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EXPONENTIAL STABILITY OF DISCRETE NEURAL NETWORKS WITH NON-INSTANTANEOUS IMPULSES, DELAYS AND VARIABLE CONNECTION WEIGHTS WITH COMPUTER SIMULATION

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Abstract: The exponential stability concept for nonlinear non-instantaneous impulsive difference equations with a single delay is studied and some criteria are derived. These results are also applied for a neural networks with switching topology at certain moments and long time lasting impulses. It is considered the general case of time varying connection weights. The equilibrium is defined and exponential stability is studied. The obtained results are illustrated on examples.

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1. Introduction

One of the most important problems in the theory and application of differential and difference equations is stability. For the basis of the stability theory of difference equations with a delay or without any delay we refer to [1], [4], [6], [9], [12], [13], [14], [15], [16]. A good overview of the basic results and methods for stability investigations of linear autonomous difference equations is given in [7]. At the same time impulses are a very useful mathematical apparatus to model some instantaneous perturbations in the process (see, for example [5], [11]). Difference equations, being a discrete version of differential equations, could have also impulses (see, for example, [8]). In the case when the acting time of the impulses is not possible to be neglected, these impulses are called non-instantaneous impulses (for continuous case, see, [2], [3]).

In this paper we study nonlinear difference equations with a constant delay in the case there are some impulses starting at initially given points and acting on finite time intervals. By utilizing the Lyapunov stability theory and discrete-time Gronwall inequality, we establish some sufficient conditions for exponential stability of the zero solution.

Neural networks have received extensive interests in recent years in connection with their potential applications in signal processing, content addressable memory, pattern recognition, combinatorial optimization. It is well known that the existence of delays in neural networks causes undesirable complex dynamical behaviors such as instability, oscillation and chaotic phenomena. In practice, for computation convenience, continuous-time neural networks are often discretized to generate discrete-time neural networks. Thus, the study of discrete-time neural networks attracts more and more interests.

In this paper, we deal with a class of discrete-time neural networks with a constant delay subject to long time lasting impulsive perturbations. The basic characteristic of these perturbations is that the time of their action is not negligible small, so we consider the so called non-instantaneous impulses. We consider the general case when the connection weights between neurons are changeable in time. We apply the obtained theoretical results to obtain exponential stability criteria and new exponential convergence rate for non-instantaneous impulsive discrete-time neural networks with delays and variable connection weights.

The main contributions of the current paper include:

(i) Some exponential stability criteria for the zero solution of nonlinear difference equations with constant delay and non-instantaneous impulses are derived.

(ii) The equilibrium point for the discrete neural network with a delay, non-instantaneous impulses and time depending connection weights is defined.

(iii) Sufficient conditions for the equilibrium point of the discrete neural network with a delay, non-instantaneous impulses and variable in time connection weights are obtained.

(iv) Computer simulations of some theoretical results for particular discrete neural networks.

The rest of this paper is organized as follows. In Section 2, nonlinear difference equations with a constant delay and non-instantaneous impulses are introduced and some preliminary lemmas are presented. In Section 3, based on the Lyapunov stability theory and the discrete-time Gronwall inequality, exponential stability criteria of the zero solution are derived. In Section 4 discrete-time neural networks with delays and long time lasting impulsive perturbations is presented. It is studied the model with a time varying connection weights. According to our knowledge it is the first discrete model in which the connection weights are time depending. Equilibrium point of the studied model is defined. It is deeply connected not only with the difference equations but also with the impulsive conditions. Based on the obtained results in the previous section, some sufficient conditions for exponential stability of equilibrium point are derived. Moreover, numerical examples are presented in Section 5 and the results are discussed. Firstly, it is discussed and illustrated the meaning and usefulness of the definition of the equilibrium point. Second, several discrete neural networks are considered and the theoretical results are applied. The examples are computer realized by the help of Wolfram Mathematica. Following the theoretical schemes for solving the problems, the corresponding algorithms are coded to calculate the values of the solution for each step. The graphs are generated by Wolfram Mathematica.

2. Statement of the problem and definition of solution

We will introduce basic notations used in this paper. Most of them are well known and used in the literature. Let $\mathbb{Z}_+$ be the set of all nonnegative integers; the increasing sequence $\{n_i\}_{i=0}^{\infty}: \ n_0 = 0, \ n_i \in \mathbb{Z}_+, \ n_i \geq n_{i-1} + 3, i = 1, 2, \dots$ and the sequence $\{d_i\}_{i=1}^{\infty}: \ d_i \in \mathbb{Z}_+, \ 1 \leq d_i \leq n_{i+1} - n_i - 2, \ i = 1, 2, \dots$ be given; $\mathbb{Z}[a, b] = \{z \in \mathbb{Z}_+ : a \leq z \leq b\}, \ a, b \in \mathbb{Z}_+, \ a < b$, $\mathbb{Z}_a = \{z \in \mathbb{Z}_+ : z \geq a\}$ and

$$I_k = \mathbb{Z}[n_k + d_k + 1, n_{k+1} - 1],\ k \in \mathbb{Z}_+,$$

and

$$J_k = \mathbb{Z}[n_k, n_k + d_k],\ k \in \mathbb{Z}_1,$$

where $d_0 = 0$.

Consider the *initial value problem (IVP)* for the system of nonlinear *difference equation with non-instantaneous impulses*

$$\begin{aligned} x(n) &= Ax(n-1) + F(n, x(n-m)) \text{ for } n \in \bigcup_{k=0}^{\infty} I_k, \\ x(n) &= P_k(n, x(n_k - 1)), \text{ for } \ n \in J_k,\ k \in \mathbb{Z}_1, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$, A is $N \times N$ square matrix, $F = (F_1, F_2, \dots, F_N)$, $F_i : \bigcup_{k=0}^{\infty} I_k \times \mathbb{R}^N \to \mathbb{R}$, $P_k = (P_{k,1}, P_{k,2}, \dots, P_{k,N})$, $P_{k,i} : J_k \times \mathbb{R}^N \to \mathbb{R}$, $i = 1, 2, \dots, N$, $k = 1, 2, \dots,$ m is a natural number.

The Cauchy problem for the system (1) is defined by the state of the system on the whole interval of the solutions previous history

$$x(k) = x_k^0,\ k = -m+1, \dots, 0, \tag{2}$$

where $x_k^0 \in \mathbb{R}^N$.

Note in the case $m = 1$ we have a system of difference equations without any delay. That is why we will assume $m \geq 2$.

