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ON THE COMPOSITION OF THE PERRON-VANNIER REPRESENTATION AND THE NATURAL MAP $P_n \mapsto P_{2n}$

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Abstract: We study the composition of F.R. Cohen's map $P_n \mapsto P_{2n}$ with the Perron and Vannier representation, where P_n is the pure braid group on n strings. We prove that the obtained representation of P_n has one of its composition factors the inverse of the Gassner representation of the pure braid group.

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Key Words: Artin representation; braid group; pure braid group; Gassner representation

1. Introduction

The pure braid group, P_n, is a normal subgroup of the braid group, B_n, on n strings. One of the most important representations of P_n is the Gassner representation which comes from embedding $P_n \mapsto Aut(\mathbb{F}_n)$, by means of Magnus representation [6]. Here $\mathbb{F}_n$ is the free group with n generators. Another type of representations, introduced by H. A. Haidar and M. N. Abdulrahim, is a representation of the pure braid group on three strands $P_3 \mapsto GL_6(\mathbb{C})$. This representation turned out to be a direct sum of irreducible subrepresentations (see [3]). We also have a representation introduced by F.R. Cohen, the map $B_n \mapsto B_{nk}$, which is defined on geometric braids by replacing each string with

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k strings. It was proven in [1] that by composing Cohen's map $B_n \mapsto B_{2n}$ with the representation introduced by M. Wada [7], we obtain a representation whose one of the composition factors is isomorphic to the Burau representation. A similar work was done on the pure braid group. In [2], it was proven that composing Cohen's map $P_n \mapsto P_{nk}$ with the Gassner representation gives us a linear representation of P_n whose composition factors are one copy of the Gassner representation of P_n and $k-1$ copies of a diagonal representation, hence a direct sum of one-dimensional representations.

There are several faithful representations of braids as automorphisms of free groups. The most popular among them is the Artin representation $B_n \mapsto Aut(\mathbb{F}_n)$, and similarly the extended Artin representation $B_n \mapsto Aut(\mathbb{F}_{n+1})$. We also have the Jones-Wenzl representation of the braid group (see [4]). Another one is the Perron and Vannier representation $B_n \mapsto Aut(\mathbb{F}_{n-1})$, and its extensions in $Aut(\mathbb{F}_n)$ and $Aut(\mathbb{F}_{n+1})$, [5]. It was proven that the extended Perron and Vannier representation in $Aut(\mathbb{F}_{n+1})$ is not equivalent to the extended Artin representation. On the other hand, the extended Perron and Vannier representation in $Aut(\mathbb{F}_n)$ is shown to be a Wada representation (see [5]).

We consider, in Section 3 of our work, Cohen's map $P_n \mapsto P_{2n}$ and compose it with the extended Perron and Vannier representation in $Aut(\mathbb{F}_n)$. We show that the obtained linear representation has a composition factor isomorphic to the inverse of Gassner representation (Theorem 1).

2. Preliminaries

The braid group on n strings, B_n, is the abstract group with generators $\sigma_1, ..., \sigma_n$ and a presentation as follows:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \qquad i = 1, 2, ..., n-2,$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \qquad |i-j| > 2.$$

The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \mapsto S_n$ defined by $\sigma_i \mapsto (i \;\; i+1)$, $1 \leq i \leq n-1$, where S_n is the symmetric group of n elements. It admits a presentation with generators:

$$A_{ij} = \sigma_{j-1} \sigma_{j-2} ... \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} ... \sigma_{j-2}^{-1} \sigma_{j-1}^{-1}, \qquad 1 \leq i < j \leq n.$$

Definition 1. The Gassner representation $\gamma_n : P_n \mapsto Aut(\mathbb{F}_n)$, where $\mathbb{F}_n$

is the free group generated by $< x_1, \dots, x_n >$, is defined by

$$\gamma_n(A_{ij}) = \begin{cases} x_r \mapsto x_r, & r < i \text{ or } r > j, \\ x_i \mapsto x_i x_j x_i x_j^{-1} x_i^{-1}, \\ x_j \mapsto x_i x_j x_i^{-1}, \\ x_r \mapsto (x_i x_j x_i^{-1} x_j^{-1}) x_r (x_i x_j x_i^{-1} x_j^{-1})^{-1}, & i < r < j. \end{cases}$$

Lemma 1. *The image of A_{ij}, under the inverse of Gassner representation, is given by*

$$\gamma_n^{-1}(A_{ij}) = \begin{cases} x_r \mapsto x_r, & r < i \text{ or } r > j, \\ x_i \mapsto x_j^{-1} x_i x_j, \\ x_j \mapsto x_j^{-1} x_i^{-1} x_j x_i x_j, \\ x_r \mapsto (x_j^{-1} x_i^{-1} x_j x_i) x_r (x_j^{-1} x_i^{-1} x_j x_i)^{-1}, & i < r < j. \end{cases}$$

We now introduce the extension of the representation given by Perron and Vannier [5]. For simplicity, we call the representation Perron-Vannier representation.

