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ON LINEAR OPERATORS GIVING HIGHER ORDER APPROXIMATION OF FUNCTIONS IN $L^p_\sigma(R^+)$

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Abstract: Numerical investigations of different authors were devoted to convergence of singular integrals and approximation of functions by linear operators. Asymptotic values of approximation of a function by linear operators were obtained here.

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1. Introduction

In this field, the works of Lebesgue, I.P. Natanson, I.P. Korovkin, F.I. Kharshiladze, A.Z. Turetskiy, R.G. Mamedov, A.D. Hajiyev and others should be mentioned.

One of the important problems in approximation theory is finding saturation classes of linear operators in various spaces. The saturation problem was first stated by Favour in 1937.

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Aleksich, Zamanskiy, Alyanchich, Kharshiladze, Turetskiy, Butzer, Berents, Nessel, Sunouchi, Mamedov and others have obtained many important results in solving the problem on definition of saturation classes. Berents and Butzer [1] determined the class and order of saturation of approximation of the function $f(t)$ $(e^{-ct}f(t) \in L_p(0,\infty),\ C > 0, p \geq 1)$ by the linear operator

$$R(f;x) = \lambda \int\limits_0^x f(x-t)K_\lambda(t)dt$$

with a positive kernel $K_\lambda(t) = \lambda K(\lambda t) > 0$ in the metric of the space $L_p(0;\infty)$. They showed that the saturation order is $0(\lambda^{-\gamma})$ $(0 < \gamma \leq 1)$.

Using the Fourier transformation, Butzer, Nessel [3], Sunouchi [4], Mamedov [6] and others determined the order and class of saturation of different singular integrals and linear operators in the space $L_p(-\infty;\infty)$.

The basic results obtained for the last years by different authors solution of the saturation problem were stated in detail in the monographs of Mamedov [6], and Butzer-Berents [2].

In this paper, we suggest a method for constructing a new linear operator on the base of the given operators that gives a high approximation order to multiply differentiable functions.

Let $L^p_\sigma(R^+)$ be a space of measurable in $R^+(0;+\infty)$ functions $f(x)$, for which $\|f\|_{L^p_\sigma(R^+)} < \infty(\sigma > 0)$, where

$$\|f(x)\|_{L^p_\sigma(R^+)} = \begin{cases} \left\{ \int\limits_0^\infty |e^{-\sigma x} f(x)|^p \, dx \right\}^{1/p} & \text{for } 1 \leq p < \infty \\ \sup vrai\ |f(x)e^{-\sigma x}| & \text{for } p = \infty \end{cases} .$$

Let the function $f(x) \in L^p_\sigma(R^+)$, then the Laplace transformation of the function $f(x)$ is determined by

$$f^\wedge(s) = \int\limits_0^\infty e^{-sx} f(x)dx \ \ (s = \sigma + i\tau,\ Res = \sigma > 0),$$

where the integral converges absolutely for $Res > 0$. Now, let $\varphi(x)$ be a function with a bounded variation on the segment $[0,r]$ for any $r > 0$, i.e. $\varphi(x) \in BV(0,r)$ and $\int\limits_0^\infty e^{-\sigma i} |d\varphi(t)| < \infty$ for each $\sigma > 0$.

Then the Laplace-Stieltjes transformation of the function $\varphi(x)$ is determined in the following way ([2])

$$\varphi^{\wedge}(s)=\int\limits_{0}^{\infty}e^{-sx}d(\varphi(x))\quad(s=\sigma+i\tau),$$

where the integral converges absolutely for $Res=\sigma>0$. Consider approximation of functions $f(x)$ by linear operators of the form

$$Q_{\lambda}^{[m],l}(f;x)=R_{\lambda,1}(R_{\lambda,2}...(R_{\lambda,e}(f;x))...))-\sum_{v=1}^{m-1}\frac{\alpha_v(l)}{\lambda^v}f^{(v)}(x),\tag{1}$$

where $l\in N,\ \alpha_v(l)\ \ (v=\overline{0,N-1})$ are real numbers and

$$R_{\lambda,n}(g;x)=\lambda\int\limits_{0}^{x}g(x-t)K_n(\lambda t)dt\quad(n=\overline{1,e}),\tag{2}$$