Denote by $\mathbb{M}_N$ the set of all quadratic $N \times N$ dimensional matrices with the spectral norm $|A| = \sqrt{\lambda_{max}(A^T A)}$, and for any vector $x \in \mathbb{R}^N$ we will use the norm $|x| = \sqrt{\sum_{i=1}^{N} x_i^2}$. Moreover, denote by $\lambda_{min}(A)$ and $\lambda_{max}(A)$ the minimum and the maximum eigenvalue of a positive definite symmetric matrix A and $\phi(A) = \frac{\lambda_{max}(A)}{\lambda_{min}(A)}$.

Description: The state of the corresponding process described by (1) starts from the initial point $(0, x_0^0)$ and it is changing on $I_0 = \mathbb{Z}[1, n_1 - 1]$ according to the first difference equation of (1). At time n_1 the state has an abrupt change (impulse) given by the second equation of (1). The time of this impulse is not reasonable negligible small and the impulse acts on the interval $J = \mathbb{Z}[n_1, n_1 + d_1]$. Then on I_1 the state changes according to the first equation of (1) and so on.

Usually, the difference equation describes the development of a certain phenomenon by recursively defining a sequence, each of whose terms is defined as a function of the preceding terms, once one or more initial terms are known (see, for example, [10]). Differently than that, we consider a difference equation in which the present state is also involved nonlinearly in the right side part.

It makes the answer of the question about the existence of the solution more complicated.

Definition 1. ([14]) The trivial solution of the system (1) is called globally exponentially stable if there exist constants $N > 0$ and $\alpha \in (0, 1)$ such that for any initial value x^0 the inequality

$$|x(n)| \leq N\alpha^n ||x^0||_m, \quad n = 1, 2, \dots$$

holds.

The constant α is called the exponential convergence rate.

In the paper, we investigate the exponential stability of (1) by the second Lyapunov method. We will apply finite-dimensional Lyapunov functions.

Consider the Lyapunov equation

$$A^T HA - H = -C, \tag{3}$$

where $A, H, C \in \mathcal{M}_N$.

3. Exponential stability of linear delay discrete equations

We will study the exponential stability of the linear system (1).

Theorem 1. (Exponential stability results) *Let*

1. *The matrix $A \in \mathbb{M}_N$ and $C \in \mathbb{M}_n$ be a positive definite matrix.*
2. *The function $F \in C(\mathbb{Z}_+ \times \mathbb{R}^N, \mathbb{R}^N)$, $F(n, 0) = 0$ for any $n \in \mathbb{Z}_+$ and there exists a constant $K > 0$ such that $|F(n, u)| \leq \sqrt{K}|u|$ for any $u \in \mathbb{R}^N$ and $n \in \mathbb{Z}_+$.*
4. *The functions $P_k \in C(\mathbb{Z}_+ \times \mathbb{R}^N, \mathbb{R}^N)$, $P_k(0) = 0$, $k = 1, 2, \dots$ and there exist constants $M_k > 0$ such that $|P_k(n, u)| \leq \sqrt{M_k}|u|$ for any $u \in \mathbb{R}^N$, $n \in J_k$ and $k = 1, 2, \dots$.*
5. *There exists a solution $H \in \mathbb{M}_n$ of the Lyapunov matrix equation (3) such that*

$$|H| \; M_k < 1, \quad \text{for} \quad k = 1, 2, \dots,$$

$$L_1(H) - L_2(H) < \lambda_{max}(H) - \lambda_{min}(H),$$

and

$$\phi(H)\Big(|A^TH|+|HA|+K\lambda_{max}(H)\Big)<\lambda_{min}(C)$$
$$<\lambda_{max}(H)+0.5\phi(H)(|A^TH|+|HA|),$$

where

$$L_1(H)=\lambda_{max}(H)-\lambda_{min}(C)+0.5\phi(H)(|A^TH|+|HA|),$$
$$L_2(H)=\lambda_{min}(H)-\phi(H)K\Big(\lambda_{max}(H)+0.5|HA|+0.5|A^TH|\Big)\Big).$$

Then the zero solution of (1) is exponentially stable.

Proof.Denote
$$\Theta=\max\Big\{\Lambda,\frac{L_1-L_2+\lambda_{min}(H)}{\lambda_{max}(H)}\Big\}<1,$$
where $\Lambda=\sup_{k\geq 1}|M_k^THM_k|$.

Consider the function $V(x)=x^THx$ for $x\in\mathbb{R}^N$. It is obviously, that
$$\lambda_{min}(H)|x|^2\leq V(x)\leq\lambda_{max}(H)|x|^2.$$

Let $x(n),\ n\in\mathbb{Z}[-m+1,\infty)$, be a solution of the IVP (1), (2).

Let $n\in\bigcup_{k=0}^p I_k$. Then we have

$$\begin{aligned}
&V(x(n))-V(x(n-1))\\
&=x^T(n)Hx(n)-x^T(n-1)Hx(n-1)\\
&=(Ax(n-1)+F(n,x(n-m)))^TH(Ax(n-1)\\
&\quad+F(n,x(n-m)))-x^T(n-1)Hx(n-1)\\
&=(x^T(n-1)A^T+F^T(n,x(n-m)))H(Ax(n-1)\\
&\quad+F(n,x(n-m)))-x^T(n-1)Hx(n-1)\\
&=-x^T(n-1)Cx(n-1)+x^T(n-1)A^THF(n,x(n-m))\\
&\quad+F^T(n,x(n-m))HAx(n-1)\\
&\quad\ +F^T(n,x(n-m))HF(n,x(n-m))\\
&\leq(-\lambda_{min}(C)+0.5|A^TH|+0.5|HA|)|x(n-1)|^2\\
&\quad+(\lambda_{max}(H)+0.5|HA|+0.5|A^TH|)K|x(n-m)|^2\\
&\leq-\lambda_{min}(C)|x(n-1)|^2+|A^THB|\ |x(n-1)|^2\\
&\quad+\Big(|A^THB|+|B^THB|\Big)|x(n-m)|^2.
\end{aligned}\tag{4}$$

Apply the inequalities

$$-|x(n-1)|^2 \leq -\frac{V(x(n-1)}{\lambda_{max}(H)}, \qquad |x(n-m)|^2 \leq \frac{V(x(n-m)}{\lambda_{min}(H)}$$

to (4) and obtain

$$\begin{aligned} V(x(n)) &\leq (1 - \frac{\lambda_{min}(C)}{\lambda_{max}(H)} + 0.5\frac{|A^T H| + |HA|}{\lambda_{min}(H)})V(x(n-1)) \\ &\quad + (\lambda_{max}(H) + 0.5|HA| + 0.5|A^T H|)\frac{KV(x(n-m))}{\lambda_{min}(H)} \\ &\leq (1 - \frac{\lambda_{min}(C)}{\lambda_{max}(H)} + \frac{|A^T HB|}{\lambda_{min}(H)})V(x(n-1)) \\ &\quad + \Big(|A^T HB| + |B^T HB|\Big)\frac{1}{\lambda_{min}(H)}V(x(n-m)). \end{aligned} \tag{5}$$