Definition 2. ([5]) The Perron-Vannier representation is defined as follows:

$$Pcp^{(1)} : B_n \mapsto Aut(\mathbb{F}_n)$$

$$\sigma_1 \mapsto Pcp^{(1)}(\sigma_1) = \begin{cases} y_1 \mapsto y_1, \\ y_2 \mapsto {y_1}^{-1} y_2, \\ y_j \mapsto y_j, & j > 2, \end{cases}$$

and for $i = 2, 3, \dots, n-1$

$$\sigma_i \mapsto Pcp^{(1)}(\sigma_i) = \begin{cases} y_{i-1} \mapsto y_{i-1} y_i, \\ y_{i+1} \mapsto {y_i}^{-1} y_{i+1}, \\ y_j \mapsto y_j, & j \neq i-1, i+1, \end{cases}$$

where $\mathbb{F}_n =< y_1, \dots, y_n >$.

We now introduce the fox derivatives as follows.

Definition 3. Let $\mathbb{F}_n$ be a free group of rank n with free basis $z_1, ..., z_n$. We define for $j = 1, 2, ..., n$ the free derivatives on the group $\mathbb{Z}[\mathbb{F}_n]$ by

(i) $\frac{\partial z_i}{\partial z_j} = \delta_{ij}$,

(ii) $\frac{\partial z_i^{-1}}{\partial z_j} = -\delta_{ij} z_i^{-1}$,
(iii) $\frac{\partial}{\partial z_j}(uv) = \frac{\partial u}{\partial z_j}\epsilon(v) + u\frac{\partial v}{\partial z_j}$, $u, v \in \mathbb{Z}[F_n]$.

Note that $\epsilon(v) = 1$ if $v \in \mathbb{F}_n$. Here δ_{ij} is the Kronecker symbol.

3. Cohen representation

Definition 4. The Cohen representation is the map $B_n \mapsto B_{nk}$ defined as follows:

$$\sigma_i \mapsto 1 \times \sigma_i$$

$$= (\sigma_{ki}\sigma_{ki+1}...\sigma_{ki+k-1})(\sigma_{ki-1}\sigma_{ki}...\sigma_{ki+k-2})(\sigma_{ki-k+1}\sigma_{ki-k+2}...\sigma_{ki}).$$

Here $1 \times \sigma_i$ is the braid obtained by replacing each string of the geometric braid, σ_i, with k parallel strings. Cohen called $1 \times \sigma_i$ a tensor product. For simplicity, we replace $1 \times \sigma_i$ by τ_i.

Here, we take the special case $k = 2$. Our objective is to construct a linear representation of P_n of degree $2n$ in the following way: Consider the following map: $P_n \mapsto P_{2n} \mapsto GL_{2n}(\mathbb{Z}[t^{\pm 1}])$, where the first map is the restriction of the Cohen representation to P_n and the second one is the restriction of the Perron-Vannier representation to P_{2n}. Next, we find a set of generators of the group $\mathbb{F}_{2n}$. We determine the action of the automorphism corresponding to τ_i on this basis of $\mathbb{F}_{2n}$. Then we find the image of A_{ij} under the Cohen map and determine the action on the basis of $\mathbb{F}_{2n}$. After applying free differential calculus to this element of $Aut(\mathbb{F}_{2n})$, we get a linear representation of degree $2n$. Given the generators of $\mathbb{F}_n$, namely $x_1, ..., x_n$, we choose a certain basis of elements z_i, each of which is a word in these x_i's. More precisely, we have, for $1 \leq i \leq n$

$$z_i = y_{2i-1}$$

and

$$z_{n+i} = y_{2i-1}.y_{2i}...y_{2n}.$$

Here,

$$y_i = x_i x_{i+1}^{-1}, \quad 1 \leq i \leq 2n-1$$

and

$$y_{2n} = x_{2n}.$$

Proposition 1. *For $1 \leq i \leq n-1$ and $1 \leq j \leq 2n$, the action of τ_i on the generators $\{z_j\}$ of $\mathbb{F}_{2n}$, under the Perron-Vannier representation, is given by*

(i) $z_i \mapsto z_{i+1}$,
(ii) $z_{i+1} \mapsto z_{i+1}^{-1} z_i z_{i+1}$,
(iii) $z_{n+i} \mapsto z_{n+i+1}$,
(iv) $z_{n+i+1} \mapsto z_{i+1}^{-1} z_{n+i} z_{n+i+1}^{-1} z_{i+1}^{-1} z_{n+i+1}$,
(v) $z_j \mapsto z_j$ *for all* $j \neq i, i+1, n+i, n+i+1$.