$K_{\lambda,n}(t)=\lambda K_n(\lambda t)$ are the functions determined on R^+ called kernels with the properties:

$$K_{\lambda,n}(t)\in L(R^+),\qquad\int\limits_{0}^{\infty}K_{\lambda,n}(t)dt=1$$

and

$$\int\limits_{0}^{x}K_{\lambda,n}(t)dt\to1\ \ (\lambda\to\infty),\ (n=\overline{1,e}).$$

Obviously, if $f^{(v)}(x)\in L_{\sigma}^{p}(R^+)$ $(1\le p<\infty,\ v=\overline{0,m-1})$, then the integral operator (1) exists almost everywhere on R^+ and $Q_{\lambda}^{[m],l}(f;x)\in L_{\sigma}^{p}(R^+)$.

The condition $D_m(\alpha(e),B(e))$: If for real numbers $\alpha_v(e)$ $(v=\overline{0,m-1}$, $\alpha_0(e)=1)$ and $B(e)\neq0$ it holds

$$\lim_{\lambda\to\infty}\left(\frac{s}{\lambda}\right)^{-m}\left[\prod_{n=1}^{e}K_n^{\wedge}\left(\frac{s}{\lambda}\right)-\sum_{v=0}^{m-1}\alpha_v(e)\left(\frac{s}{\lambda}\right)^{v}\right]=B(e)\neq0,$$

for some fixed $s(Res>0)$, it is said that the kernel $K_n(t)$ operator (2) satisfies the condition $D_m(\alpha(e),B(e))$, where

$$\alpha(e)\ =(\alpha_0(e),\ \alpha_1(e),...,\alpha_{m-1}(e)).$$

Denote by $a_p(s^\nu; f)$ a class of functions $f(x)$, having derivatives $f^{(\nu)}(x) \in L^p_\sigma(R^+)$ and $f^{(\nu)}(x) \in AC_{loc}(R^+)$ $(\nu = \overline{1,\, N-1})$.

Introduce the class of functions

$$b_p(s^v; f)$$

$$= \begin{cases} f(x) \in a_1(s^v; f)/s^m f^\wedge(s) = h_1^\vee(s), \text{ if } h_1(t) \in BV_\sigma(R^+) \\ \int_0^\infty e^{-\sigma t}\, |dh_1(t)| < +\infty, \qquad \text{if } p = 1 \\ f(x) \in a_p(s^v; f)/s^m f^\wedge(s) = h_2^\vee(s), \; if\; h_2(t) \in L^p_\sigma(R^+),\; p > 1 \end{cases}.$$

2. On linear operators giving higher order approximation of functions in $L^p_\sigma(R^+)$

Theorem 1. *Let the kernel $K_{\lambda,n}(t) = \lambda K_n(\lambda t)$ $(n = \overline{1, e})$ of operator (2) satisfy the condition $D_m(\alpha(e), B(e))$ and $f(t) \in a_p(s^v, f)$. Then, if for the function $g(t)$ $(g(t) \in L^p_\sigma(R^+), p \geq 1)$ the condition*

$$\Bigg\| \lambda^{-m} \left[R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f : x))...)) \right.$$
$$\left. - \sum_{v=0}^{m-1} \frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x) - g(x) \right\|_{L^p_\sigma(R^+)} = 0(1) \tag{3}$$

is fulfilled as $\lambda \to \infty$, the function $f(x)$ has a m order derivative and $g(x) = B(e) f^{(m)}(x)$ almost everywhere.

