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SOME RESULTS ON UNIVALENT HOLOMORPHIC FUNCTIONS BASED ON q-ANALOGUE OF NOOR OPERATOR

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Abstract: The main object of this paper is to define a new sublclass of univalent holomorphic functions along with the recently defined q–analogue of Noor operator. We obtained a number of useful properties such as: coefficient bounds, extreme points, radii of starlikeness, convexity and close-to-convexity and weighted mean.

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1. Preliminaries

Let $\mathcal{A}$ be the class of all functions $f(z)$ which are analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and have the following Taylor series representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Let us denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions with negative coefficients of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geqslant 0). \tag{2}$$

For functions f and g which are analytic in $\mathbb{U}$ and have the form (2), we define

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the convolution (or Hadamard product) of f and g by:

$$(f * g)(z) = z - \sum_{k=2}^{\infty} a_k b_k z^k, \qquad (z \in \mathbb{U}). \tag{3}$$

Now, we provide some notations regarding the q–calculus used in this article, see [1, 3] and [4].

For $0 < q < 1$, the q–derivative of f is defined by:

$$D_q f(z) = \frac{f(zq) - f(z)}{z(q-1)} \qquad (z \neq 0). \tag{4}$$

We can easily conclude that:

$$D_q\Big(\sum_{k=2}^{\infty} a_k z^k\Big) = \sum_{k=2}^{\infty} [k,q] a_z z^{k-1} \quad (k \in \mathbb{N}, \quad z \in \mathbb{U}), \tag{5}$$

where

$$[k,q] = \frac{1-q^k}{1-q} = 1 + \sum_{t=1}^{k-1} q^t \qquad ([0,q] = 0), \tag{6}$$

and

$$[k,q] = \begin{cases} 1 & , \quad k = 0, \\ [1,q][2,q]\cdots[k,q] & , \quad k \in \mathbb{N}. \end{cases} \tag{7}$$

Also, the q–generalization of the Pochhammer symbol for $y > 0$ is defined by:

$$[y,q]_k = \begin{cases} [y,q][y+1,q]\cdots[y+k-1,q] & , \quad k \in \mathbb{N}, \\ 1 & , \quad k = 0. \end{cases} \tag{8}$$

For $\mu > -1$ and $f(z) \in \mathcal{T}$, we consider the q–analogue of Noor integral operator as follows:

$$\mathcal{N}_q^{\mu} f(z) = \mathcal{T}_{q,\mu+1}^{-1}(z) * f(z) = z - \sum_{k=2}^{\infty} \Psi_{k-1} a_k z^k \quad (z \in \mathbb{U}), \tag{9}$$

where

$$\mathcal{T}_{q,\mu+1}^{-1}(z) * \mathcal{T}_{q,\mu+1}(z) = z D_q f(z), \tag{10}$$

$$\mathcal{T}_{q,\mu+1}(z) = z - \sum_{k=2}^{\infty} \frac{[\mu+1,q]_{k-1}}{[k,q]!} z^k, \tag{11}$$

and

$$\Psi_{k-1} = \frac{[k,q]!}{[\mu+1,q]_{k-1}}, \tag{12}$$

see [2].

It is clear that $\mathcal{N}_q^0 f(z) = zD_q f(z)$, $\mathcal{N}_q^1 f(z) = f(z)$ and

$$\lim_{q\to 1^-} \mathcal{N}_q^\mu f(z) = z - \sum_{k=2}^{\infty} \frac{k!}{(\mu+1)_{k-1}} a_k z^k, \tag{13}$$

which is the familiar Noor integral operator, see [5] and [6].