Proof. The action of $\sigma_i' s$ on the generators y_i's is given by Perron-Vannier (see Definition 2).

(i) We have that $\tau_i = \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}$. So

$$\begin{aligned}
\tau_i(z_i) &= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}(y_{2i-1}) \\
&= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}(y_{2i-1}y_{2i}) \\
&= \sigma_{2i}\sigma_{2i+1}(y_{2i-1}y_{2i-1}^{-1}y_{2i}) \\
&= \sigma_{2i}(y_{2i}y_{2i+1}) \\
&= y_{2i}y_{2i}^{-1}y_{2i+1} \\
&= y_{2i+1} \\
&= z_{i+1}.
\end{aligned}$$

(ii)
$$\begin{aligned}
\tau_i(z_{i+1}) &= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}(y_{2i+1}) \\
&= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}(y_{2i}^{-1}y_{2i+1}) \\
&= \sigma_{2i}\sigma_{2i+1}(y_{2i}^{-1}y_{2i-1}y_{2i+1}) \\
&= \sigma_{2i}(y_{2i+1}^{-1}y_{2i}^{-1}y_{2i-1}y_{2i+1}) \\
&= y_{2i+1}^{-1}y_{2i}y_{2i}^{-1}y_{2i-1}y_{2i}y_{2i}^{-1}y_{2i+1} \\
&= y_{2i+1}^{-1}y_{2i-1}y_{2i+1} \\
&= z_{i+1}^{-1}z_i z_{i+1}.
\end{aligned}$$

(iii)
$$\begin{aligned}
\tau_i(z_{n+i}) &= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}(y_{2i-1}y_{2i}y_{2i+1}\cdots y_{2n}) \\
&= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}(y_{2i-1}y_{2i}y_{2i}y_{2i}^{-1}y_{2i+1}\cdots y_{2n}) \\
&= \sigma_{2i}\sigma_{2i+1}(y_{2i-1}y_{2i-1}^{-1}y_{2i}y_{2i+1}y_{2i+2}\cdots y_{2n}) \\
&= \sigma_{2i}(y_{2i}y_{2i+1}y_{2i+1}y_{2i+1}^{-1}y_{2i+2}\cdots y_{2n}) \\
&= y_{2i}y_{2i}^{-1}y_{2i+1}\cdots y_{2n} \\
&= y_{2i+1}\cdots y_{2n} \\
&= z_{n+i+1}.
\end{aligned}$$

(iv)
$$\begin{aligned}
\tau_i(z_{n+i+1}) &= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}\sigma_{2i}(y_{2i+1}y_{2i+2}\cdots y_{2n}) \\
&= \sigma_{2i}\sigma_{2i+1}\sigma_{2i-1}(y_{2i}^{-1}y_{2i+1}\cdots y_{2n}) \\
&= \sigma_{2i}\sigma_{2i+1}(y_{2i}^{-1}y_{2i-1}y_{2i+1}y_{2i+2}\cdots y_{2n})
\end{aligned}$$

$$
\begin{aligned}
&= \sigma_{2i}(y_{2i+1}^{-1}y_{2i}^{-1}y_{2i-1}y_{2i+1}y_{2i+1}^{-1}y_{2i+2}...y_{2n})\\
&= y_{2i+1}^{-1}y_{2i}y_{2i}^{-1}y_{2i-1}y_{2i}y_{2i+2}...y_{2n}\\
&= y_{2i+1}^{-1}y_{2i-1}y_{2i}y_{2i+2}...y_{2n}\\
&= z_{i+1}^{-1}z_{n+i}z_{n+i+1}^{-1}z_{i+1}^{-1}z_{n+i+1}.
\end{aligned}
$$

(v) if $1 \leq j \leq n$ with $j \neq i, i+1$, then $2j-1 \leq 2i-3$ or $2j-1 \geq 2i+3$ and in both cases z_j is fixed under the action of τ_i.
if $n+1 \leq j \leq 2n$ with $j \neq n+i, n+i+1$, then $j = n+s$ for some $1 \leq s \leq n$ with $s \neq i, i+1$, and we can see that $2s-1$ is not $2i-1, 2i, 2i+1, 2i+2$ which implies that $z_{n+s} \mapsto z_{n+s}$ under τ_i.