From equalities

$$\begin{aligned} &1 - \frac{\lambda_{min}(C)}{\lambda_{max}(H)} + 0.5\frac{|A^T H| + |HA|}{\lambda_{min}(H)} \\ &= \frac{1}{\lambda_{max}(H)}(\lambda_{max}(H) - \lambda_{min}(C) + 0.5\phi(H)(|A^T H| + |HA|)) \end{aligned}$$

and

$$\begin{aligned} &\Big(\lambda_{max}(H) + 0.5|HA| + 0.5|A^T H|\Big)\frac{K}{\lambda_{min}(H)} \\ &= \phi(H)\Big(\lambda_{max}(H) + 0.5|HA| + 0.5|A^T H|\Big)\frac{K}{\lambda_{max}(H)} \end{aligned}$$

and inequality (5) we get

$$\begin{aligned} V(x(n)) &\leq \frac{L_1(H)}{\lambda_{max}(H)}V(x(n-1)) \\ &\quad + \frac{1}{\lambda_{max}(H)}\Big(\lambda_{min}(H) - L_2(H)\Big)V(x(n-m)). \end{aligned} \tag{6}$$

Let $n = 1$. Then from inequality (6) we obtain

$$\begin{aligned} V(x(1)) &\leq \frac{L_1(H)V(x(0))}{\lambda_{max}(H)} + \frac{\lambda_{min}(H) - L_2(H)}{\lambda_{max}(H)}V(x(-m+1)) \\ &\leq \frac{1}{\lambda_{max}(H)}\Big(L_1(H) - L_2(H) + \lambda_{min}(H)\Big)v_0 \\ &\leq \Theta(H)v_0 < \sqrt[1+m]{\Theta(H)}v_0. \end{aligned} \tag{7}$$

Let $n = 2$. Then from inequality (6) we get

$$\begin{aligned} V(x(2)) &\leq \frac{L_1(H)}{\lambda_{max}(H)} V(x(1)) \\ &\quad + \frac{\lambda_{min}(H) - L_2(H)}{\lambda_{max}(H)} V(x(2-m)) \\ &\leq \frac{L_1(H)}{\lambda_{max}(H)} \sqrt[1+m]{\Theta(H)} v_0 + \frac{\lambda_{min}(H) - L_2(H)}{\lambda_{max}(H)} v_0 \\ &\leq \frac{L_1(H) \sqrt[1+m]{\Theta(H)} - L_2(H) + \lambda_{min}(H)}{\lambda_{max}(H)} v_0 \\ &\leq \Theta(H) v_0 < \sqrt[1+m]{\Theta^2(H)} v_0. \end{aligned} \tag{8}$$

Consider the following two possible cases:

Case 1. Let $m \geq n_1 - 1$. Then using induction, the inequalities $\Theta < \sqrt[p]{\Theta}$ for $p > 1$, $n \leq n_1 - 1 < m + 1$, i.e. $\frac{m+1}{n} > 1$ for $n \in I_0$ and inequality (6), we prove that

$$V(x(n)) \leq \sqrt[1+m]{\Theta^n(H)} v_0, \quad \text{for} \;\; n \in I_0 = \mathbb{Z}[1, n_1 - 1].$$

Case 2. Let $m < n_1 - 1$. Then using induction and inequality (6) we prove that

$$V(x(n)) \leq \sqrt[1+m]{\Theta^n(H)} v_0, \text{ for } n = 1, 2, \ldots, m.$$

Then,

$$\begin{aligned} &V(x(m+k)) \\ &\leq \frac{L_1(H)}{\lambda_{max}(H)} V(x(m)) + \frac{\lambda_{min}(H) - L_2(H)}{\lambda_{max}(H)} V(x(m+k-m)) \\ &\leq \frac{L_1(H)}{\lambda_{max}(H)} \sqrt[1+m]{\Theta(H)^m} v_0 + \frac{\lambda_{min}(H) - L_2(H)}{\lambda_{max}(H)} \sqrt[1+m]{\Theta^k(H)} v_0 \\ &\leq \Theta(H) \sqrt[1+m]{\Theta^k(H)} v_0 < \sqrt[1+m]{\Theta^{m+k}(H)} v_0, \\ &\qquad \text{for } k = 1, 2, \ldots, n_1 - 1 - m. \end{aligned} \tag{9}$$

For $n = n_1$ apply the inequalities $1 + \frac{n_1 - 1}{m+1} = \frac{m+n_1}{m+1} > \frac{n_1}{m+1}$ and $\Theta^{\frac{m+n_1}{m+1}} < \Theta^{\frac{n_1}{m+1}}$ to (9) and obtain

$$\begin{aligned} V(x(n_1)) &= x^T(n_1)Hx(n_1) \\ &= (P_1(x(n_1-1)))^T HP_1(x(n_1-1)) \\ &\le |H|\ |P_1(x(n_1-1))|^2 \le |H|M_1\ |x(n_1-1)|^2 \\ &\le |H|M_1 \frac{V(x(n_1-1))}{\lambda_{min}(H)} \\ &\le (|H|M_1)\ \sqrt[1+m]{\Theta^{n_1-1}(H)}v_0 \\ &\le \Theta\ \sqrt[1+m]{\Theta^{n_1-1}}v_0 = \sqrt[1+m]{\Theta^{n_1+m}}v_0 < \sqrt[1+m]{\Theta^{n_1}}v_0. \end{aligned} \tag{10}$$

For $n = n_1 + 1$ apply the inequalities $1 + \frac{n_1}{m+1} = \frac{m+n_1+1}{m+1} > \frac{n_1+1}{m+1}$ and $\Theta^{\frac{m+n_1+1}{m+1}} < \Theta^{\frac{n_1+1}{m+1}}$ to (9) and get

$$V(x(n_1+1)) \le \Theta\ \sqrt[1+m]{\Theta^{n_1}}v_0 < \sqrt[1+m]{\Theta^{n_1+1}}v_0. \tag{11}$$