For $0 \leqslant \alpha \leqslant 1$ and $0 \leqslant \beta < 1$, the function $f(z) \in \mathcal{T}$ is in the class $\mathcal{N}_q^\mu(\alpha,\beta)$ if it satisfies:

$$\operatorname{Re}\left\{ \frac{zD_q\big(\mathcal{N}_q^\mu(\alpha,\beta)\big) + \alpha z^2 D_q^2\big(\mathcal{N}_q^\mu f(z)\big)}{\alpha z D_q\big(\mathcal{N}_q^\mu f(z)\big) + (1-\alpha)\mathcal{N}_q^\mu f(z)} \right\} > \beta, \tag{14}$$

where D_q and $\mathcal{N}_q^\mu$ are defined in (4) and (9) respectively. Also $D_q^2\big(\mathcal{N}_q^\mu f(z)\big)$ means $D_q\big[D_q\big(\mathcal{N}_q^\mu f(z)\big)\big]$.

2. Main results

In this section, we obtain coefficient bounds for functions in the class $\mathcal{N}_q^\mu(\alpha,\beta)$ and show that this class is a convex set.

Theorem 1. *$f(z) \in \mathcal{T}$ is in the class $\mathcal{N}_q^\mu(\alpha,\beta)$ if and only if:*

$$\sum_{k=2}^{\infty} \Psi_{k-1}\Big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big) + \beta(1-\alpha)\Big)\alpha_k \leqslant 1-\beta, \tag{15}$$

where Ψ_{k-1} and $[k,q]$ are given by (12) *and* (6)*, respectively.*

Proof. By making use of (4) and (5), we obtain:

$$D_q\big(\mathcal{N}_q^\mu(\alpha,\beta)\big) = 1 - \sum_{k=2}^{\infty} [k,q]\Psi_{k-1} a_k z^{k-1}, \tag{16}$$

$$D_q^2\big(\mathcal{N}_q^{\mu} f(z)\big) = -\sum_{k=2}^{\infty} [k,q]^2 \Psi_{k-1} a_k z^{k-2}, \tag{17}$$

where $[k,q]$ and Ψ_{k-1} are defined in (6) and (12), respectively.

By replacing (16) and (17) in (14) we have:

$$\operatorname{Re}\left\{ \frac{z - \sum\limits_{k=2}^{\infty} [k,q]\Psi_{k-1} a_k z^k - \sum\limits_{k=2}^{\infty} \alpha [k,q]^2 \Psi_{k-1} a_k z^k}{A} \right\} > \beta$$

where

$$\begin{aligned} A = {} & \alpha z - \sum_{k=2}^{\infty} \alpha [k,q] \Psi_{k-1} a_k z^k \\ & + (1-\alpha) z - \sum_{k=2}^{\infty} (1-\alpha) \Psi_{k-1} a_k z^k, \end{aligned}$$

or

$$\operatorname{Re}\left\{ \frac{z - \sum\limits_{k=2}^{\infty} [k,q]\Psi_{k-1}\big(1 + \alpha[k,q]\big) a_k z^k}{z - \sum\limits_{k=2}^{\infty} \Psi_{k-1}\big(\alpha([k,q]-1)+1\big) a_k z^k} \right\} > \beta.$$

By choosing the values of z on the real axis and then letting $z \to 1^-$ through real values, we get:

$$\begin{aligned} 1 - \beta - \sum_{k=2}^{\infty} & \Big[[k,q]\Psi_{k-1}\big(1+\alpha[k,q]\big) \\ & - \beta \Psi_{k-1}\big(\alpha([k,q]-1)+1\big) \Big] a_k \geqslant 0, \end{aligned}$$

or

$$\sum_{k=2}^{\infty} \Psi_{k-1} \Big[[k,q]\big(1 + \alpha[k,q] - \alpha\beta\big) - \beta(1-\alpha) \Big] a_k \leqslant 1 - \beta.$$