□

We can see that for $1 \leq i \leq n-1$ and $1 \leq j \leq 2n$, the action of τ_i^{-1} on the generators $\{z_j\}$ of $\mathbb{F}_{2n}$ is given by

(i) $z_i \mapsto z_i z_{i+1} z_i^{-1}$,
(ii) $z_{i+1} \mapsto z_i$,
(iii) $z_{n+i} \mapsto z_i z_{n+i+1} z_{n+i}^{-1} z_i z_{n+i}$,
(iv) $z_{n+i+1} \mapsto z_{n+i}$,
(v) $z_j \mapsto z_j$ for all $j \neq i, i+1, n+i, n+i+1$.

Lemma 2. *The image of A_{ij} under Cohen's map acts on the generators $\{z_s\}$ as follows:*

(i) $z_i \mapsto z_j^{-1} z_i z_j$,
(ii) $z_j \mapsto z_j^{-1} z_i^{-1} z_j z_i z_j$,
(iii) $z_r \mapsto z_r$ *if* $r < i$ *or* $r > j$,
(iv) $z_r \mapsto (z_j^{-1} z_i^{-1} z_j z_i) z_r (z_j^{-1} z_i^{-1} z_j z_i)^{-1}$ *if* $i < r < j$,
(v) $z_{n+i} \mapsto z_j^{-1} z_{n+i} z_{n+j}^{-1} z_j^{-1} z_{n+j}$,
(vi) $z_{n+j} \mapsto z_{n+j}^{-1} z_j z_{n+j} z_{n+i}^{-1} z_j$,
(vii) $z_{n+r} \mapsto z_{n+r}$ *if* $r < i$ *or* $r > j$,
(viii) $z_{n+r} \mapsto (z_j^{-1} z_i^{-1} z_j z_i) z_{n+r} (z_{n+i}^{-1} z_i z_{n+i})(z_{n+j}^{-1} z_j z_{n+j})$
$(z_{n+i}^{-1} z_i z_{n+i})^{-1} (z_{n+j}^{-1} z_j z_{n+j})^{-1}$ *if* $i < r < j$.

Proof. We consider the image of the generators of the pure braid group under Cohen's map and still call it A_{ij},

$$A_{ij} = \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}\tau_{j-1}^{-1}, \quad 1 \leq i \leq j \leq n.$$

That is, we need to consider A_{ij} as right automorphisms acting on the generators of $\mathbb{F}_{2n}$ from the right.

(i)
$$\begin{aligned}
A_{ij}(z_i) &= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}\tau_{j-1}^{-1}(z_i)\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}(z_i)\\
&\vdots\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2(z_i)\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i(z_{i+1})\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}(z_{i+1}^{-1}z_iz_{i+1})\\
&\vdots\\
&= \tau_{j-1}(z_{j-1}^{-1}z_iz_{j-1})\\
&= z_j^{-1}z_iz_j.
\end{aligned}$$

(ii)
$$\begin{aligned}
A_{ij}(z_j) &= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}\tau_{j-1}^{-1}(z_j)\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}(z_{j-1})\\
&\vdots\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2(z_{i+1})\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i(z_{i+1}^{-1}z_iz_{i+1})\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+1}(z_{i+1}^{-1}z_i^{-1}z_{i+1}z_{i+1}z_{i+1}^{-1}z_iz_{i+1})\\
&= \tau_{j-1}\tau_{j-2}...\tau_{i+2}(z_{i+2}^{-1}z_i^{-1}z_{i+2}z_iz_{i+2})\\
&\vdots\\
&= \tau_{j-1}(z_{j-1}^{-1}z_i^{-1}z_{j-1}z_iz_{j-1})\\
&= z_j^{-1}z_i^{-1}z_jz_iz_j.
\end{aligned}$$

(iii) Suppose $r < i$ or $r > j$. Then τ_s fixes z_r for all $i \leq s \leq j-1$ and therefore, A_{ij} fixes z_r.