By induction we prove that

$$V(x(n_1+k)) < \sqrt[1+m]{\Theta^{n_1+k}}v_0, \quad k = 0, 1, \ldots, d_1.$$

Let $n = n_1 + d_1 + 1$. Then using the inequalities $n_1 + d_1 + 1 - m > 0$ and (6) we get

$$\begin{aligned} V(x(n_1+d_1+1)) &\le \frac{L_1(H)}{\lambda_{max}(H)}V(x(n_1+d_1)) \\ &\quad + \frac{1}{\lambda_{max}(H)}\Big(\lambda_{min}(H) - L_2(H)\Big)V(x(n_1+d_1+1-m)) \\ &\le \frac{1}{\lambda_{max}(H)}L_1(H)\ \sqrt[1+m]{\Theta^{n_1+d_1}}v_0 \\ &\quad + \frac{1}{\lambda_{max}(H)}\Big(-L_2(H) + \lambda_{min}(H)\Big)\ \sqrt[1+m]{\Theta^{n_1+d_1+1-m}}v_0 \\ &\le \frac{1}{\lambda_{max}(H)}\Big(L_1(H) - L_2(H) + \lambda_{min}(H)\Big)\Theta^{\frac{n_1+d_1+1-m}{m+1}}v_0 \\ &= \Theta^{\frac{n_1+d_1+2}{m+1}}v_0 < \Theta^{\frac{n_1+d_1+1}{m+1}}v_0. \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} V(x(n_1+d_1+2)) &\le \Theta\Theta^{\frac{n_1+d_1+2-m}{m+1}}v_0 = \Theta^{\frac{n_1+d_1+3}{m+1}}v_0 \\ &< \Theta^{\frac{n_1+d_1+2}{m+1}}v_0. \end{aligned} \tag{13}$$

By induction process we prove the validity of the inequality

$$V(x(n)) < \Theta^{\frac{n}{m+1}} v_0 \qquad \text{for all } n \in \mathbb{Z}_1. \tag{14}$$

Therefore, by $v_0 \leq \lambda_{max}(H)|x^0|_m^2$ and inequality (14) we get

$$|x(n)| < N|x^0|_m \alpha^n, \qquad \text{for all } n \in \mathbb{Z}_1$$

with $\alpha = \sqrt[2(m+1)]{\Theta} < 1$, and $N = \sqrt{\phi(H)}$. □

4. Exponential stability of delay discrete neural networks with non-instantaneous impulses and time variable connection weights

We will apply the ideas in the previous section to study the exponential stability of discrete neural networks with a delay. We will consider the case when the connections of the network are subject to a long lasting impulses at certain moments which will be modeled by the so-called non-instantaneous impulses. Also, we will study the general case of time variable connection weights.

Consider the following model which is a generalization of the studied one in [7]

$$\begin{aligned} u_i(n) &= a_i u_i(n-1) + \sum_{j=1}^{n} \Psi_{ij}(n) f_j(u_j(n-m)) + G_i \\ &\qquad \text{for } n \in \bigcup_{k=0}^{\infty} I_k, i = 1, 2, \dots, N, \\ u_i(n) &= M_{ik} u_i(n_k - 1) + \sum_{j=1}^{n} \Phi_{ij}^k(n) S_j(u_j(n_k - 1)) + Q_{ik}, \\ &\qquad \text{for } n \in J_k,\ k \in \mathbb{Z}_1, \\ u_i(n) &= \phi_i(n), \quad n \in \mathbb{Z}[-m, 0], \end{aligned} \tag{15}$$

where $u_i(n)$ denotes the state of the i-th neuron at discrete time n, a_i, $i \in \mathbb{Z}[1, N]$, represents the passive decay rate, f_j is the neuron activation function with $f_j(0) = 0$, G_i is the exogenous input, P_j is the neuron output signal function which is a continuous function, $\Psi_{ij}(n)$ and $\Phi_{ij}(n)$ denote the connection weight from the neuron j to the neuron i at time n, $m \in \mathbb{Z}(1)$ is the transmission delay, $\phi_i(n), n \in \mathbb{Z}[-m, 0]$ is the initial function for the i-th neuron.

Let us denote $f(u) = (f_1(u_1), f_2(u_2), \dots, f_N(u_N))$, $S(u) = (S_1(u_1), S_2(u_2), \dots, S_N(u_N))$ with $u = (u_1, u_2, \dots, u_N)$.

We will introduce the following assumptions:

A1. The functions $f_i \in C(\mathbb{R},\mathbb{R})$ and there exist positive constants L_i, $i = 1, 2, \dots, N$, such that $|f_i(u) - f_i(v)| \leq L_i|u - v|,\ u, v \in \mathbb{R}$.

A2. The functions $S_i \in C(\mathbb{R},\mathbb{R})$ and there exist positive constants K_i, $i = 1, 2, \dots, N$, such that $|S_i(u) - S_i(v)| \leq K_i|u - v|,\ u, v \in \mathbb{R}$.

A3. The functions $\Psi_{ij} : \bigcup_{k=0}^{\infty} I_k \to \mathbb{R}$ and $\Phi_{ij}^k : J_k \to \mathbb{R}$ are bounded, i.e. there exists constants $\beta_{ij}^k > 0, \gamma_{ij} > 0$ such that $|\Psi_{ij}(n)| \leq \beta_{ij}$ for $n \in \bigcup_{k=0}^{\infty} I_k$ and $|\Phi_{ij}^k(n)| \leq \gamma_{ij}^k$ for $n \in J_k,\ k = 1, 2, \dots$ and $i, j = 1, 2 \dots, N$.

In the non-homogeneous case we will define an equilibrium of the model (15).

Definition 2. A vector $u^* \in \mathbb{R}^N : \ u^* = (u_1^*, u_2^*, \dots, u_N^*)$ is said to be an equilibrium point of the impulsive discrete-time neural network (15) if it satisfies the equalities

$$\begin{aligned} u_i^* &= a_i u_i^* + \sum_{j=1}^{N} \Psi_{ij}(n) f_j(u_j^*) + G_i \\ &\qquad \text{for } n \in \bigcup_{k=0}^{\infty} I_k, i = 1, 2, \dots, N, \\ u_i^* &= M_{ik} u_i^* + \sum_{j=1}^{N} \Phi_{ij}^k(n) S_j(u_j^*) + Q_{ik}, \ \text{for} \ \ n \in J_k,\ k \in \mathbb{Z}_1, \end{aligned} \tag{16}$$

Remark 1. In the case of instantaneous impulses, i.e. $d_k = 0$, $k = 1, 2, \dots,$ we have the model of neural networks with impulses. For example, in [17] a neural network with impulses and constant weights is studied but the definition of the equilibrium point is not connected with the impulses. This is changing the meaning of the equilibrium (see Examples 1 and 2).

