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# THE ESTIMATION OF SOLUTIONS NONDIVERGENT ELLIPTIC-PARABOLIC EQUATIONS

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**Abstract:** The estimations of solutions of nondivergent elliptic-parabolic equations are obtained. We consider strong solution of the problem and prove the boundedness of the solution. Also we show that the solution of the problem belongs to Holder classes.

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**Key Words:** degenerate, elliptic-parabolic equations

#### 1. Introduction

The degenerate elliptic-parabolic equations arise as mathematical models of various applied problems of mechanics, for instance in reaction drift diffusion processes of electrically charged species phase transition processes and transport processes in porous media. Investigations of boundary value problems for second order degenerate elliptic-parabolic equations ascend to the work by Keldysh [1], where correct statements for boundary value problems were considered for the case of one space variable as well as existence and uniqueness of solutions. In the work by Fichera [2] boundary value problems were given for multidimentional case. He proved existence of generalized solutions to these boundary value problems.

Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  and  $Q_T = \Omega \times (0,T), T > 0$  be a cylinder. We consider the following initial boundary value problem

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$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \left( x, t \right) \frac{\partial u}{x_j} \right) - \psi \left( x, t \right) \frac{\partial^2 u}{\partial t^2}$$

$$+\sum_{i=1}^{n} b_i(x,t) \frac{\partial u}{\partial x_i} + c(t,x) u = 0, \quad (x,t) \in Q_T,$$

$$\tag{1}$$

$$u|_{\Gamma(Q_T)} = 0, (2)$$

where  $\Gamma(Q_T) = (\partial \Omega \times [0, T]) \cup \{(x, t) : x \in \Omega, t = 0\}$  is parabolic boundary of  $Q_T$ .

The equation (1) is degenerate elliptic. Also the function  $\psi(x,t)$  and coefficient  $a_{ij}(x,t)$  can tend to zero. Initial boundary problems for degenerate parabolic equations have been studied by many authors (see for example [3, 4, 5, 6]). But the structure of the equation (1) is different from that one considered in these papers. Boundary value problems for the degenerate equation also were studied in the stationary case in [7]. Similar result for this type of equation in the case of coefficients satisfying the Cordes condition is obtained in adjiev, Gasimova [6].

We consider problem (1)-(2) under standard conditions for the functions  $a_{ij}(x,t)$  and some conditions for the function  $b_i(t,x)$ , c(x,t) are considered. Let  $\partial\Omega\subset C^2$ .

Let the coefficients from (1)-(2) satisfy the following assumptions:  $||a_{ij}(x,t)||$  a real symmetrical matrix and for any  $(x,t) \in Q_T$  and  $\xi \in \mathbb{R}^n$  the following inequality is true

$$\gamma \omega(x) |\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x,t) \, \xi_i \xi_j \le \gamma^{-1} \omega(x) |\xi|^2,$$
 (3)

where  $\gamma \in (0,1]$   $a_{ij}(x,t)$ , c(x,t)  $b_i(x,t)$ ,  $i,j=\overline{1,n}$  are measurable functions with respect to t,x for every  $(t,x) \in Q_T$ . Also,

$$c(x,t) \le 0, \ c(x,t) \in L_{n+1}(Q_T),$$
 (4)

$$|b_i(x,t)| \in L_{n+2}(Q_T), |b_i(x,t)|^2 + Kc(x,t) \le 0.$$
 (5)

Assume that the following conditions are true for the weighted functions

$$\psi(x,t) = \omega(x) \cdot \lambda(p) \cdot \varphi(T-t),$$

where  $\omega(x) \in A_p$  satisfies the Muckenhoupt condition (see [8])

$$\lambda(\rho) \ge 0, \ \lambda(\rho) \in C^1[0, diam\Omega],$$

$$|\lambda'(\rho)| \le k\sqrt{\lambda(\rho)}, \text{ where } \rho = dist(x, \partial\Omega),$$

$$\varphi(z) \ge 0, \ \varphi'(z) \ge 0, \ \varphi(z) \in C^1[0, T]$$

$$\varphi(z) \ge \beta \cdot z \cdot \varphi'(z), \ \varphi(0) = \varphi'(0) = 0,$$
(6)

where  $\rho, \beta$  are positive constants.

