

CONDITIONAL SOLVABILITY OF THE BOUNDARY VALUE
PROBLEM OF A SELF-ADJOINT OPERATOR-DIFFERENTIAL
EQUATIONS IN A SOBOLEV-TYPE SPACE

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Abstract: In this paper, in the space of Sobolev type $W_2^5(R; H)$ obtained the sufficient conditions of regular solvability of initial-boundary value problem of fifth order operator-differential equations with complicated characteristics on the real axis, these conditions depend only on the operator coefficients of the considered equation. The exact values of norms of the intermediate derivatives operators of the essential part of the investigated equation are obtained.

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1. Introduction

The theory of initial-boundary value problems of operator- differential equations in Banach or Hilbert space is useful because it offers the possibility of looking at ordinary as well as partial differential operators (see [1]). It should be noted that the essential part of the investigated equation (1.1) has complicated characteristics, thus, in this paper, the investigated equations are of interest, for instance, they arise in modeling the dynamics problems of arches and rings (see [2]). The solvability of initial boundary value problems for higher

order operator differential equations has been researched by many authors. For example, Aydin A. Gasymov, Araz R. Aliev, V.I. Gorbachuk, M.L. Gorbachuk, Yakubov S. Ya, V.N. Pilipchuk and their followers.

We shall study the following initial-boundary value Problem In a separable Hilbert space H

$$\prod_{k=1}^5 \left(\frac{d}{dx} - \mu_k A \right) u(x) + \sum_{j=1}^4 A_j \frac{d^{5-j}}{dx^{5-j}} u(x) = f(x),$$

$$x \in R = (-\infty, +\infty), \tag{1.1}$$

$$\frac{d^s u(0)}{dx^s} = 0, \quad s = 0, 1, 2, 3, \tag{1.2}$$

where, A is a self-adjoint positively defined operator; $\mu_1 = 1, \mu_2 = \mu_3 = \mu_4 = \mu_5 = -1$ and $A_j; j = 1, 2, 3, 4$ are linear unbounded operators. From now on, the derivatives are accepted according to the distributions theory, [1]. We define the following subspaces.

We consider $f(x) \in L_2(R; H)$, and $u(x) \in W_2^5(R; H)$, where

$$L_2(R; H) = \left\{ f(x) : \|f(x)\|_{L_2(R;H)} = \left(\int_{-\infty}^{+\infty} \|f(x)\|_H^2 dx \right)^{\frac{1}{2}} < +\infty \right\},$$

$$W_2^5(R; H) = \left\{ u(t) : \frac{d^5 u(x)}{dx^5} \in L_2(R; H), A^5 u(x) \in L_2(R; H) \right\},$$

with the norm (see [1]-[5])

$$\|u\|_{W_2^5(R;H)} = \left(\left\| \frac{d^5 u}{dx^5} \right\|_{L_2(R;H)}^2 + \|A^5 u\|_{L_2(R;H)}^2 \right)^{\frac{1}{2}}.$$

Definition 1.1. If for any $f(x) \in L_2(R; H)$ there exists a vector function $u(x) \in W_2^5(R; H)$ that satisfies (1.1) almost everywhere in R , then it is known as a regular solution of (1.1).

Definition 1.2. If for any function $f(x) \in L_2(R; H)$ there exists a regular solution $u(x) \in W_2^5(R; H)$ of (1.1) satisfying the initial boundary conditions (1.2) in the sense that

$$\lim_{x \rightarrow 0} \left\| A^{\frac{9}{2}-i} \frac{d^i u(x)}{dx^i} \right\|_H = 0, \quad i = 0, 1, 2, 3,$$

and the inequality

$$\|u\|_{W_2^5(R;H)} \leq \text{const} \|f\|_{L_2(R;H)},$$

holds, then problem (1.1), (1.2) will be regularly solvable.