Let (15) have an equilibrium $u^* \in \mathbb{R}^N$. Substitute $x = u - u^* \in \mathbb{R}^N$ in (15) and obtain

$$
\begin{aligned}
x_i(n) &= a_i x_i(n-1) + \sum_{j=1}^{N} \Psi_{ij}(n)\mathcal{F}_j(x_j(n-\tau)) \\
&\text{for } n \in \bigcup_{k=0}^{\infty} I_k, \\
x_i(n) &= M_{ik} x_i(n_k-1)) + \sum_{j=1}^{N} \Phi_{ij}^{k}(n) r_j(x_j(n_k-1)), \\
&\text{for } \ n \in J_k, \ k \in \mathbb{Z}_1,
\end{aligned}
\tag{17}
$$

where $\mathcal{F}_i(y) = f_i(y-u_i^*) - f_i(u_i^*)$ and $r_i(y) = S_i(y-u_i^*) - S_i(u_i^*), \ i = 1, 2, \ldots, N$ for $y \in \mathbb{R}$.

Remark 2. The stability behavior of the equilibrium of (15) is equivalent to the stability behavior of zero solution of (17).

The system (17) could be written in the matrix form (1), where

$$
A = \begin{bmatrix} a_1 & 0 & 0 & \ldots\ldots & 0 \\ 0 & a_2 & 0 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & a_N \end{bmatrix},
$$

$$
\mathcal{B}(n) = \begin{bmatrix} \Psi_{11}(n) & \Psi_{12}(n) & \Psi_{13}(n) & \ldots\ldots & \Psi_{1N}(n) \\ \Psi_{21}(n) & \Psi_{22}(n) & \Psi_{23}(n) & \ldots\ldots & \Psi_{2N}(n) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \Psi_{N1}(n) & \Psi_{N2}(n) & \Psi_{N3}(n) & \ldots\ldots & \Psi_{NN}(n) \end{bmatrix},
$$

$$
\mathcal{F}(u) = \begin{bmatrix} \mathcal{F}_1(u_1) \\ \mathcal{F}_2(u_2) \\ \ldots \\ \mathcal{F}_N(u_N) \end{bmatrix}, \qquad r(u) = \begin{bmatrix} r_1(u_1) \\ r_2(u_2) \\ \ldots \\ r_N(u_N) \end{bmatrix},
$$

$$
\mathcal{M}_k = \begin{bmatrix} M_{1k} & 0 & 0 & \ldots\ldots & 0 \\ 0 & M_{2k} & 0 & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots & M_{Nk} \end{bmatrix},
$$

$$\mathcal{G}_k(n) = \begin{bmatrix} \Phi^k_{11}(n) & \Phi^k_{12}(n) & \gamma^k_{13}(n) & \dots\dots & \Phi^k_{1N}(n) \\ \Phi^k_{21}(n) & \Phi^k_{22}(n) & \Phi^k_{23}(n) & \dots\dots & \Phi^k_{2N}(n) \\ \dots & \dots & \dots & \dots & \dots \\ \Phi^k_{N1}(n) & \Phi^k_{N2}(n) & \Phi^k_{N3}(n) & \dots\dots & \Phi^k_{NN}(n) \end{bmatrix},$$

$u = (u_1, u_2, \dots, u_N)$, $F = (F_1, F_2, \dots, F_N)$, $F(n, u) = \mathcal{B}(n)\mathcal{F}(u)$, $P_k = (P_{k,1}, P_{k,2}, \dots, P_{k,N})$, $P_k(n, u) = \mathcal{G}_k(n)r(u) + \mathcal{M}_k u^T$.

From assumption (A1) and the inequality

$$\left(\sum_{j=1}^N \gamma_j u_j\right)^2 \le N \sum_{j=1}^N (\gamma_j u_j)^2$$

we have

$$\begin{aligned} |F(n,u)| &\le \sqrt{\sum_{i=1}^N \Big(\sum_{j=1}^N \beta_{ij}\mathcal{F}_j(u_j)\Big)^2} \\ &= \sqrt{\sum_{i=1}^N \Big(\sum_{j=1}^N \beta_{ij}(f_j(u_j - u_j^*) - f_j(u_j^*))\Big)^2} \\ &\le \sqrt{\sum_{i=1}^N N \sum_{j=1}^N \beta_{ij}^2 (f_j(u_j - u_j^*) - f_j(u_j^*))^2} \\ &\le \sqrt{N \sum_{i=1}^N \sum_{j=1}^N \beta_{ij}^2 L_j^2 u_j^2} \\ &\le \sqrt{N \sum_{i=1}^N \max_j (L_j \beta_{ij})^2} \sqrt{\sum_{j=1}^N u_j^2}, \end{aligned} \tag{18}$$

i.e. the condition 3 of Theorem 1 is satisfied with

$$K = N \sum_{i=1}^N \max_j (L_j \beta_{ij})^2.$$

From assumption (A2) and the inequality

$$\Big(\sum_{j=1}^N \gamma_j u_j\Big)^2 \le N \sum_{j=1}^N (\gamma_j u_j)^2$$

we have

$$
\begin{aligned}
|P_k(n,u)| &= \sqrt{\sum_{i=1}^{N}\Big(P_{k,i}(n,u)\Big)^2} \\
&\le \sqrt{\sum_{i=1}^{N}\Big(\sum_{j=1}^{N}\gamma_{ij}^k r_j(u_j) + M_{ik}u_i\Big)^2} \\
&= \sqrt{\sum_{i=1}^{N}\Big(\sum_{j=1}^{N}\gamma_{ij}^k (S_j(u_j-u_j^*) - S_j(u_j^*)) + M_{ik}u_i\Big)^2} \\
&\le \sqrt{2\sum_{i=1}^{N}\Big(\sum_{j=1}^{N}\gamma_{ij}^k (S_j(u_j-u_j^*) - S_j(u_j^*))\Big)^2 + 2\sum_{i=1}^{N}M_{ik}^2u_i^2} \\
&\le \sqrt{2\sum_{i=1}^{N}N\sum_{j=1}^{N}\Big(\gamma_{ij}^k (S_j(u_j-u_j^*) - S_j(u_j^*))\Big)^2 + 2\sum_{i=1}^{N}M_{ik}^2u_i^2} \\
&\le \sqrt{2\sum_{i=1}^{N}N\sum_{j=1}^{N}\Big(\gamma_{ij}^k K_j\Big)^2 u_j^2 + 2\sum_{i=1}^{N}M_{ik}^2u_i^2} \\
&\le \sqrt{2\Big(\max_i M_{ik}^2 + \sum_{i=1}^{N}N\max_j\Big(\gamma_{ij}^k K_j\Big)^2\Big)\sum_{i=1}^{N}u_i^2},
\end{aligned}
\tag{19}
$$

i.e. the condition 4 of Theorem 1 is satisfied with

$$M_k = 2\Big(\max_i M_{ik}^2 + \sum_{i=1}^{N}N\max_j\big(\gamma_{ij}^k K_j\big)^2\Big).$$

