & 0 & -1 & 0 & 0 \\ & & & 0 & 0 & q & \\ \hline & & & & & & -1 & 0 \\ 0 & & & & & & 0 & q \\ \hline & & & & & & & 0 \\ 0 & & & & 0 & & & q \end{array} \right),$$

$$\rho_4^{(3,8)}(\sigma_2) = \left(\begin{array}{cc|cc|cc} A & 0 & & & & \\ 0 & q & & & & \\ \hline & & -1 & 0 & & \\ 0 & & 0 & A & & \\ \hline & & & & A & 0 \\ \hline 0 & & & & 0 & q \end{array} \right),$$

and

$$\rho_4^{(3,8)}(\sigma_3) = \left(\begin{array}{cc|cc|cc} q & 0 & & & & \\ 0 & B & & & & \\ \hline & & B & 0 & & \\ 0 & & 0 & -1 & & \\ \hline & & & & -1 & 0 \\ 0 & & 0 & & 0 & q \\ \hline 0 & & 0 & & 0 & q \end{array} \right).$$

Here the matrices A and B are given by

$$A = \begin{pmatrix} q - \frac{1+q}{2+\sqrt{2}} & -(1+q)\sqrt{\frac{1}{\sqrt{2}}(1-\frac{1}{\sqrt{2}})} \\ -(1+q)\sqrt{\frac{1}{\sqrt{2}}(1-\frac{1}{\sqrt{2}})} & q - \frac{1+q}{\sqrt{2}} \end{pmatrix}$$

and

$$B = \begin{pmatrix} q - (1+q)\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}} & -(1+q)\sqrt{\frac{2}{2+\sqrt{2}}(1-\frac{2}{2+\sqrt{2}})} \\ -(1+q)\sqrt{\frac{2}{2+\sqrt{2}}(1-\frac{2}{2+\sqrt{2}})} & q - \frac{2(1+q)}{2+\sqrt{2}} \end{pmatrix}.$$

5. Faithfulness of $\rho_4^{(k,r)} : B_4 \rightarrow GL_9(\mathbb{C}^*)$

For each partition of $n = 4$, we get a matrix whose size depends on the number of the standard tableaux corresponding to the considered partition. The direct sum of the obtained matrices is the 9×9 matrix which is the image of the generator σ_i ($i = 1, 2, 3$).

We get two results concerning Jones-Wenzl representation $B_4 \rightarrow GL_9(\mathbb{C}^*)$. The first one is Theorem 2, which excludes a family of words from belonging to the kernel of the representation. The other result is Theorem 3, which finds a condition under which Jones-Wenzl representation is not faithful.

Lemma 2. *For any (k, r) -admissible Young diagram of shape $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k] = [4, 0, \dots, 0]$, $\rho_\lambda^{(k,r)}(\sigma_i) = q$ where $q = e^{\pm \frac{2\pi i}{r}}$ ($i = 1, 2, 3$).*

Proof. We assign the integers $\{1, 2, 3, 4\}$ into the boxes of the considered Young diagram of shape $\lambda = [4, 0, \dots, 0]$ so that the entries in the row are increasing.

Let t be the obtained standard tableau of shape $\lambda = [4, 0, \dots, 0]$. Then, $t^{(i)}$ is the standard tableau obtained from t by deleting boxes with entries $4, 3, \dots, 4 - i + 1$, where $i = 1, 2, 3$.

Since $r - k \geq 4$, it follows that $\lambda_1 - \lambda_k \leq r - k$. Thus, each tableau $t^{(i)}$ is a (k, r) -admissible Young diagram. This implies that the standard tableau t is a (k, r) -admissible standard tableau of shape $\lambda = [4, 0, \dots, 0]$.

Let $s_i(t)$ be the tableau obtained from t by interchanging the entries i and $i + 1$. Clearly, the entries $\{1, 2, 3, 4\}$ in $s_i(t)$ are not increasing. Therefore, $s_i(t)$ are not standard tableaux for $i = 1, 2, 3$ and so they are not (k, r) -admissible.

Consequently, we set $\beta_{t,i} = 0$ in $\pi_\lambda^{(k,r)}(e_i)(v_t) = \alpha_{t,i}v_t + \beta_{t,i}v_{s_i(t)}$, where $\{v_t\}$ is the basis of the complex vector space $V_\lambda^{(k,r)}$.

We have $\alpha_{t,i} = \frac{[d_{t,i,i+1}+1]}{[2][d_{t,i,i+1}]}$ if $[d_{t,i,i+1}] \neq 0$. Here $d_{t,m_1,m_2} = (c_1 - c_2) - (r_1 - r_2)$ where m_i is the entry that appears in row r_i and column c_i of the standard tableau t . By simple computations, we get $d_{t,i,i+1} = -1$ for $i = 1, 2, 3$. This implies that $\alpha_{t,i} = 0$. Thus, $\pi_\lambda^{(k,r)}(e_i) = 0$.