We consider the problem (1)-(2) with data such that

$$f\left(x,t\right)\in L_{\infty}\left(Q_{T}\right)\cap L_{\infty}\left(0,T;W_{2}^{1}\left(\Omega\right)\right)\cap L_{1}\left(0,T;W_{\infty}^{1}\left(\Omega\right)\right)$$

$$\frac{\partial f}{\partial t} \in L_1(0, T; L_\infty(\Omega)), \tag{7}$$

$$h\left(x\right) \in L_{\infty}\left(\Omega\right),\tag{8}$$

$$u(x,t) \in L^2(0,T; W_{2,\psi}^{1,2}(Q_T)).$$

We introduce some space of functions in  $Q_T$  with finite norm

$$||u||_{W_{2,\psi(Q_T)}^{2,2}} = \left( \left( \int_{Q_T} \omega(x) \left( u^2 + \sum_{i=1}^n u_{x_i}^2 + \sum_{i,j=1}^n u_{x_i x_j}^2 + u_t^2 \right) \right)$$

$$+\psi^{2}(x,t)u_{tt}^{2} + \psi(x,t)\sum_{i=1}^{n}u_{x_{i}t}^{2}dxdt$$

 $W_{2,\psi}^{0,1,2}(Q_T)$ -subspace of space  $W_{2,\psi}^{1,2}(Q_T)$  is closure to all functions from  $C^{\infty}(\bar{Q}_T)$ , vanishing to zero on  $\Gamma(Q_T)$ .

We consider a strong solution  $u(x,t) \in L^2\left(0,T;W^{1,2}_{2,\Psi}(Q_T)\right)$  of the problem (1)-(2) for almost every  $\tau \in (0,T)$ 

$$u(x,t) - f(x,t) \in L^{2}\left(0, T; W_{2,\omega}^{0}\right).$$

We consider the case  $\psi(z) > 0$  at z > 0. If  $\psi(z) \equiv 0$ , then the equation (1) is parabolic.

We understand the solution of the auxiliary problem (1)-(2) with weight  $\omega_{\varepsilon}(x)$ ,  $\psi_{\varepsilon}(x,t)$  in the sense of defined solution after replacing  $\omega(x)$  and  $\psi(x,t)$  by  $\omega_{\varepsilon}(x)$ ,  $\psi_{\varepsilon}(x,t)$ , where  $\omega_{\varepsilon}(x)$  and  $\psi_{\varepsilon}(x,t)$  regularize the functions.

### 2. Main results

**Theorem 1.** Let the conditions (3)-(8) be satisfied. Then there exists a constant  $M_1$  depending only on the known parameters and independent of  $\varepsilon \in (0,1]$  such that each solution u of the problem (1)-(2) with weight  $\omega_{\varepsilon}(x), \psi_{\varepsilon}(x,t)$  satisfies

$$ess \sup_{t \in (0,T)} \int_{\Omega} \left\{ \Lambda_{1} \left( u \left( x,t \right) \right) + \Lambda_{2} \left( u \left( x,t \right) \right) \right\} dx$$

$$+ \int_{Q_T} \omega_{\varepsilon}(x) \left| \frac{\partial u}{\partial x} \right|^2 dx dt + \int_{Q_T} \psi_{\varepsilon}(x) \left| \frac{\partial^2 u}{\partial t^2} \right|^2 dx dt \le M_1, \tag{9}$$

where

$$\Lambda_{1}(u) = \int_{0}^{u} s \cdot \omega(s) ds, \quad \Lambda_{2}(u) = \int_{0}^{u} s \cdot \psi(x, s) ds, \tag{10}$$

a.e.  $t \in (0,T)$ .

*Proof.* Let u(t,x) be the solution to the regularized problem (1)-(2). We extend the function u(t,x) by setting  $u(t,x) = \varphi(x)$  for  $t < 0, x \in \Omega$ . Denote

$$\bar{u}(t,x) = u(t,x) - f(t,x).$$

Immediately from the definition of  $\Lambda_1(u(x,t))$ ,  $\Lambda_2(u(x,t))$ , we deduce

$$u < \varepsilon \left( \Lambda_1 \left( u \right) + \Lambda_2 \left( u \right) \right) + C_{\varepsilon_1} \tag{11}$$