(iv) Suppose $i < r < j$. Then,
$$\begin{aligned}
A_{ij}(z_r) &= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{r-1}^{-1}\tau_r^{-1}\tau_{r+1}^{-1}...\tau_{j-2}^{-1}\tau_{j-1}^{-1}(z_r)\\
&= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{r-1}^{-1}\tau_r^{-1}\tau_{r+1}^{-1}...\tau_{j-2}^{-1}(z_r)\\
&\vdots\\
&= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{r-1}^{-1}\tau_r^{-1}(z_r)\\
&= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{r-1}^{-1}(z_rz_{r+1}z_r^{-1})\\
&= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{r-2}^{-1}(z_{r-1}z_{r+1}z_{r-1}^{-1})\\
&\vdots\\
&= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}\tau_i^2(z_{i+1}z_{r+1}z_{i+1}^{-1})\\
&= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}\tau_i(z_{i+1}^{-1}z_iz_{i+1}z_{r+1}z_{i+1}^{-1}z_i^{-1}z_{i+1})
\end{aligned}$$

$$= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+1}(z_{i+1}^{-1}z_i^{-1}z_{i+1}z_{i+1}z_{i+1}^{-1}z_iz_{i+1}z_{r+1}z_{i+1}^{-1}$$
$$z_i^{-1}z_{i+1}z_{i+1}^{-1}z_{i+1}^{-1}z_iz_{i+1})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}...\tau_{i+2}(z_{i+2}^{-1}z_i^{-1}z_{i+2}z_iz_{i+2}z_{r+1}z_{i+2}^{-1}z_i^{-1}z_{i+2}^{-1}$$
$$z_iz_{i+2})$$
$$\vdots$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r\tau_{r-1}(z_{r-1}^{-1}z_i^{-1}z_{r-1}z_iz_{r-1}z_{r+1}z_{r-1}^{-1}z_i^{-1}$$
$$z_{r-1}^{-1}z_iz_{r-1})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{r+1}\tau_r(z_r^{-1}z_i^{-1}z_rz_iz_rz_{r+1}z_r^{-1}z_i^{-1}z_r^{-1}z_iz_r)$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{r+1}(z_{r+1}^{-1}z_i^{-1}z_{r+1}z_iz_{r+1}z_{r+1}^{-1}z_rz_{r+1}z_{r+1}^{-1}z_i^{-1}$$
$$z_{r+1}^{-1}z_iz_{r+1})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{r+2}(z_{r+2}^{-1}z_i^{-1}z_{r+2}z_iz_rz_i^{-1}z_{r+2}^{-1}z_iz_{r+2})$$
$$\vdots$$
$$= \tau_{j-1}\tau_{j-2}(z_{j-2}^{-1}z_i^{-1}z_{j-2}z_iz_rz_i^{-1}z_{j-2}^{-1}z_iz_{j-2})$$
$$= \tau_{j-1}(z_{j-1}^{-1}z_i^{-1}z_{j-1}z_iz_rz_i^{-1}z_{j-1}^{-1}z_iz_{j-1})$$
$$= z_j^{-1}z_i^{-1}z_jz_iz_rz_i^{-1}z_j^{-1}z_iz_j.$$

(v) $A_{ij}(z_{n+i}) = \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}\tau_{j-1}^{-1}(z_{n+i})$

$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}(z_{n+i})$$
$$\vdots$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}(z_{n+i})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2(z_{n+i})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i(z_{n+i+1})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}(z_{i+1}^{-1}z_{n+i}z_{n+i+1}^{-1}z_{i+1}^{-1}z_{n+i+1})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+2}(z_{i+2}^{-1}z_{n+i}z_{n+i+2}^{-1}z_{i+2}^{-1}z_{n+i+2})$$
$$\vdots$$
$$= \tau_{j-1}(z_{j-1}^{-1}z_{n+i}z_{n+j-1}^{-1}z_{j-1}^{-1}z_{n+j-1})$$
$$= z_j^{-1}z_{n+i}z_{n+j}^{-1}z_j^{-1}z_{n+j}.$$

(vi) $A_{ij}(z_{n+j}) = \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}\tau_{j-1}^{-1}(z_{n+j})$

$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}...\tau_{j-2}^{-1}(z_{n+j-1})$$
$$\vdots$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i^2(z_{n+i})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}\tau_i(z_{n+i+1})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+1}(z_{n+i+1}^{-1}z_{i+1}z_{n+i+1}z_{n+i}^{-1}z_{i+1})$$
$$= \tau_{j-1}\tau_{j-2}...\tau_{i+2}(z_{n+i+2}^{-1}z_{i+2}z_{n+i+2}z_{n+i}^{-1}z_{i+2})$$

$$\begin{aligned}
&\vdots \\
&= \tau_{j-1}(z_{n+j-1}^{-1} z_{j-1} z_{n+j-1} z_{n+i}^{-1} z_{j-1}) \\
&= z_{n+j}^{-1} z_j z_{n+j} z_{n+i}^{-1} z_j.
\end{aligned}$$

(vii) Suppose $r < i$ or $r > j$. Then τ_s fixes z_{n+r} for all $i \leq s \leq j-1$ and therefore, A_{ij} fixes z_{n+r}.