Conversely, suppose that (15) holds true. We will show that (14) is satisfies and so $f \in \mathcal{N}_q^{\mu}(\alpha,\beta)$. Using the fact that $\operatorname{Re}\{W\} > \beta$ if and only if $|W - (1-\beta)| < |W - (1-\beta)|$, it is enough to show that:

$$L = \left| \frac{z D_q\big(\mathcal{N}_q^{\mu} f(z)\big) + \alpha z^2 D_q^2\big(\mathcal{N}_q^{\mu} f(z)\big)}{\alpha z D_q\big(\mathcal{N}_q^{\mu} f(z)\big) + (1-\alpha)\mathcal{N}_q^{\mu} f(z)} - 1 - \beta \right|$$

$$< \left| \frac{zD_q\big(\mathcal{N}_q^\mu f(z)\big) + \alpha z^2 D_q^2\big(\mathcal{N}_q^\mu f(z)\big)}{\alpha z D_q\big(\mathcal{N}_q^\mu f(z)\big) + (1-\alpha)\mathcal{N}_q^\mu f(z)} + 1 - \beta \right| = R.$$

But, if $\alpha z D_q\big(\mathcal{N}_q^\mu f(z)\big) + (1-\alpha)\mathcal{N}_q^\mu f(z) = J$, then we have:

$$L = \frac{1}{|J|}\Big[zD_q\big(\mathcal{N}_q^\mu f(z)\big) + \alpha z^2 D_q^2\big(\mathcal{N}_q^\mu f(z)\big) - (1+\beta)J\Big].$$

By (16) and (17) we get:

$$\begin{aligned} L &= \frac{1}{|J|}\left[\beta z - \sum_{k=2}^{\infty} \Psi_{k-1}\Big[[k,q]\big(1+\alpha[k,q]+(1-\beta)\big) \right. \\ &\quad \left. + (1-\beta)(1-\alpha)\Big] a_k z^k \right] \\ &< \frac{|z|}{|J|}\left[\beta + \sum_{k=2}^{\infty} \Psi_{k-1}\Big[[k,q]\big(1+\alpha[k,q]+(1-\beta)\big) \right. \\ &\quad \left. + (1-\beta)(1-\alpha) a_k |z|^{k-1} \right], \end{aligned}$$

and

$$\begin{aligned} R &= \frac{1}{|J|}\left|zD_q\big(\mathcal{N}_q^\mu f(z)\big) + \alpha z^2 D_q^2\big(\mathcal{N}_q^\mu f(z)\big) + (1-\beta)J\right| \\ &= \frac{1}{|J|}\left|(2-\beta)z - \sum_{k=2}^{\infty} \Psi_{k-1}\Big([k,q]\big(1+\alpha[k,q]+(1-\beta)\big) \right. \\ &\quad \left. + (1-\beta)(1-\alpha)\Big) a_k z^k \right| \\ &\geqslant \frac{|z|}{|J|}\left[(2-\beta) - \sum_{k=2}^{\infty} \Psi_{k-1}\Big([k,q]\big(1+\alpha[k,q]+(1-\beta)\big) \right. \\ &\quad \left. + (1-\beta)(1-\alpha)\Big) a_k \, |z|^{k-1} \right]. \end{aligned}$$

when $z \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, it is easy to verify that $R - L > 0$, if (15) holds and so the proof is complete. □

Remark. The result (15) is sharp for the function $F(z)$ given by:

$$F(z) = z - \frac{1-\beta}{\Psi_1\Big([2,q][k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\Big)}z^2, \tag{18}$$

where $\Psi_1 = \frac{[2,q]!}{[\mu+1,q]_1}$ and $[2,q] = \frac{1-q^2}{1-q} = 1+q$.

Corollary 2. *If $f(z) \in \mathcal{N}_q^{\mu}(\alpha,\beta)$, then for $k = 1, 2, \ldots$, we have:*

$$a_k \leqslant \frac{1-\beta}{\Psi_{k-1}\Big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\Big)}. \tag{19}$$

Theorem 3. *$\mathcal{N}_q^{\mu}(\alpha,\beta)$ is a convex set.*

Proof. We must show that, if the functions $f_t(z)$, $t = 1, 2, \ldots, m$, be in the class $\mathcal{N}_q^{\mu}(\alpha,\beta)$, then the function $h(z) = \sum_{t=1}^{m}\lambda_t f_t(z)$ for λ_t and $\sum_{t=1}^{m}\lambda_t = 1$, is also in $\mathcal{N}_q^{\mu}(\alpha,\beta)$.