for  $u \geq 0$  with arbitrary positive number  $\varepsilon$  and a constant  $c_{\varepsilon}$  depending only on  $\varepsilon_1$  and the functions  $\omega(x)$ ,  $\psi(x,t)$ . Using the conditions (3)-(5), (7),(8) and the conditions on  $\omega(x)$ ,  $\psi(x,t)$  and the inequality (11), we obtain for arbitrary positive number  $\varepsilon_1$  and some functions  $\mu(t) \in L_1(0,T)$ 

$$\left| \int_{0}^{\tau} \int_{\Omega} \omega_{\varepsilon}(x) \left| \frac{\partial u}{\partial x} \right|^{2} \frac{\partial f(x,t)}{\partial x_{j}} dx dt \right|$$

$$+ \left| \int_{0}^{\tau} \int_{\Omega} \psi_{\varepsilon} \left( x, t \right) \left| \frac{\partial^{2} u}{\partial t^{2}} \right|^{2} \left| \frac{\partial f}{\partial x_{j}} \right| dx dt \right|$$

$$\leq \varepsilon_{1} \int_{0}^{\tau} \int_{\Omega} \omega_{\varepsilon}(x) \left| \frac{\partial u}{\partial x} \right|^{2} dx dt + \varepsilon_{1} \int_{0}^{\tau} \int_{\Omega} \psi_{\varepsilon}(x, t) \left| \frac{\partial^{2} u}{\partial t^{2}} \right|^{2} dx dt + \frac{C}{\varepsilon_{1}} \int_{0}^{\tau} \int_{\Omega} \left( \Lambda_{1}(u) + \Lambda_{2}(u) \right) \mu(t) dx dt, \qquad (12)$$

$$\int_{0}^{\tau} \int_{\Omega} u \frac{\partial f}{\partial t} dx dt \leq C \left\{ 1 + \int_{0}^{\tau} \int_{\Omega} \left( \Lambda_{1}(u) + \Lambda_{2}(u) \right) \mu(t) dx dt \right\},$$

$$\int_{\Omega} u(x, \tau) f(x, \tau) dx \leq C_{2} \left( \varepsilon_{1} \int_{\Omega} \left( \Lambda_{1}(u(x, \tau)) + \Lambda_{2}(u(x, \tau)) \right) dx \right) + C_{\varepsilon_{1}}.$$

We estimate the terms involving the function  $\alpha$  in standard way by using (3)-(5), (7),(8). Now from (12) and the evidently estimations for the other terms, we obtain

$$\int_{\Omega} \left( \Lambda_{1} \left( u \left( x, \tau \right) \right) + \Lambda_{2} \left( u \left( x, \tau \right) \right) \right) dx$$

$$+ \int_{0}^{\tau} \left[ \int_{\Omega} \omega_{\varepsilon} \left( x \right) \left| \frac{\partial u}{\partial x} \right|^{2} + \psi_{\varepsilon} \left( x, t \right) \left| \frac{\partial^{2} u}{\partial t^{2}} \right|^{2} \right] dx dt$$

$$\leq C \left( 1 + \int_{0}^{\tau} \int_{\Omega} \left[ 1 + \mu(t) \right] \left( \Lambda_{1}(u) + \Lambda_{2}(u) \right) dx dt \right). \tag{13}$$

Now the last inequality and Gronwall's lemma complete the proof of Theorem 1.  $\Box$ 

**Theorem 2.** Let the assumptions of Theorem 1 be satisfied. Then the estimates

$$||u||x,t||_{L^{\infty}(Q_{T})} \leq M_{2},$$

$$|u(t,x')\omega(x') - u(t,x'')\omega(x'')| \leq M_{3}|x'-x''|^{\eta}$$
(14)

hold for arbitrary  $t \in [0, \tau]$ , x',  $x'' \in \Omega$  with  $\eta \in (0, 1)$  and constants  $M_2$ ,  $M_3$ , depending only on the known parameters and independent of  $\varepsilon$ .

**Theorem 3.** Let the conditions (3)-(5), (6)-(8) and growth condition be satisfied. Then there exists a constant  $M_4$ , depending only on known parameters and independent of  $\varepsilon \in \left[0, \frac{1}{M_4}\right]$ , such that each solution of the problem (1)-(2) satisfies

$$ess \sup \{|u(x,t)| : (x,t) \in Q_T\} \le M_4. \tag{15}$$

**Theorem 4.** Let the conditions of Theorem 3 be satisfied. Then the initial-boundary value problem (1)-(2) has at least one strong solution.