(viii)
$$\begin{aligned}
A_{ij}(z_{n+r}) &= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}\ldots\tau_{r-1}^{-1}\tau_r^{-1}\tau_{r+1}^{-1} \\
&\ldots\tau_{j-2}^{-1}\tau_{j-1}^{-1}(z_{n+r}) \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}\ldots\tau_{r-1}^{-1}\tau_r^{-1}\tau_{r+1}^{-1} \\
&\ldots\tau_{j-2}^{-1}(z_{n+r}) \\
&\vdots \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}\ldots\tau_{r-1}^{-1} \\
&\tau_r^{-1}(z_{n+r}) \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}\ldots \\
&\tau_{r-1}^{-1}(z_r z_{n+r+1} z_{n+r}^{-1} z_r z_{n+r}) \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1}\tau_i^2\tau_{i+1}^{-1}\ldots \\
&\tau_{r-2}^{-1}(z_{r-1} z_{n+r+1} z_{n+r-1}^{-1} z_{r-1} z_{n+r-1}) \\
&\vdots \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1} \\
&\tau_i^2(z_{i+1} z_{n+r+1} z_{n+i+1}^{-1} z_{i+1} z_{n+i+1}) \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1} \\
&\tau_i(z_{i+1}^{-1} z_i z_{i+1} z_{n+r+1} z_{n+i+1}^{-1} z_{i+1} z_{n+i+1} z_{n+i}^{-1} z_{i+1} z_{i+1}^{-1} \\
&z_i z_{i+1} z_{i+1}^{-1} z_{n+i} z_{n+i+1}^{-1} z_{i+1}^{-1} z_{n+i+1}) \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+1}(z_{i+1}^{-1} z_i^{-1} z_{i+1} \\
&z_{i+1} z_{i+1}^{-1} z_i z_{i+1} z_{n+r+1} z_{n+i+1}^{-1} z_{i+1} z_{n+i+1} z_{n+i}^{-1} \\
&z_{i+1} z_{i+1}^{-1} z_i z_{i+1} z_{i+1}^{-1} z_{n+i} z_{n+i+1}^{-1} z_{i+1}^{-1} z_{n+i+1} z_{n+i+1}^{-1} \\
&z_{i+1} z_{n+i+1} z_{n+i+1}^{-1} z_{i+1} z_{n+i+1} z_{n+i}^{-1} z_{i+1} z_{i+1}^{-1} z_i^{-1} \\
&z_{i+1} z_{i+1}^{-1} z_{n+i} z_{n+i+1}^{-1} z_{i+1}^{-1} z_{n+i+1}) \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r\tau_{r-1}\ldots\tau_{i+2}(z_{i+2}^{-1} z_i^{-1} z_{i+2} z_i \\
&z_{i+2} z_{n+r+1} z_{n+i+2}^{-1} z_{i+2} z_{n+i+2} z_{n+i}^{-1} z_i z_{n+i} z_{n+i+2}^{-1} \\
&z_{i+2} z_{n+i+2} z_{n+i}^{-1} z_i^{-1} z_{n+i} z_{n+i+2}^{-1} z_{i+2}^{-1} z_{n+i+2}) \\
&\vdots \\
&= \tau_{j-1}\tau_{j-2}\ldots\tau_{r+1}\tau_r(z_r^{-1} z_i^{-1} z_r z_i z_r z_{n+r+1} z_{n+r}^{-1} \\
&z_r z_{n+r} z_{n+i}^{-1} z_i z_{n+i} z_{n+r}^{-1} z_r z_{n+r} z_{n+i}^{-1} z_i^{-1} z_{n+i} z_{n+r}^{-1} \\
&z_r^{-1} z_{n+r})
\end{aligned}$$