By definition of $h(z)$, we conclude:

$$\begin{aligned} h(z) &= \sum_{t=1}^{m}\lambda_t\Big(z - \sum_{k=2}^{\infty}a_{k,t}z^k\Big) \\ &= z - \sum_{k=2}^{\infty}\Big(\sum_{t=1}^{m}\lambda_t a_{k,t}\Big)z^k. \end{aligned}$$

But from Theorem 1, we have:

$$\begin{aligned} &\sum_{k=2}^{\infty}\Psi_{k-1}\Big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\Big)\Big(\sum_{t=1}^{m}\lambda_t a_{k,t}\Big) \\ &= \sum_{t=1}^{m}\lambda_t\Big\{\sum_{k=2}^{\infty}\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)a_{k,t}\Big\} \\ &\leqslant \sum_{t=1}^{m}\lambda_t(1-\beta) = 1-\beta, \end{aligned}$$

which completes the proof. $\square$

3. Extereme points and some properties of $\mathcal{N}_q^{\mu}(\alpha,\beta)$

In the last section, we obtain extreme points of $\mathcal{N}_q^{\mu}(\alpha,\beta)$ and investigate some properties of the some class.

Theorem 4. *Let $f_1(z)=z$ and*

$$f_k(z)=z-\frac{(1-\beta)z^k}{\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)},$$

where $k=2,3,\ldots$. Then $f\in\mathcal{N}_q^{\mu}(\alpha,\beta)$ if and only if it can be expressed in the form $f(z)=\sum_{k=1}^{\infty}t_kf_k(z)$, where $t_k\geqslant 0$ and $\sum_{k=1}^{\infty}t_k=1$. In particular, the extreme points of $\mathcal{N}_q^{\mu}(\alpha,\beta)$ are functions $f_1(z)$ and $f_k(z)$, where $k=2,3,\ldots$.

Proof. Let f be expressed as in the above. This means that we can write:

$$\begin{aligned}f(z)&=\sum_{k=1}^{\infty}t_kf_k(z)=t_1f_1(z)+\sum_{k=2}^{\infty}t_kf_k(z)\\&=t_1z+\sum_{k=2}^{\infty}t_kz\\&-\sum_{k=2}^{\infty}\frac{(1-\beta)t_k}{\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)}z^k\\&=z\sum_{k=1}^{\infty}t_k-\sum_{k=2}^{\infty}d_kz^k,\end{aligned}$$

where

$$d=\frac{(1-\beta)t_k}{\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)}.$$

Therefor $f\in\mathcal{N}_q^{\mu}(\alpha,\beta)$ since by Theorem 1, we have:

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)}{1-\beta}d_k\\&=\sum_{k=2}^{\infty}t_k=1-t_1<1.\end{aligned}$$

Conversely, suppose that $f\in\mathcal{N}_q^{\mu}(\alpha,\beta)$. Theny by (19), for $k=2,3,\ldots$, we have:

$$a_k\leqslant\frac{1-\beta}{\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)}.$$

By putting

$$t_k = \frac{\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)}{1-\beta}, \quad (k \geqslant 2),$$

we have $t_k \geqslant 0$ and if $t_1 = 1-\sum_{k=2}^{\infty} t_k$, we get the required result. So the proof is complete. □

Theorem 5. *Let the function $f(z)$ by (2) be in the class $\mathcal{N}_q^{\mu}(\alpha,\beta)$, then:*