By substituting $\pi_\lambda^{(k,r)}(e_i)$ in $\rho_\lambda^{(k,r)}(\sigma_i) = q - (1 + q)\pi_\lambda^{(k,r)}(e_i)$, we conclude that $\rho_\lambda^{(k,r)}(\sigma_i) = q$. □

Theorem 2. *Given a word u of the form $\sigma_{\alpha_1}^{\beta_1} \dots \sigma_{\alpha_n}^{\beta_n}$, where $\alpha_i = 1, 2, 3$, $\beta_i \in \mathbb{Z}$. Let β_a and β_b be the sums of the exponents of (σ_{α_i}) 's and $(\sigma_{\alpha_i}^{-1})$'s respectively. If $\beta_a - \beta_b \neq rw$, $w \in \mathbb{Z}$, then the words u don't belong to the kernel of Jones-Wenzl representation $\rho_4^{(k,r)}$ for all pairs of integers (k, r) .*

Proof. We consider the (k, r) -admissible Young diagram of shape $\lambda = [4, 0, \dots, 0]$. Assume, for contradiction, that the words of the form $\sigma_{\alpha_1}^{\beta_1} \dots \sigma_{\alpha_n}^{\beta_n}$ belong to the kernel of $\rho_4^{(k,r)}$.

By Lemma 2, $\rho_\lambda^{(k,r)}(\sigma_i)$ is a 1×1 block matrix (q) for $i = 1, 2, 3$, and consequently, $\rho_\lambda^{(k,r)}(\sigma_i^{-1})$ is a 1×1 block matrix ($1/q$) for $i = 1, 2, 3$.

Thus, $\rho_4^{(k,r)}(\sigma_{\alpha_1}^{\beta_1} \dots \sigma_{\alpha_n}^{\beta_n})$ has a 1×1 block matrix of entry $= \frac{q^{\beta_a}}{q^{\beta_b}} = q^{\beta_a - \beta_b}$ on its diagonal where β_a and β_b are the sums of the exponents of (σ_i) 's and (σ_i^{-1}) 's respectively.

We have $q^{\beta_a - \beta_b} = (e^{\pm \frac{2\pi i}{r}})^{\beta_a - \beta_b} = 1$. Thus, $\beta_a - \beta_b = rw$, $w \in \mathbb{Z}$, a contradiction. □

We approach the question of the faithfulness of Jones-Wenzl representation $\rho_4^{(k,r)}$ of the braid group B_4 . First, we prove the following lemma:

Lemma 3. *For $n = 4$ and for each (k, r) -admissible standard tableau t , the tableaux $s_1(t)$ are not (k, r) -admissible standard tableaux.*

Proof. Let λ be a Young diagram of a partition of $n = 4$ such that $\lambda = [\lambda_1, \dots, \lambda_k]$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0$. $\sum_{i=1}^k \lambda_i = 4$. We consider all the possible cases.

1. For $\lambda = [3, 1, 0, \dots, 0]$, we have 3 standard tableaux t of shape λ in which the entries 1 and 2 are either in the same row or in the same column.
2. For $\lambda = [2, 2, 0, \dots, 0]$, we have 2 standard tableaux t of shape λ in which the entries 1 and 2 are either in the same row or in the same column.
3. For $\lambda = [2, 1, 1, 0, \dots, 0]$, we have 3 standard tableaux t of shape λ in which the entries 1 and 2 are either in the same row or in the same column.
4. For $\lambda = [4, 0, \dots, 0]$, we have 1 standard tableau t of shape λ in which the entries 1 and 2 are in the same row.
5. For $\lambda = [1, 1, 1, 1, 0, \dots, 0]$, we have 1 standard tableau t of shape λ in which the entries 1 and 2 are in the same column.

In all the above cases, we consider the (k, r) -admissible standard tableaux t of shape λ . By interchanging the entries 1 and 2 in each t , we obtain the tableau $s_1(t)$. Clearly, the entries in $s_1(t)$ are not in increasing order. Consequently, $s_1(t)$ are not standard tableaux. \square

Using Lemma 3, we get the following theorem:

Theorem 3. *For all integers k and for each even integer r , Jones-Wenzl representation $\rho_4^{(k,r)}$ of the braid group B_4 is not faithful.*

Proof. By Lemma 3, and for all (k, r) -admissible standard tableaux t , $s_1(t)$ are not (k, r) -admissible standard tableaux. Consequently, we set $\beta_{t,1} = 0$ in $\pi_\lambda^{(k,r)}(e_1)(v_t) = \alpha_{t,1}v_t + \beta_{t,1}v_{s_1(t)}$, where $\{v_t\}$ is the basis of the complex vector space $V_\lambda^{(k,r)}$.

We have $\alpha_{t,1} = \frac{[d_{t,1,2}+1]}{[2][d_{t,1,2}]}$ if $[d_{t,1,2}] \neq 0$. Here $d_{t,1,2} = (c_1 - c_2) - (r_1 - r_2)$, where 1 appears in row r_1 and column c_1 , 2 appears in row r_2 and column c_2 of each standard tableau t .

By simple computations, we get $d_{t,1,2} = -1$ or 1 . This implies that $\alpha_{t,1} = 0$ or 1 .

By substituting $\pi_\lambda^{(k,r)}(e_1)$ in $\rho_\lambda^{(k,r)}(\sigma_1) = q - (1+q)\pi_\lambda^{(k,r)}(e_1)$, we conclude that $\rho_\lambda^{(k,r)}(\sigma_1)$ is either q or -1 . Therefore, $\rho_4^{(k,r)}(\sigma_1)$ is a diagonal matrix having entries equal q or -1 .

If r is an even integer, then $\rho_4^{(k,r)}(\sigma_1^r)$ is an identity matrix. This implies that for any even integer r , the word having the form σ_1^r belongs to the kernel of Jones-Wenzl representation $\rho_4^{(k,r)}$ of the braid group B_4 . Consequently, the considered representation is not faithful. \square

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