$$= \tau_{j-1}\tau_{j-2}...\tau_{r+1}(z_{r+1}^{-1}z_i^{-1}z_{r+1}z_iz_{r+1}z_{r+1}^{-1}z_{n+r}$$
$$z_{n+r+1}^{-1}z_{r+1}^{-1}z_{n+r+1}z_{n+r+1}^{-1}z_{r+1}z_{n+r+1}z_{n+i}^{-1}z_iz_{n+i}$$
$$z_{n+r+1}^{-1}z_{r+1}z_{n+r+1}z_{n+i}^{-1}z_i^{-1}z_{n+i}z_{n+r+1}^{-1}z_{r+1}^{-1}z_{n+r+1})$$
$$\vdots$$
$$= \tau_{j-1}(z_{j-1}^{-1}z_i^{-1}z_{j-1}z_iz_{n+r}z_{n+i}^{-1}z_iz_{n+i}z_{n+j-1}^{-1}z_{j-1}$$
$$z_{n+j-1}z_{n+i}^{-1}z_i^{-1}z_{n+i}z_{n+j-1}^{-1}z_{j-1}^{-1}z_{n+j-1})$$
$$= z_j^{-1}z_i^{-1}z_jz_iz_{n+r}z_{n+i}^{-1}z_iz_{n+i}z_{n+j}^{-1}z_jz_{n+j}z_{n+i}^{-1}z_i^{-1}z_{n+i}$$
$$z_{n+j}^{-1}z_j^{-1}z_{n+j}.$$

□

To find the linear representation obtained by composing the Cohen map $P_n \mapsto P_{2n}$ with the Perron-Vannier representation, we let Ψ be a homomorphism from $\mathbb{F}_{2n}$ to $\mathbb{C}^*$ defined by $\Psi(z_s) = t_s$ for $1 \leq s \leq 2n$. Let $D_s = \Psi\frac{\partial}{\partial z_s}$. We now determine the Jacobian matrix of the image of the generator A_{ij} under Cohen's map,

$$J(A_{ij}) = \begin{pmatrix} D_1(A_{ij}(z_1)) & \dots & D_{2n}(A_{ij}(z_1)) \\ \vdots & & \vdots \\ D_1(A_{ij}(z_{2n})) & \dots & D_{2n}(A_{ij}(z_{2n})) \end{pmatrix}.$$

The construction used here is the Magnus representation of P_{2n}. Let us prove our main theorem.

Theorem 1. *The linear representation obtained by composing the Cohen map $P_n \mapsto P_{2n}$ with the the Perron-Vannier representation, namely $P_n \mapsto GL_{2n}(\mathbb{Z}[t_1^{\pm 1}, \dots, t_{2n}^{\pm 1}])$, has a subrepresentation isomorphic to the inverse of Gassner representation. That is, the image of A_{ij} is*

$$\left(\begin{array}{c|c} \gamma_n^{-1}(t_s)(A_{ij}) & 0 \\ \hline C_n & H_n \end{array}\right), \qquad \textit{where}$$

C_n is the $n \times n$ matrix given by

$$C_n = \left(\begin{array}{c|cccccc|c} 0_{i-1} & & & & 0 & & & 0 \\ \hline & 0 & 0 & \dots & \dots & 0 & -t_j^{-1} - t_j^{-2}t_{n+i}t_{n+j}^{-1} & \\ & m_{i+1} & 0 & 0 & \dots & 0 & n_{i+1} & \\ 0 & m_{i+2} & 0 & \ddots & & \vdots & n_{i+2} & 0 \\ & \vdots & \vdots & & \ddots & & \vdots & \\ & m_{j-1} & 0 & \dots & 0 & 0 & n_{j-1} & \\ & 0 & 0 & \dots & \dots & 0 & t_{n+j}^{-1} + t_jt_{n+i}^{-1} & \\ \hline 0 & & & & 0 & & & 0_{n-j} \end{array}\right),$$

and H_n is the $n \times n$ matrix given by

$$H_n = \left(\begin{array}{c|cccccc|c} I_{i-1} & & & & 0 & & & 0 \\ \hline & t_j^{-1} & 0 & \dots & \dots & 0 & -t_j^{-1}t_{n+i}t_{n+j}^{-1} + t_j^{-2}t_{n+i}t_{n+j}^{-1} & \\ & p_{i+1} & 1 & 0 & \dots & 0 & q_{i+1} & \\ 0 & p_{i+2} & 0 & \ddots & & \vdots & q_{i+2} & 0 \\ & \vdots & \vdots & & \ddots & & \vdots & \\ & p_{j-1} & 0 & \dots & 0 & 1 & q_{j-1} & \\ & -t_{n+i}^{-1}t_j & 0 & \dots & \dots & 0 & -t_{n+j}^{-1} + t_{n+j}^{-1}t_j & \\ \hline 0 & & & & 0 & & & I_{n-j} \end{array}\right).$$

For $i < r < j$, the entries of the matrices C_n and H_n are given by

$m_r = -t_j^{-1}t_i^{-1} + t_i^{-1} + t_{n+r}t_{n+i}^{-1} - t_j t_{n+r}t_{n+i}^{-1}$,
$n_r = -t_j^{-1} + t_j^{-1}t_i^{-1} + t_i t_{n+r}t_{n+j}^{-1} - t_{n+r}t_{n+j}^{-1}$,
$p_r = t_{n+r}t_{n+i}^{-1}(-1 + t_i + t_j - t_i t_j)$,
$q_r = t_{n+r}t_{n+j}^{-1}(1 - t_i - t_j + t_i t_j)$.