2. Main results

From the theorem of intermediate derivatives (see [6], [7]) if $u(x) \in W_2^5(R;H)$, then $A^{5-j} \frac{d^j u(x)}{dx^j} \in L_2(R;H)$, $j = \overline{1,4}$, and the following inequalities:

$$\left\| A^{5-j} \frac{d^j u(x)}{dx^j} \right\|_{L_2(R;H)} \leq c_j \|u\|_{W_2^5(R;H)}, \quad j = 1, 2, 3, 4, \tag{2.1}$$

hold.

Equation (1.1) has the following operator form:

$$Qu(x) \equiv Q_0 u(x) + Q_1 u(x) = f(x), \text{ where}$$

$$Q_0 = \prod_{k=1}^5 \left(\frac{d}{dx} - \mu_k A \right) u(x) \text{ and } Q_1 = \sum_{j=1}^4 A_s \frac{d^{5-j}}{dx^{5-j}}.$$

The following theorem provides the association between the norms of operators of intermediate derivatives and the solvability conditions of the problem (1.1), (1.2).

Theorem 2.1. *The operator Q_0 isomorphically maps the space $W_2^5(R;H)$ onto the space $L_2(R;H)$, moreover, for $f(x) \in L_2(R;H)$ and equation (1.1) has a solution*

$$u(x) = \int_{-\infty}^{+\infty} G(x-s) f(s) ds + u_0(x),$$

where

$$G(x-s) = \frac{1}{16} \left\{ \begin{array}{l} e^{A(x-s)} A^{-4}, \text{ if } x-s > 0, \\ \left(E + 2A(x-s) + 2A^2(x-s)^2 + \frac{4}{3}A^3(x-s)^3 \right) \\ \times e^{-A(x-s)} A^{-4}, \text{ if } x-s < 0, \end{array} \right\}$$

$$u_0(x) = \frac{-1}{16} \left\{ \begin{array}{l} (E + 2Ax + 2A^2x^2 + \frac{4}{3}A^3x^3) \\ \times \int_{-\infty}^0 e^{-A(x+s)} A^{-4} f(s) ds \\ + (E - 2As + 2A^2s^2 - \frac{4}{3}A^3s^3) \\ \times \int_0^{+\infty} e^{-A(x-s)} A^{-4} f(s) ds \\ + 2Ax (E - 2As + 2A^2s^2) \int_0^{+\infty} e^{-A(x-s)} A^{-4} f(s) ds \\ + 2A^2x^2 (E - 2As) \int_0^{+\infty} e^{-A(x-s)} A^{-4} f(s) ds \\ + \frac{4}{3}A^3x^3 \int_0^{+\infty} e^{-A(x-s)} A^{-4} f(s) ds. \end{array} \right.$$

Proof. First, we find Green’s function of equation (1.1) using Cauchy integral where, $A_j = 0, j = 1, 2, 3, 4$, and from inequality (2.1), it is simple to prove that Q_0 which acts from $W_2^5(R; H)$ to $L_2(R; H)$ is bounded (see [8]). When applying Fourier transform to the equation $Q_0u(x) = f(x)$, we obtain

$$(i\xi E - A)(i\xi E + A)^4 \tilde{u}(\xi) = \tilde{f}(\xi),$$

where E -the identity operator and $\tilde{u}(\xi), \tilde{f}(\xi)$ are the Fourier transforms of the functions $u(x), f(x)$, respectively.

Thus the operator pencil $(i\xi E - A)(i\xi E + A)^4$ is invertible and moreover,

$$\tilde{u}(\xi) = (i\xi E - A)^{-1} (i\xi E + A)^{-4} \tilde{f}(\xi). \tag{2.2}$$

Hence,

$$u(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\xi E - A)^{-1} (i\xi E + A)^{-4} \tilde{f}(\xi) e^{i\xi t} d\xi.$$

Now we show that $u(x) \in W_2^5(R; H)$.