Proof. Let $p = 1$, since $f(x) \in a_1(s^v; f)$, then

$$\lambda^m \left[\prod_{n=1}^{e} K_n^\wedge\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1} \alpha_v(e) \left(\frac{s}{\lambda}\right)^v \right] f^\wedge(s) - g^\wedge(s)$$
$$= \int_0^\infty e^{-xs} \left\{ \lambda^m \left[R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f; x))...)) \right. \right.$$
$$\left. \left. - \sum_{v=0}^{m-1} \frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x) \right] - g(x) \right\} dx. \tag{4}$$

By the conditions of the theorem, from the last equality we find

$$\left| \lambda^m \left[\prod_{n=1}^{e} K_n^\wedge\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1} \alpha_v(e) \left(\frac{s}{\lambda}\right)^v \right] f^\wedge(s) - g^\wedge(s) \right| = 0(1)$$

as $\lambda \to \infty$.

Since $K_n(t)$ $(n = \overline{1, e})$ satisfies the condition $D_m(\alpha(e), B(e))$, then

$$B(e)s^m f^\wedge(s) = g^\wedge(s), \tag{5}$$

for each s $(Res > 0)$, or

$$g(x) = B(e)f^{(m)}(x)$$

almost everywhere.

Let $1 < p < \infty$. For $0 < \sigma' < \sigma$ by means of the Hölder inequality, from (4) we get

$$\left|\lambda^m \left[\prod_{n=1}^{e} K_n^\wedge\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1} \alpha_v(e)\left(\frac{s}{\lambda}\right)^v\right] f^\vee(s) - g^\wedge(s)\right|$$

$$\leq \left\| e^{-\sigma' t} \{ [R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;x))...))\right.$$

$$\left. - \sum_{v=0}^{m-1} \frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x)\right] \lambda^m - g(x) \Bigg\|_{L^p(R^+)} \cdot \left\{\frac{1}{(\sigma - \sigma') \cdot q}\right\}^{\frac{1}{q}}, \tag{6}$$

where $pq = p + q$.

The further reasoning is conducted similarly to the proof of the case $p = 1$.

Theorem 2. *Let the kernel $K_{\lambda,n}(t) = \lambda K_n(\lambda t)$ $(n = \overline{1, e})$ of operator (2) satisfy the condition $D_m(\alpha(e), B(e))$. If for the function $f(t) \in L^p_\sigma(R^+)$ the condition*

$$\left\| R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;x)...)) - \sum_{v=0}^{m-1} \frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x) \right\|_{L^p_\sigma(R^+)} = 0(\lambda^{-m}) \tag{7}$$

is fulfilled as $\lambda \to \infty$, then $f(x) \in b_p(s^v; f)$ $(v = \overline{0, m},\ p \geq 1)$.

Proof. Let $p = 1$. Consider the partial integral

$$S_{\tau,\lambda}(x)$$

$$= \frac{1}{2\pi i} \int_{\sigma - i\tau}^{\sigma + i\tau} e^{sx} \left[\prod_{n=1}^{e} K_n^\wedge\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1} \alpha_v(e)\left(\frac{s}{\lambda}\right)^v\right] f^\wedge(s)ds, \tag{8}$$

where $\sigma > 0$ and $\tau > 0$ are any numbers.

Feier's mean values $\sigma_{r,\lambda}(x)$ of partial integrals $S_{\tau,\lambda}(x)$ equal

$$\sigma_{r',\lambda}(x) = \frac{1}{r'}\int_0^{r'} S_{\tau,\lambda}(x)d\tau = \frac{1}{2\pi i}\int_{\sigma-ir'}^{\sigma+ir'} (1-\frac{|\tau|}{r'})e^{sx}\cdot\prod_{n=1}^{e} K_n^{\wedge}\left(\frac{s}{\lambda}\right)$$

$$-\sum_{v=0}^{m-1}\alpha_v(e)\left(\frac{s}{\lambda}\right)^v ds = \frac{1}{2\pi i}\int_{-r'}^{r'}(1-\frac{|\tau|}{r'})e^{(\sigma+i\tau)x}\left\{\int_0^{\infty} e^{-(\sigma+i\tau)u}\right.$$

$$\left.\times R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;u)...)) - \sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v}f^{(v)}(u)\right] du\Bigg\} d\tau.$$