Proof. It is easy to see that the action of A_{ij} on the elements of the basis $\{z_1, ..., z_n\}$ coincides with the inverse of the Gassner representation of degree n (see Lemma 1 and (i),(ii),(iii), and (iv) in Lemma 2).

Now,

$D_{n+s}(A_{ij})(z_r) = 0, \quad 1 \le r \le n, \quad 1 \le s \le n.$

<u>For C_n</u>:
$D_i(A_{ij})(z_{n+i}) = 0.$
$D_j(A_{ij})(z_{n+i}) = -t_j^{-1} - t_j^{-2}t_{n+i}t_{n+j}^{-1}.$
$D_i(A_{ij})(z_{n+j}) = 0.$
$D_j(A_{ij})(z_{n+j}) = t_{n+j}^{-1} + t_j t_{n+i}^{-1}.$
$D_i(A_{ij})(z_{n+r}) = -t_j^{-1}t_i^{-1} + t_i^{-1} + t_{n+r}t_{n+i}^{-1} - t_j t_{n+r}t_{n+i}^{-1}, \quad i < r < j.$
$D_j(A_{ij})(z_{n+r}) = -t_j^{-1} + t_j^{-1}t_i^{-1} + t_i t_{n+r}t_{n+j}^{-1} - t_{n+r}t_{n+j}^{-1}, \quad i < r < j.$
$D_s(A_{ij})(z_{n+r}) = 0, \quad r < i$ or $r > j$ and $1 \le s \le n$.
$D_s(A_{ij})(z_{n+r}) = 0, \quad s \ne i, j$ and $i \le r \le j$.

<u>For H_n</u>:
$D_{n+i}(A_{ij})(z_{n+i}) = t_j^{-1}.$
$D_{n+j}(A_{ij})(z_{n+i}) = -t_j^{-1}t_{n+i}t_{n+j}^{-1} + t_j^{-2}t_{n+i}t_{n+j}^{-1}.$

$D_{n+i}(A_{ij})(z_{n+j}) = -t_{n+i}^{-1} t_j.$
$D_{n+j}(A_{ij})(z_{n+j}) = -t_{n+j}^{-1} + t_{n+j}^{-1} t_j.$
$D_{n+i}(A_{ij})(z_{n+r}) = t_{n+r} t_{n+i}^{-1}(-1 + t_i + t_j - t_i t_j), \quad i < r < j.$
$D_{n+j}(A_{ij})(z_{n+r}) = t_{n+r} t_{n+j}^{-1}(1 - t_i - t_j + t_i t_j), \quad i < r < j.$
$D_{n+r}(A_{ij})(z_{n+r}) = 1, \quad r < i$ or $r > j$.
$D_{n+r}(A_{ij})(z_{n+r}) = 1, \quad i < r < j.$
$D_{n+s}(A_{ij})(z_{n+r}) = 0$ in case $s \neq i, j, r \neq i, j$, and $s \neq r$. □

References

[1] M.N. Abdulrahim, Wada's representation and the natural map $B_n \mapsto B_{2n}$, *Int. J. Pure Appl. Math.*, **18**, No 2 (2005), 137-142.

[2] M.N. Abdulrahim, Pure braids as automorphisms of free groups, *J. Algebra Appl.*, **4**, No 4 (2005), 1-6.

[3] H.A. Haidar, M.N. Abdulrahim, On the irreducibility of the representation of the pure braid group on three strands, *Int. J. Appl. Math.*, **32**, No 2 (2019), 249-257; DOI: 10.12732/ijam.v32i2.7.

[4] M.M. Dally, M.N. Abdulrahim, On the faithfulness of Jones-Wenzl representation of the braid group B_4, *International Journal of Applied Mathematics*, **32**, No 3 (2019), 423-431; DOI: 10.12732/ijam.v32i3.4.

[5] V. Bardakov, G. Bellingeri, On representation of braids as automorphisms of free groups and corresponding linear representations, *Knot Theory and Its Applications*, In: *Contemp. Math.*, **670**, Amer. Math. Soc., Providence (2016), 285-298.

[6] J.S. Birman, *Braids, Links and Mapping Class Groups*, Annals of Mathematical Studies, Princeton University Press, New Jersey **82** (1975).

[7] M. Wada, Group invariants of links, *Topology*, **31** (1992), 399-406.