Using the Parseval equality and (2.2), we obtain

$$\begin{aligned} \|u\|_{W_2^5(R;H)}^2 &= \left\| \frac{d^5 u}{dx^5} \right\|_{L_2(R;H)}^2 + \|A^5 u\|_{L_2(R;H)}^2 \\ &= \left\| i\xi^5 u(\xi) \right\|_{L_2(R;H)}^2 + \left\| A^5 u(\xi) \right\|_{L_2(R;H)}^2 \\ &= \left\| i\xi^5 (i\xi E - A)^{-1} (i\xi E + A)^{-4} \tilde{f}(\xi) \right\|_{L_2(R;H)}^2 \\ &+ \left\| A^5 (i\xi E - A)^{-1} (i\xi E + A)^{-4} \tilde{f}(\xi) \right\|_{L_2(R;H)}^2 \end{aligned}$$

$$\begin{aligned} &\leq \sup_{\zeta \in R} \left\| i\zeta^5 (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \right\|_{H \rightarrow H}^2 \left\| \tilde{f}(\zeta) \right\|_{L_2(R;H)}^2 \\ &+ \sup_{\zeta \in R} \left\| A^5 (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \right\|_{H \rightarrow H}^2 \left\| \tilde{f}(\zeta) \right\|_{L_2(R;H)}^2. \end{aligned} \tag{2.3}$$

From the spectral decomposition of the operator A ($\sigma(A)$ —the spectrum of operator A) for $\zeta \in R$ we have

$$\begin{aligned} &\left\| i\zeta^5 (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \right\|_{H \rightarrow H} \\ &= \sup_{\sigma \in \sigma(A)} \left| i\zeta^5 (i\zeta - \sigma)^{-1} (i\zeta + \sigma)^{-4} \right| \\ &\leq \sup_{\sigma \in \sigma(A)} \frac{|\zeta|^5}{(\zeta^2 + \sigma^2)^{\frac{5}{2}}} \leq 1, \end{aligned} \tag{2.4}$$

$$\begin{aligned} &\left\| A^5 (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \right\|_{H \rightarrow H} \\ &= \sup_{\sigma \in \sigma(A)} \left| \sigma^5 (i\zeta E - \sigma)^{-1} (i\zeta E + \sigma)^{-4} \right| \\ &\leq \sup_{\sigma \in \sigma(A)} \frac{\sigma^5}{(\zeta^2 + \sigma^2)^{\frac{5}{2}}} \leq 1. \end{aligned} \tag{2.5}$$

From (2.4) and (2.5) into (2.3) we obtain:

$$\|u\|_{W_2^5(R;H)}^2 \leq 2 \left\| \tilde{f}(\zeta) \right\|_{L_2(R;H)}^2 = 2 \|f(x)\|_{L_2(R;H)}^2.$$

Hence, $u(x) \in W_2^5(R;H)$.

Using the Banach theorem of the inverse operator, then the operator Q_0 is an isomorphism from $W_2^5(R;H)$ to $L_2(R;H)$. □

We formulate exact conditions on regular solvability of problem (1.1), (1.2). Expressed only by its operator coefficients, we must estimate the norms of intermediate derivative operators participating in the perturbed part of the given equation. It follows from Theorem 2.1. that the norm $\|Q_0 u\|_{L_2(R;H)}$ is equivalent to the norm $\|u\|_{W_2^5(R;H)}$ in the space $W_2^5(R;H)$. Therefore by the norm $\|Q_0 u\|_{L_2(R;H)}$, the theorem on intermediate derivatives is valid as well.

Theorem 2.2. *When the function $u(x) \in W_2^5(R; H)$, then it keeps the following inequalities:*

$$\left\| A^{5-j} \frac{d^j u(x)}{dx^j} \right\|_{L_2(R;H)} \leq b_j \|Q_0 u\|_{L_2(R;H)}, \quad j = \overline{1, 4}, \tag{2.6}$$

true, where $b_1 = b_4 = \frac{16}{25\sqrt{5}}$, $b_2 = b_3 = \frac{6\sqrt{3}}{25\sqrt{5}}$ (see [9] – [11]).