Since by Lemma A (see [1]), the internal integral converges uniformly on $-r' \leq \tau < r'$, then

$$\sigma_{r',\lambda}(x) = \frac{2}{\pi r'}\int_0^{\infty} e^{\sigma(x-u)}\cdot\frac{\sin^2 r'\cdot\frac{x-u}{2}}{(x-u)^2}\{R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;u))...)$$

$$\left.-\sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v}f^{(v)}(u)\right\} du. \tag{9}$$

Further, taking into account (7), we get

$$\|\sigma_{r;\lambda}(x)\|_{L_\sigma(-\infty;+\infty)} \leq \|R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;u))...))$$

$$\left.-\sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v}f^{(v)}(x)\right\|_{L^p_\sigma(R^+)} = 0(\lambda^{-m}) \tag{10}$$

as $\lambda \to \infty$.

Consequently,

$$\left\|\frac{1}{2\pi i}\int_{\sigma-ir'}^{\sigma+ir'}\left(1-\frac{|\tau|}{r'}\right)e^{sx}\left[\prod_{n=1}^{e}K_n^{\wedge}\left(\frac{s}{\lambda}\right)\right.\right.$$

$$\left.\left.-\sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v}s^v\right] f^{\wedge}(s)ds\right\|_{L_\sigma(-\infty;+\infty)} = O(\lambda^{-m}). \tag{11}$$

Furthermore, by the condition $D_m(\alpha(e), B(e))$, we have

$$\left|\lambda^m \left[\prod_{n=1}^{e} K_n^{\wedge}\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1} \alpha_v(e)\left(\frac{s}{\lambda}\right)^v\right]\right| \le B(e)\,|s^m|$$

and

$$\left|e^{-\sigma x}(1-\frac{|\tau|}{r'})e^{sx}\left[\prod_{n=1}^{e} K_n^{\wedge}\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1} \alpha_v(e)\left(\frac{s}{\lambda}\right)^v\right] f^{\wedge}(s)\cdot\lambda^m\right|$$

$$\le 2B(e)\cdot|s^m|\cdot\left|f^{\wedge}(s)\right|$$

for each $\lambda > \lambda_0$ $|\tau| < r'$, where $s = \sigma + i\tau,\ \sigma > 0$.

Consequently,

$$2B(e)\int_{-r'}^{r'} |\sigma+i\tau|^m \left|f^{\wedge}(\sigma+i\tau)\right| d\tau \le 2B(e)\int_{-r'}^{r'} (\sigma^2+\tau^2)^{\frac{m}{2}}$$

$$\times\left(\int_0^\infty e^{-\sigma t}\,|f(t)|\,dt\right) d\tau \le M_1 < \infty.$$

Since $e^{-\sigma x}\left|\sigma_{r',\lambda}(x)\right|\lambda^m > 0$ and estimation (9) is valid, then

$$\int_{-\infty}^{\infty} e^{-\sigma x}\left|\sigma_{r',\lambda}(x)\right|\lambda^m dx < \infty.$$

Thus, the requirements of the Fatou lemma are fulfilled and taking into account (9), we find:

$$\left\|\frac{1}{2\pi i}\int_{\sigma-ir'}^{\sigma+ir'}(1-\frac{|\tau|}{r'})e^{sx}B(e)s^m f^{\wedge}(s)ds\right\|_{L_\sigma(-\infty;+\infty)} = 0(1).$$

To complete the proof of $f(x) \in b_p(s^v; f)$, $(v = \overline{0,m})$ for $p = 1$ we should apply Lemma A in [1].