Theorem 2. *Let the conditions (A1)–(A3) be satisfied and:*

1. *The discrete model (15) has an equilibrium* $u^*(n): N_{-m+1} \to \mathbb{R}^n$.
2. *The constants* $a_i, M_{ik} \in \mathbb{R},\ \beta_{ij}, \gamma_{ij}^k > 0,\ G_i, Q_{ik} \in \mathbb{R}, i,j = 1,2,\dots,N,\ k \in \mathbb{Z}_1$.
3. *The inequalities*

$$\max_i M_{ik}^2 + N\sum_{i=1}^{N}\max_j\big(\gamma_{ij}^k K_j\big)^2 < 1, \quad \textit{for}\ \ k = 1,2,\dots,$$

$$\max_i a_i^2 + \max_i |a_i| + N(1 - \max_i |a_i|) \sum_{i=1}^{N} \max_j (L_j \beta_{ij})^2) < 1,$$

and

$$\max_i a_i^2 + \max_i |a_i| + K\Big(1 + \max_i |a_i|\Big) < 1.$$

Then the equilibrium point of the difference neural network with non-instantaneous impulses (15) is exponentially stable with a rate $\alpha = \sqrt[2(m+1)]{\Theta} < 1$, *where*

$$\Theta = \max\Big\{\Lambda, \max_i a_i^2 + \max_i |a_i| + K\Big(1 + \max_i |a_i|\Big)\Big\} < 1$$

with $\Lambda = \sup_{k \geq 1} \Big(\max_i M_{ik}^2 + \sum_{i=1}^{N} N \max_j \big(\gamma_{ij}^k K_j\big)^2\Big)$.

Proof.Let $H = E$, where $E \in \mathbb{M}_n$ is the unit matrix. Then the matrix equation (3) is satisfied with $C \in \mathbb{M}_n : \ c_{ii} = 1 - a_i^2$ and $c_{ij} = 0$ for $i \neq j$. Then $\lambda_{max}(H) = \lambda_{min}(H) = \phi(H) = 1$ and $\lambda_{min}(C) = 1 - \max_i a_i^2$, $|A| = \max_i |a_i|$. According to Theorem 1 the zero solution of (17) is exponentially stable. Therefore, the equilibrium u^* of (15) is exponentially stable. □

5. Application

In this section we will consider some particular examples to illustrate the obtained results in the paper. The examples are computer realized by the help of Wolfram Mathematica. Following the theoretical schemes for solving the problems, the corresponding algorithms are coded to calculate the values of the solution for each step. The graphs are generated by Wolfram Mathematica.

First we will illustrate the importance of Definition 2 and the meaningful connection of the equilibrium with both the difference equation and the impulsive equations.

Example 1. (*Neural networks with impulses*) Consider a system with three agent with constant connection weights modeled by the following discrete model of neural network with impulses

$$\begin{aligned} u_1(n) &= \frac{1}{2}u_1(n-1) + \frac{1}{8}\sin(u_1(n-3)) - \frac{1}{4}\sin(u_2(n-2)) \\ &\quad + \frac{1}{16}u_3(n-2) + 1 \\ u_2(n) &= \frac{1}{3}u_2(n-1) + \frac{1}{4}\sin(u_1(n-3)) \\ &\quad + \frac{1}{8}\sin(u_2(n-2)) + 2 \\ u_3(n) &= \frac{1}{4}u_3(n-1) + \frac{1}{16}\sin(u_1(n-3)) - \frac{1}{8}\sin(u_2(n-2)) \\ &\quad + \frac{1}{16}u_3(n-2) + 1 \text{ for } n \neq 4m+1, \end{aligned} \tag{20}$$

with impulsive conditions

$$\begin{aligned} u_1(4m+1) &= \frac{1}{3}u_1(4m), \\ u_2(4m+1) &= \frac{1}{3}u_2(4m), \\ u_3(4m+1) &= \frac{1}{3}u_3(4m), \quad m = 1, 2, 3, \dots. \end{aligned} \tag{21}$$

and initial conditions

$$\begin{aligned} u_1(n) &= -n^3 + 2, \\ u_2(n) &= -n^3 + 2, \\ u_3(n) &= -n^3 + 2, \ \text{ for } \ n = -2, -1, 0. \end{aligned} \tag{22}$$

If we use the definition for the equilibrium which is given in [17], then the model (20), (21) has an equilibrium point $u^* = (2.4057, 3.2344, 1.5324)$. Note this point does not satisfy the impulsive conditions (21). From the graph of the solution (see Figure 1) it could be seen the equilibrium is not exponentially stable. Therefore, the obtaining of the equilibrium point only by the difference equations (20) (as it is defined in [17]) is not appropriate. □

Example 2. (*Neural networks with impulses*) Consider the same discrete model of neural network with impulses (20) but we will change the impulsive conditions and we will use Definition 2 to obtain the equilibrium point, i.e. we consider the impulsive conditions

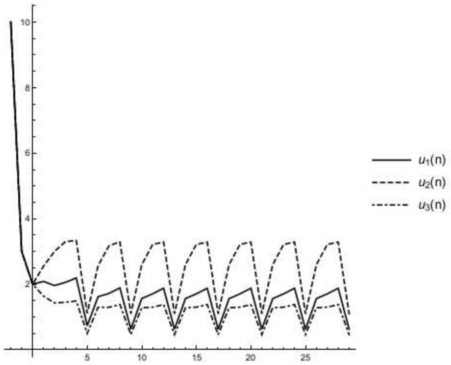

Figure 1. Graph of the solution of (20), (21).