Proof. To establish the validity of inequalities (2.6) we take $Q_0 u(x) = f(x)$ and apply the Fourier transformation as follow

$$\begin{aligned} & \left\| A^{5-j} (i\zeta)^j (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \tilde{f}(\zeta) \right\|_{L_2(R;H)} \\ & \leq \sup_{\zeta \in R} \left\| A^{5-j} (i\zeta)^j (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \right\|_{H \rightarrow H} \left\| \tilde{f}(\zeta) \right\|_{L_2(R;H)}, \\ & \qquad \qquad \qquad j = 1, 2, 3, 4. \end{aligned} \tag{2.7}$$

For $\zeta \in R$ we have:

$$\begin{aligned} & \left\| A^{5-j} (i\zeta)^j (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \right\|_{H \rightarrow H} \\ & \leq \sup_{\sigma \in \sigma(A)} \left| \sigma^{5-j} (i\zeta)^j (i\zeta E - \sigma)^{-1} (i\zeta E + \sigma)^{-4} \right| \\ & = \sup_{\sigma \in \sigma(A)} \left| \sigma^{-j} (i\zeta)^j \left(i\frac{\zeta}{\sigma} - 1 \right)^{-1} \left(i\frac{\zeta}{\sigma} + 1 \right)^{-4} \right| \\ & \leq \sup_{\eta = \frac{\zeta^2}{\sigma^2} \geq 0} \frac{\eta^{\binom{j}{2}}}{(\eta + 1)^{\frac{5}{2}}} = \frac{1}{25\sqrt{5}} j^{\binom{j}{2}} (5 - j)^{\binom{5-j}{2}} = b_j, \\ & \qquad \qquad \qquad j = 1, 2, 3, 4. \end{aligned}$$

Using inequalities (2.7), we have

$$\begin{aligned} & \left\| A^{5-j} (i\zeta)^j (i\zeta E - A)^{-1} (i\zeta E + A)^{-4} \tilde{f}(\zeta) \right\|_{L_2(R;H)} \\ & \leq b_j \left\| \tilde{f}(\xi) \right\|_{L_2(R;H)}, \quad j = 1, 2, 3, 4. \end{aligned}$$

□

Lemma. *The operator Q_1 continuously acts from $W_2^5(R; H)$ to $L_2(R; H)$ provided that the operators $A_j A^{-j}$, $j = 1, 2, 3, 4$ are bounded in H .*

Considering the results found till now (see [12]), we get the possibility to establish regular solvability conditions of the problem (1.1), (1.2).

Theorem 2.3. *Let $|k| < 2\lambda_0$ ($A = A^* \geq \lambda_0 E$, $\lambda_0 > 0$) for any $u(t) \in W_2^5(R; H)$, then holds the inequality (see [13])*

$$\alpha(k) \left(c_1(k) \|A_1 A^{-1}\|_{H \rightarrow H} + c_2(k) \|A_2 A^{-2}\|_{H \rightarrow H} + c_3(k) \|A_3 A^{-3}\|_{H \rightarrow H} + c_4(k) \|A_4 A^{-4}\|_{H \rightarrow H} \right) < 1,$$

where

$$c_1(k) = \left[1 + \frac{4\lambda_0|\lambda_0+k|}{(2\lambda_0+k)^2} \right]^{\frac{3}{2}}, \quad c_2(k) = \frac{2\lambda_0}{2\lambda_0+k} \left[1 + \frac{4\lambda_0|\lambda_0+k|}{(2\lambda_0+k)^2} \right],$$

$$c_3(k) = \frac{4\lambda_0^2}{(2\lambda_0+k)^2} \left[1 + \frac{4\lambda_0|\lambda_0+k|}{(2\lambda_0+k)^2} \right]^{\frac{1}{2}}, \quad c_4(k) = \frac{8\lambda_0^3}{(2\lambda_0+k)^3},$$

and

$$\alpha(k) = \begin{cases} \frac{\lambda_0}{2^{\frac{1}{2}}(2\lambda_0^2-k^2)^{\frac{1}{2}}}, & \text{if } 0 \leq \frac{k^2}{4\lambda_0^2} < \frac{1}{3}, \\ \frac{2\lambda_0|k|}{4\lambda_0^2-k^2}, & \text{if } \frac{1}{3} \leq \frac{k^2}{4\lambda_0^2} < 1. \end{cases}$$