Let $1 < p < \infty$. By (4) and the theorem on weak compactness in the space $L^p(R^+)$, there exists a sequence $\{\lambda_j\}$ $\left(\lim\limits_{j\to\infty}\lambda_j = \infty\right)$ and a function

$q_e^{[m]}(x) \in L^p(R^+)$ such that

$$\lim_{\lambda\to\infty} \int_0^\infty e^{-\varepsilon x} g(x)\lambda_j^m \left[Q_{\lambda_j}^{[m],e}(f;x) - f(x)\right]\, dx$$

$$= \int_0^\infty g(x) q_e^{[m]}(x)dx, \qquad (12)$$

for each $g(t) \in L^p(R^+)$, $pq = p + q$ ($\varepsilon > 0$ is any number), where $Q_{\lambda_j}^{[m],e}(f;t)$ is determined from formula (12). Then, it is obvious that $h(t) = e^{\varepsilon t} q_e^{[m]}(t)$.

Now, if in equality (12) instead of $g(x)$ we take the function of the form

$$g(x) = e^{-[(\sigma-\varepsilon)+iv]x} \ \ (\sigma > 0,\ -\infty < \tau < \infty),$$

we have

$$\lim_{\lambda_j\to\infty} \lambda_j^m \int_0^\infty e^{-sx}[Q_{\lambda_j}^{[m],e}(f;x) - f(x)]dx = \int_0^\infty e^{-sx} g(x)dx \ \ (Res > 0).$$

Hence, by the condition $D_m(\alpha(e), B(e))$, we find $B(e)s^m f^\wedge(s) = g^\wedge(s)$, i.e. $f(x) \in b_p(s^v; f)$. This is the required relation for $1 < p < \infty$. In the case $p = \infty$, the required one is proved in the same way.

Theorem 3. *Let the kernel $K_{\lambda,n}(t) = \lambda K_n(\lambda t)$ $(n = \overline{1,e})$ satisfy the condition $D_m(\alpha(e), B(e))$ and be such that the function*

$$\theta_{m,e}\left(\frac{s}{\lambda}\right) = \frac{1}{B(e)}\left(\frac{s}{\lambda}\right)^{-m}\left[\prod_{n=1}^{e} K_n^\wedge\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1}\alpha_v(e)\left(\frac{s}{\lambda}\right)^v\right],$$

is the Laplace-Stielties transformation of the normalized function with bounded variation on R^+ for $p = 1$ or the Laplace transformation of functions in the space $L(R^+)$ for $p > 1$. Then from $f(x) \in b_p(s^v; f)$ $(v = \overline{0,m})$, the relation

$$\left\| R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;x))...)) - \sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x)\right\|_{L_\sigma^p(R^+)} = O(\lambda^{-m})$$

is fulfilled as $\lambda \to \infty$.

Proof. Let $p = 1$. Since $f(x) \in b_1(s^v; f)$ $(v = \overline{0, m})$ and $\theta_{m,e}\left(\frac{s}{\lambda}\right)$ is a Laplace-Stielties transformation of the normalized function with bounded variation on R^+ i.e. $\theta_{m,e}\left(\frac{s}{\lambda}\right) = N^v_{m,e}\left(\frac{s}{\lambda}\right)$, $(N_{m,e}(x) \in BV(0, r))$.

$\int\limits_0^\infty e^{-\sigma x} |dN_{m,e}(x)| < \infty$, we apply Lemma A in [1] and find

$$\int\limits_0^\infty e^{-sx}\left[R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;x)...)) - \sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v}f^{(v)}(x)\right]\lambda^m dx$$

$$= \lambda^m f^\wedge(s)\left[\prod_{n=1}^{e} K_n^\wedge\left(\frac{s}{\lambda}\right) - \sum_{v=0}^{m-1}\alpha_v(e)\left(\frac{s}{\lambda}\right)^v\right]$$

$$= \overset{\vee}{\bigcup_{\lambda,m,e}}(s) \equiv \int\limits_0^\infty e^{-sx} d\bigcup_{\lambda,m,e}(x), \tag{13}$$

where

$$U_{\lambda,m,e}(x) = \int\limits_0^x h_1(x-t)dN_{m,e}(\lambda t), \quad U_{\lambda,m,e}(x) \in BV(0, r)$$

for each $r > 0$ and the relation

$$\int\limits_0^\infty e^{-\sigma x}|dU_{\lambda,m,e}(x)| \leq M_2 < \infty \tag{14}$$

is satisfied for all $\lambda > 0$.