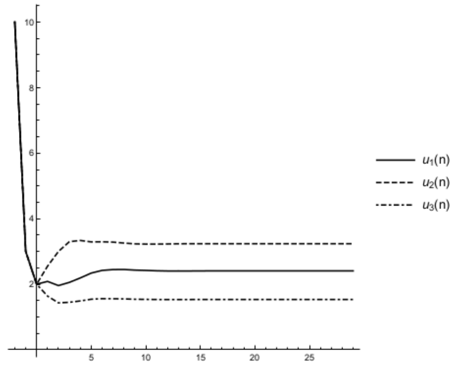

Figure 2. Graph of the solution of (20), (23).

$$
\begin{aligned}
u_1(4m+1) &= \frac{1}{2}u_1(4m) + \frac{1}{8}\sin(u_1(4m)) - \frac{1}{4}\sin(u_2(4m)) \\
&\quad + \frac{1}{16}u_3(4m) + 1, \\
u_2(4m+1) &= \frac{1}{3}u_2(4m) + \frac{1}{4}\sin(u_1(4m)) + \frac{1}{8}\sin(u_2(4m)) + 2, \\
u_3(4m+1) &= \frac{1}{4}u_3(4m) + \frac{1}{16}\sin(u_1(4m)) - \frac{1}{8}\sin(u_2(4m)) \\
&\quad + \frac{1}{16}u_3(4m) + 1, \quad m = 1, 2, 3, \dots .
\end{aligned} \tag{23}
$$

The neural network model (20), (23) has an equilibrium point $u^* = (2.4057, 3.2344, 1.5324)$ defined by Definition 2.

In this particular case we have $K_i = L_i = 1,\ i = 1, 2, 3,$ and

$$
A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \qquad \mathcal{B} = \mathcal{G}_k = \begin{bmatrix} \frac{1}{8} & -\frac{1}{4} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{16} & -\frac{1}{8} & \frac{1}{16} \end{bmatrix}, \qquad M_k = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}.
$$

The conditions of Theorem 2 are reduced to $0.25 + 3(\frac{1}{16} + \frac{1}{16} + \frac{1}{64}) = 0.680556 < 1$, and $0.25+0.5+3(1-0.5)(\frac{1}{16}+\frac{1}{16}+\frac{1}{64}) = 0.965278 < 1$. According to Theorem 2 with $d_k = 0,\ k \in \mathbb{Z}_1$ the equilibrium $u^* = (1.5786, 1.8355, 1.2322)$ of (20), (23) is exponentially stable with a rate $\alpha = \sqrt[8]{0.965278} \approx 0.995592$ (see the Table 1).

The graph of the solution $u(n)$ of (20), (23) and some values are given on Figure 2 and Table 1. From the graph and the table it can be seen the solution exponentially approaches the equilibrium u^*. $\square$

Example 3. (*Neural networks with non-instantaneous impulses*). Consider the discrete model of neural network with impulses (20) on $n \in \bigcup_{k=0}^{\infty} I_k$ with non-instantaneous impulses

$$\begin{aligned}
u_1(n) &= \frac{1}{2}u_1(n_k-1) + \frac{1}{8}\sin(u_1(n_k-1)) \\
&\quad - \frac{1}{4}\sin(u_2(n_k-1)) + \frac{1}{16}u_3(n_k-1) + 1, \\
u_2(n) &= \frac{1}{3}u_2(n_k-1) + \frac{1}{4}\sin(u_1(n_k-1)) \\
&\quad + \frac{1}{8}\sin(u_2(n_k-1)) + 2, \\
u_3(n) &= \frac{1}{4}u_3(n_k-1) + \frac{1}{16}\sin(u_1(n_k-1)) \\
&\quad - \frac{1}{8}\sin(u_2(n_k-1)) + \frac{1}{16}u_3(n_k-1) + 1, \\
&\quad \text{for } \ n \in J_k, \ k \in \mathbb{Z}_1.
\end{aligned} \tag{24}$$

The neural network model (20), (24) has an equilibrium point $u^* = (2.4057, 3.2344, 1.5324)$ defined by Definition 2.

As in Example 2 the conditions of Theorem 2 are satisfied and therefore, the equilibrium u^* is exponentially stable with a rate $\alpha = \sqrt[8]{0.965278} \approx 0.995592$.

Consider the partial case $n_0 = 0$, $d_0 = 0$, $n_1 = 5, d_1 = 5$, $n_2 = 19, d_2 = 6$, $n_3 = 33$, $d_3 = 7$. In this particular case $I_0 = \mathbb{Z}[1,4]$, $I_1 = \mathbb{Z}[11,18]$, $I_2 = \mathbb{Z}[17,19]$, $J_1 = \mathbb{Z}[5,10]$, $J_2 = \mathbb{Z}[19,25]$, $J_3 = \mathbb{Z}[33,40]$. The graph of the solution $\tilde{u}(n)$ of the discrete model (20), (24) and some values are given on Figure 3 and Table 1. From both it could be seen the equilibrium point U^* is exponentially stable. $\square$

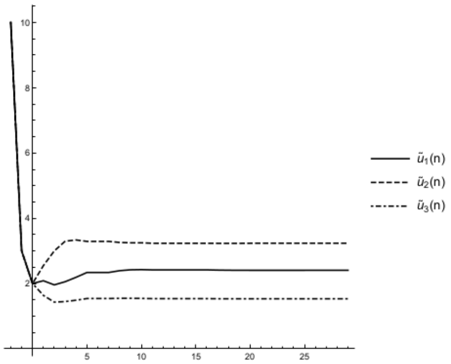

Figure 3. Graph of the solution of (20), (24).

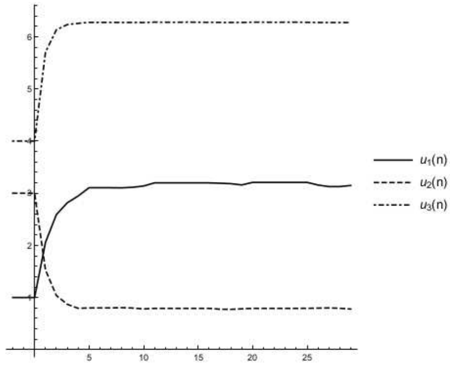

Figure 4. Graph of the solution of (25), (26).