Theorem 2.4. *Suppose that the operators $A_j A^{-j}$, $j = \overline{1, 4}$, are bounded in H and they satisfy the inequality*

$$\sum_{j=1}^4 C_j(k) \alpha(k) \|A_{5-j} A^{-(5-j)}\|_{H \rightarrow H} < 1,$$

where the numbers $C_j(k)$, $j = 1, 2, 3, 4$, and $\alpha(k)$ are determined in Theorem 2.3. Then the problem (1.1), (1.2) is regularly solvable.

Proof. $f(x) \in L_2(R; H)$, $u(x) \in W_2^5(R; H)$ and by Theorem (1.1), there exist a bounded inverse operator to Q_0 , which acts from $L_2(R; H)$ to $W_2^5(R; H)$, then after replacing $Q_0 u(x) = w(x)$ in equation (1.1), it can be written as $(E + Q_1 Q_0^{-1})w(x) = f(x)$.

Now we prove under the theorem conditions (see [13], [14]) that

$$\|Q_1 Q_0^{-1}\|_{L_2(R;H) \rightarrow L_2(R;H)} < 1.$$

By Theorem (2.3) we have:

$$\begin{aligned} \|Q_1 Q_0^{-1} w\|_{L_2(R;H)} &= \|Q_1 u\|_{L_2(R;H)} \leq \sum_{j=1}^4 \left\| A_j \frac{d^{5-j} u}{dx^{5-j}} \right\|_{L_2(R;H)} \\ &\leq \sum_{j=1}^4 \|A_j A^{-j}\|_{H \rightarrow H} \left\| A_j \frac{d^{5-j} u}{dx^{5-j}} \right\|_{L_2(R;H)} \\ &\leq \sum_{j=1}^4 C_j(k) \alpha(k) \|A_j A^{-j}\|_{H \rightarrow H} \|Q_0 u\|_{L_2(R;H)} \\ &= \sum_{j=1}^4 C_j(k) \alpha(k) \|A_j A^{-j}\|_{H \rightarrow H} \|v\|_{L_2(R;H)}. \end{aligned}$$

Consequently,

$$\|Q_1 Q_0^{-1}\|_{L_2(R;H) \rightarrow L_2(R;H)} \leq \sum_{j=1}^4 C_j(k) \alpha(k) \|A_j A^{-j}\|_{H \rightarrow H} \prec 1.$$

Thus, the operator $E + Q_1 Q_0^{-1}$ is invertible in $L_2(R;H)$, therefore $u(x)$ can be determined by $u(x) = Q_0^{-1} (E + Q_1 Q_0^{-1})^{-1} f(x)$, and moreover

$$\begin{aligned} \|u\|_{W_2^5(R;H)} &\leq \|Q_0^{-1}\|_{L_2(R;H) \rightarrow W_2^5(R;H)} \\ &\times \left\| \left((E + Q_1 Q_0^{-1}) \right)^{-1} \right\|_{L_2(R;H) \rightarrow L_2(R;H)} \|f\|_{L_2(R;H)} \\ &\leq \text{const} \|f\|_{L_2(R;H)}. \end{aligned}$$

□

3. Conclusion

In the real axis, for the fifth order self-adjoint differential operator with complicated characteristics, we demonstrated the association between the coefficients of the differential operator and conditions of the regular solvability of problem (1.1)-(1.2). We estimated the norms of intermediate derivative operators which appear in the essential part of the investigated equation. The norms of the linear operators ($A_j, j = 1, 2, 3, 4$) participating in the second part estimated and used to formulate the exact solvability conditions.

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