According to the uniqueness of the Laplace-Stielties transformation ([2], p. 62), from (13) we have:

$$U_{\lambda,m,e}(x) = \lambda^m \int\limits_0^x \left[R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;t))...)) - \sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v}f^{(v)}(t)\right] dt.$$

Consequently, by (15) we find

$$\lambda^m \int_0^x e^{-\sigma x}|R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;x))...)) - \sum_{v=0}^{m-1}\frac{\alpha_v(e)}{\lambda^v}f^{(v)}(x)dx = 0(1)$$

as $\lambda \to \infty$. This is relation (4) for $p=1$. Let $1<p<\infty$. Since $f(x) \in b_p(s^v; f)$ and $\theta_{m,e}\left(\frac{s}{\lambda}\right)$ is a Laplace transformation of the function from the space $L(R^+)$, then ([4], p. 92), we have

$$\left[R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;x))...)) - \sum_{v=0}^{m-1} \frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x)\lambda^m\right]^{\wedge}(s)$$

$$= \left[\lambda B(e) \int_0^x f^{(m)}(x-u) h_{2,m,e}(\lambda u) du\right]^{\wedge}(s),$$

for $Res > 0$, where $Q_{m,e}\left(\frac{s}{\lambda}\right) = h^{\wedge}_{r,m,e}\left(\frac{s}{\lambda}\right), \quad (h_{r,m,e}(t) \in L(R^+))$.

By the theorem on uniqueness of the Laplace transformation ([6], p. 63), from the last relation we find

$$\left[R_{\lambda,1}(R_{\lambda,2}(...(R_{\lambda,e}(f;x))...)) - \sum_{v=0}^{m-1} \frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x)\right]\lambda^m$$

$$= \lambda B(e) \int_0^x f^{(m)}(x-t) h_{r,m,e}(\lambda t) dt. \tag{15}$$

Furthermore, by the Hölder inequality, we have

$$\left\|\lambda B(e) \int_0^x f^{(m)}(x-t) h_{2,m,e}(t) dt\right\|^p_{L^p_\sigma(R+)}$$

$$\le B(e) \cdot \|h_{r,m,e}(t)\|^{\frac{p}{q}+1}_{L(R^+)} \cdot \left\|f^{(m)}(x)\right\|^p_{L^p_\sigma(R^+)} = O(1)$$

as $\lambda \to \infty$.

Then, the validity of relation (4) follows from equality (15). Notice that for $p=\infty$ the reasonings are conducted similarly.

It follows from Theorem 2 that the family of operators (1) is saturated in the space $L^p_\sigma(R^+)$ $(p \ge 1)$ with order $O(\lambda^{-m})$ and $b_p(s^v; f)$ is a saturation class.

Apply the obtained results to one concrete linear operator:

$$p^{[m],e}_\lambda(f;x) = R_\lambda(R_\lambda(...(R_\lambda(f;x))...)) - \sum_{v=0}^{m-1} \frac{\alpha_v(e)}{\lambda^v} f^{(v)}(x), \tag{16}$$

where

$$R_\lambda(f;x) = \lambda \int_0^x f(x-t)e^{-\lambda t}dt$$

and

$$\alpha_v(e) = \frac{(-1)^v e(e+1)...(e+v-1)}{v!}.$$

It is easily verified that the operator (16) is saturated in the space $L^p_\sigma(R^+)$ $(p \geq 1)$ with order $O(\lambda^{-\nu})$ and $b_p(s^v; f)$ $(v = \overline{0,m})$, is its saturation order.

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