1. *$f(z)$ is starlike of order δ_1 for $0 \leqslant \delta_1 < 1$ in $|z| < R_1$,*

$$R_1 = \inf_k \left[\frac{B}{(k-\sigma_1)(1-\beta)}\right]^{\frac{1}{k-1}}, \tag{20}$$

where

$$B = (1-\delta_1)\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big).$$

2. *$f(z)$ is convex of order δ_2 for $0 \leqslant \delta_2 < 1$ in $|z| < R_2$, where:*

$$R_2 = \inf_k \left[\frac{C}{k(k-2\delta_2)(1-\beta)}\right]^{\frac{1}{k-1}}, \tag{21}$$

where

$$C = (1-\delta_2)\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big).$$

3. *$f(z)$ is close-to-convex of order δ_3 for $0 \leqslant \delta_3 < 1$ in $|z| < R_3$, where:*

$$R_3 = \inf_k \left[\frac{D}{k(1-\beta)}\right]^{\frac{1}{k-1}}, \tag{22}$$

where

$$D = (1-\delta_3)\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big).$$

Proof. To establish the required result, it is sufficient to prove that:

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant 1-\delta_1, \qquad (|z| \leqslant R_1).$$

But

$$\left|\frac{zf'(z)}{f(z)}-1\right| = \left|\frac{z-\sum\limits_{k=2}^{\infty} ka_k z^k}{z-\sum\limits_{k=2}^{\infty} a_k z^k}-1\right|$$

$$= \left|\frac{-\sum\limits_{k=2}^{\infty}(k-1)a_k z^k}{z-\sum\limits_{k=2}^{\infty} a_k z^k}\right| \leqslant \frac{\sum\limits_{k=2}^{\infty}(k-1)a_k\,|z|^{k-1}}{1-\sum\limits_{k=2}^{\infty} a_k\,|z|^{k-1}} \leqslant 1-\delta_1.$$

Thus $\sum_{k=2}^{\infty}\left(\frac{k-\delta_1}{1-\delta_0}\right)a_k\,|z|^{k-1} \leqslant 1$.

Since $f(z)\in\mathcal{N}_q^{\mu}(\alpha,\beta)$, the last inequality holds, if:

$$|z|^{k-1} \leqslant \left[\frac{D}{(k-\delta_1)(1-\beta)}\right],$$

where

$$D = (1-\delta_1)\Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big).$$

□

In the last theorem, we investigate the weighted mean concept.

Theorem 6. *If f and g belong to $\mathcal{N}_q^{\mu}(\alpha,\beta)$, then the weighted mean of f and g is also in the some class.*

Proof. We have to prove that $h_t(z) = \left[\frac{(1-t)f(z)+(1+t)g(z)}{2}\right]$ is in the class $\mathcal{N}_q^{\mu}(\alpha,\beta)$.

Since $f(z) = z-\sum_{k=2}^{\infty}a_k z^k$ and $g(z) = \sum_{k=2}^{\infty}b_k z^k$, so:

$$h_t(z) = z-\sum_{k=2}^{\infty}\left\{\frac{(1-t)a_k+(1+t)b_k}{2}\right\}z^k.$$

To prove $h_t(z)\in\mathcal{N}_q^{\mu}(\alpha,\beta)$, by (15) we need to show that:

$$\sum_{k=2}^{\infty}\frac{E}{2(1-\beta)} < 1,$$

where

$$E = \Psi_{k-1}\big([k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)\big)$$

$$\times \left[(1-t)a_k + (1-t)b_k\right].$$

For this, we have:

$$\begin{aligned}
F &= \sum_{k=2}^{\infty} \frac{E}{2(1-\beta)} \\
&= \frac{(1-t)}{2} \sum_{k=2}^{\infty} \frac{\Psi_{k-1}[k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)}{1-\beta} a_k \\
&+ \frac{(1+t)}{2} \sum_{k=2}^{\infty} \frac{\Psi_{k-1}[k,q]\big(1+\alpha[k,q]-\alpha\beta\big)+\beta(1-\alpha)}{1-\beta} b_k
\end{aligned}$$

and by (15), we have:

$$F < \frac{(1-t)}{2} + \frac{(1+t)}{2} = 1.$$

Hence the result follows. □

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