**Theorem 5.** Let the conditions of Theorem 3 be satisfied and assume additionally that the functions  $a_{ij}(x,t)$ ,  $b_i(x,t)$ , c(x,t) are locally Lipschitzian with respect to x. Then the initial-boundary value problem (1)-(2) has a unique solution.

Proof.

For proving the uniqueness of the solution for the problem (1)-(2) we assume that there exists two solutions  $u_1, u_2$ . By Theorems 1,2, we have for j = 1, 2

$$\|u_j\|_{L_{\infty}(Q_T)} + \left\|\frac{\partial u_j}{\partial x}\right\|_{L_{2,\omega}(Q_T)} + \left\|\frac{\partial^2 u_j}{\partial t^2}\right\|_{L_{2,\psi}(Q_T)}^2 \le M \tag{16}$$

with some constant M.

The proof of the theorem will proceed in four steps corresponding to four different choices of the test functions.

Applying Cauchy's inequality to the term involving the derivative of  $u_1$  and choosing a suitable value of R, we obtain

$$\int_{\Omega} |u_1(\tau, x) - u_2(\tau, x)|^2 dx$$

$$+ \int_{Q_T} \left( \omega(x) \left| \frac{\partial (u_1 - u_2)}{\partial x} \right|^2 + \psi(x, t) \left| \frac{\partial^2 (u_1 - u_2)}{\partial t^2} \right|^2 \right) dx dt$$

$$\leq C \int_{Q_T} (1 + |\alpha| + |f|) |u_1 - u_2|^2 dx dt. \tag{17}$$

We estimate the integral on the right hand site of (17) by Holder's inequality and use condition on  $\alpha$ , to get

$$ess \sup_{\tau \in (0,\theta)} |u_1(\tau, x) - u_2(\tau, x)|^2 dx$$

$$+ \int_{Q_{\theta}} \left( \omega(x) \left| \frac{\partial (u_1 - u_2)}{\partial x} \right|^2 + \psi(x, t) \left| \frac{\partial^2 (u_1 - u_2)}{\partial t^2} \right|^2 \right) dx dt$$

$$\leq C \left\{ \int |u_1 - u_2|^{2p_1'} dx dt \right\}^{\frac{1}{p_1'}} + C \int_{\theta}^{\theta} \left\{ \int |u_1 - u_2|^{2p_2'} dx \right\}^{\frac{1}{p_2'}} dt \qquad (18)$$

for an arbitrary  $\theta \in (0,T)$ . Estimating the first integral on the right hand site of (18) by Holder's inequality, using the embedding  $L_{\frac{2(n+2)}{n}}(Q_T) \subset L_2(Q_T)$  (comp. with [5]) and setting  $q_1 = n + 2 - p_1'n$ , we find for arbitrary  $\varepsilon \in (0,1)$ :

$$\int_{0}^{\theta} \left\{ \int_{\Omega} |u_{1} - u_{2}|^{2p_{2}'} dx \right\}^{\frac{1}{p_{1}'}} dt \leq C \left\{ \varepsilon^{-\frac{2p_{1}'}{\gamma}} \int_{Q_{\theta}} |u_{1} - u_{2}|^{2} dx dt \right.$$

$$+ \varepsilon^{\frac{2p_{1}'}{2p_{1}' - \gamma}} ess \sup_{\tau \in (0, \theta)} \int_{\Omega} |u_{1} - u_{2}|^{2} dx$$

$$+ \int_{\Omega} \left( \omega(x) \left| \frac{\partial (u_{1} - u_{2})}{\partial x} \right|^{2} + \psi(x, t) \left| \frac{\partial^{2} (u_{1} - u_{2})}{\partial t^{2}} \right|^{2} \right) dx dt \right\}. \tag{19}$$

The inequalities (18),(19) imply with suitable  $\varepsilon$ 

$$\int_{\Omega} |u_1(\theta, x) - u_2(\theta, x)|^2 dx \le C \int_{\Omega_2} |u_1 - u_2|^2 dx dt$$
 (20)

for arbitrary  $\theta \in (0,\tau)$ . Finally, Gronwall's lemma yields  $u_1 = u_2$ .

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