n	$u_1(n)$	$\tilde{u}_1(n)$	$u_2(n)$	$\tilde{u}_2(n)$	$u_3(n)$	$\tilde{u}_3(n)$
1	2.08422	2.08422	2.5483	2.5483	1.63586	1.63586
2	1.95742	1.95742	2.99838	2.99838	1.42912	1.42912
3	2.05484	2.05484	3.29667	3.29667	1.44647	1.44647
4	2.18994	2.18994	3.3345	3.3345	1.48754	1.48754
5	2.33767	2.33767	3.29113	3.29113	1.53972	1.53972
6	2.42037	2.33767	3.29436	3.29113	1.55718	1.53972
7	2.44546	2.33767	3.28309	3.29113	1.55505	1.53972
8	2.44811	2.39232	3.25536	3.25844	1.55011	1.54479
9	2.42922	2.41965	3.23074	3.24755	1.53855	1.54606
10	2.42003	2.42553	3.22303	3.24796	1.53579	1.54264
11	2.40833	2.41778	3.22302	3.23349	1.53119	1.53637
12	2.40220	2.41778	3.22758	3.23349	1.5298	1.53637
13	2.40241	2.41778	3.23359	3.23349	1.53091	1.53637
14	2.40195	2.41778	3.23445	3.23349	1.53091	1.53637
15	2.40385	2.41778	3.23513	3.23349	1.53201	1.53637
16	2.40500	2.41778	3.23521	3.23349	1.53238	1.53637
17	2.40561	2.41063	3.23466	3.23192	1.53254	1.53298
18	2.40603	2.40706	3.23469	3.2314	1.53263	1.53213
19	2.40600	2.40468	3.23456	3.23142	1.53254	1.53151
20	2.40594	2.40448	3.23439	3.23393	1.5325	1.53181
21	2.40580	2.40448	3.23431	3.23393	1.53243	1.53181
22	2.40573	2.40448	3.23428	3.23393	1.53241	1.53181
23	2.40567	2.40448	3.23429	3.23393	1.5324	1.53181
24	2.40565	2.40448	3.23432	3.23393	1.53239	1.53181
25	2.40566	2.40448	3.23435	3.23393	1.5324	1.53181
26	2.40566	2.40505	3.23436	3.23449	1.5324	1.53223
27	2.40567	2.40533	3.23436	3.23467	1.53241	1.53233
28	2.40568	2.40564	3.23436	3.23467	1.53241	1.53245
29	2.40568	2.40579	3.23436	3.23454	1.53241	1.53248

Table 1. Values of the solutions $u_i(n)$ and $\tilde{u}_i(n)$, $i = 1, 2, 3$, of (20), (23) and (20), (24), respectively

Example 4. (*Neural networks with time variable connection weights*). Consider the discrete model of neural network with three agents and variable connection weights between them:

$$\begin{aligned} u_1(n) &= \frac{1}{2}u_1(n-1) + 0.0275\sin(n)\sin(u_1(n-3)) \\ &\quad - 0.03\sin(n)\cos(u_2(n-3)) \\ &\quad + 0.025u_3(n-3) + 0.45\pi, \\ u_2(n) &= \frac{1}{3}u_2(n-1) + 0.03\sin(n)\sin(u_1(n-3)) \\ &\quad + 0.02\sin(n)\cos(u_2(n-3)) + \frac{\pi}{6}, \\ u_3(n) &= \frac{1}{4}u_3(n-1) + 0.01\sin(n)\sin(u_1(n-3)) \\ &\quad - 0.0095\cos(u_2(n-3)) \\ &\quad + 0.01u_3(n-3) + 1.48\pi, \quad \text{for } n \in \bigcup_{k=0}^{\infty} I_k, \end{aligned} \tag{25}$$

with non-instantaneous impulses, starting at times n_k, $k = 1, 2, \ldots$, and acting on intervals J_k, $k = 1, 2, \ldots$:

$$\begin{aligned} u_1(n) &= \frac{1}{2}u_1(n_k-1) + \frac{0.0125n}{n+1}\sin(u_1(n_k-1)) \\ &\quad - \frac{0.03n}{n+1}\sin(u_2(n_k-1)) \\ &\quad + 0.025u_3(n_k-1) + 0.475\pi, \\ u_2(n) &= \frac{1}{3}u_2(n_k-1) + \frac{0.03n}{n+1}\sin(u_1(n_k-1)) \\ &\quad + \frac{0.0125n}{n+1}\sin(u_2(n_k-1)) + \frac{\pi}{6}, \\ u_3(n) &= \frac{1}{4}u_3(n_k-1) + 0.01\cos(n)\sin(u_1(n_k-1)) \\ &\quad - 0.005\cos^2(n)\sin(u_2(n_k-1)) \\ &\quad + 0.005u_3(n_k-1) + 1.49\pi, \quad \text{for } n \in J_k,\ k \in \mathbb{Z}_1. \end{aligned} \tag{26}$$

The model (25), (26) has an equilibrium $u^* = (\pi, 0.5\pi, 2\pi)$ according to Definition 2.

In this particular case we have $K_i = L_i = 1,\ i = 1, 2, 3$, and

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \ \mathcal{B}(n) = \begin{bmatrix} 0.0275|\sin(n)| & 0.03|\sin(n)| & 0.025 \\ 0.03|\sin(n)| & 0.02|\sin(n)| & 0 \\ 0.01|\sin(n)| & 0.0095||\sin(n)| & 0.01 \end{bmatrix}$$

$$M_k = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}, \qquad \mathcal{G}(n) = \begin{bmatrix} \frac{0.00125n}{n+1} & \frac{0.03n}{n+1} & 0.025 \\ \frac{0.03n}{n+1}\sin(n) & \frac{0.0125n}{n+1} & 0 \\ 0.01|\cos(n)| & 0.005 & 0.005 \end{bmatrix}.$$

The conditions of Theorem 2 are reduced to $0.5^2+3(0.03+0.03+0.01)^2 < 1$, $0.25 + 0.5 + 3(1 - 0.5)(0.03^2 + 0.03^2 + 0.01^2) < 1$, $\sqrt{K} = 0.01$.

Consider the partial case $n_0 = 0, d_0 = 0,\ n_1 = 5, d_1 = 2,\ n_2 = 11, d_2 = 5$, $n_3 = 20, d_3 = 5,\ n_4 = 30$. In this particular case $I_0 = \mathbb{Z}[1, 4], I_1 = \mathbb{Z}[8, 10], I_2 = \mathbb{Z}[17, 19], I_3 = \mathbb{Z}[26, 29],\ n_k = 5, 11, 20,\ k = 1, 2, 3,\ J_1 = \mathbb{Z}[5, 7],\ J_2 = \mathbb{Z}[11, 16]$, $J_3 = \mathbb{Z}[20, 25]$ and the initial conditions

$$u_1(n) = 1,\ u_2(n) = 3,\ u_3(n) = 4,\ \text{for}\ \ n = -2, -1, 0. \tag{27}$$

The graph of the solution of the discrete model (25), (26), (27) is given on Figure 4.

It could be seen the equilibrium $(\pi, 0.5\pi, 2\pi)$ is exponentially stable. □

Acknowledgments

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