

**CONTROLLABILITY OF THE BURGERS
EQUATION UNDER THE INFLUENCE OF
IMPULSES, DELAY AND NONLOCAL CONDITIONS**

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Abstract: In the case of the Burgers equation, this work proves the following conjecture: impulses, delays, and nonlocal conditions, under some assumptions, do not destroy some posed system qualitative properties since they are themselves intrinsic to it. We verified that the property of controllability is robust under this type of disturbances. Specifically, we prove that the interior approximate controllability of the linear heat equation is not destroyed if we add impulses, nonlocal conditions, and a nonlinear perturbation with delay in the state. This is done by using new techniques avoiding fixed point theorems employed by A.E. Bashirov et al. In this case the delay helps us to prove the approximate controllability of this system by pulling back the control solution to a fixed curve in a short time interval, and from this position, we are able to reach a neighborhood of the final state in time τ by using the fact that the corresponding linear heat equation is approximately controllable on any interval $[t_0, \tau]$, $0 < t_0 < \tau$.

AMS Subject Classification: 93B05, 93C10

Received: January 15, 2020

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Key Words: interior approximate controllability; impulsive Burgers equation with delays and nonlocal conditions; strongly continuous semigroups; new technique

1. Introduction

In this paper, we prove the interior approximate controllability of the Burgers equation with delays, impulses, and nonlocal conditions; this is done by using new techniques avoiding fixed point theorems employed by A.E. Bashirov et al. [3, 4, 5]. In this case, the delay helps us to prove the approximate controllability by pulling back the control solution to a fixed curve in a short time interval, and from this position, we are able to reach a neighborhood of the final state in time τ by using the approximate controllability of the heat equation in any positive interval. In this regard, we proved the interior approximate controllability of the following semilinear PDE equation with delays, impulses, and nonlocal conditions, which represents a perturbation of the Burgers equation,

$$\begin{cases} \frac{\partial z}{\partial t} = \nu z_{xx} + 1_\omega u(t) - z(t-r, x)z_x(t-r, x) + f(t, z(t-r), u(t), \\ z(t, 0) = z(t, 1) = 0, \quad t \in [0, \tau], \\ z(s, x) - h(z(\tau_1 + s, x), \dots, z(\tau_q + s, x)) = \phi(s, x), \quad s \in [-r, 0], \\ z(t_k^+, x) = z(t_k^-, x) + I_k(t_k, z(t_k, x)u(t_k, x), x) \in \Omega, k = 1, \dots, p, \end{cases} \quad (1)$$

where $\Omega = [0, 1]$, $\phi \in C([-r, 0]; H_0^1)$, ω is an open nonempty subset of $(0, 1)$, 1_ω denotes the characteristic function of the set ω and the distributed control u belongs to $L_2([0, \tau]; L_2[0, 1])$, $\phi : [-r, 0] \times \Omega \rightarrow \mathbb{R}$ is a continuous function and $0 < \tau_1 < \tau_2 < \dots < \tau_q < \tau$ and $0 < t_1 < \dots < t_p < \tau$.

Our main hypotheses are that the nonlinear functions $f : [0, \tau] \times \mathbb{R} \rightarrow \mathbb{R}$, $I_k : [0, \tau] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $h : \mathbb{R}^q \rightarrow \mathbb{R}$ are smooth enough such that the system (1) admits a unique mild solutions for each control u according to [15, 16, 17], and satisfy the following estimates for $z, w, u, v \in \mathbb{R}$:

$$|f(t, z, u) - f(t, w, v)| \leq L\{|z - w| + |u - v|\}, \quad t \in [0, \tau], \quad (2)$$

$$|f(t, z, u) \leq a(t)|z| + b(t), \quad t \in [0, \tau], \quad z, u \in \mathbb{R}, \quad a(\cdot), b(\cdot) \in L_\infty[0, \tau], \quad (3)$$

$$|h(z_1, \dots, z_q) - h(w_1, \dots, w_q)| \leq K \max_{1 \leq i \leq q} |z_i - w_i|, \quad z_i, w_i \in \mathbb{R}. \quad (4)$$

We shall denote $Z = L_2(\Omega)$, $Z^1 = H_0^1(\Omega)$, and by C the space of continuous functions

$$C = \{\phi : [-r, 0] \rightarrow Z^1 : \phi \text{ is continuous}\},$$

endowed respectively with the norms,

$$\|\phi\|_{Z^1} = \sqrt{\|\phi\|_Z^2 + \|\phi_x\|_Z^2}, \quad \phi \in Z^1,$$

$$\|\phi\| = \sup_{-r \leq s \leq 0} \|\phi(s)\|_{Z^1}, \text{ and } \phi(s)(x) = \phi(s, x), \quad x \in \Omega.$$

Also, we assume that:

$$z(t_k, x) = z(t_k^+) = \lim_{t \rightarrow t_k^+} z(t, x), \quad z(t_k^-, x) = \lim_{t \rightarrow t_k^-} z(t, x).$$

In this regard, we also consider the following natural Banach space to set evolution equations with delay, impulses and nonlocal conditions;

$$PC_{t_1 \dots t_p}([-r, \tau]; Z) = \{z : J = [-r, \tau] \rightarrow Z : z \in C(J'; Z), \exists z(t_k^+, \cdot), z(t_k, \cdot) = z(t_k^+, \cdot)\},$$

where $J' = [-r, \tau] \setminus \{t_1, \dots, t_p\}$, endowed with the norm

$$\|z\|_{PC} = \sup_{t \in [-r, \tau]} \|z(t, \cdot)\|_Z,$$

with

$$\|z\|_Z = \left(\int_{\Omega} |z(x)|^2 dx \right)^{1/2}, \quad z \in Z.$$

Definition 1. (*Approximate Controllability*) The system (1) is said to be approximately controllable on $[0, \tau]$ if for every $\phi \in C$, $z^1 \in Z = U = L_2(\Omega)$ and $\varepsilon > 0$, there exists $u \in C(0, \tau, U)$ such that the mild solution $z(t)$ of (1) corresponding to u verifies:

$$z(0) + h(z_{\tau_1}, \dots, z_{\tau_q})(0) = \phi(0), \quad \text{and} \quad \|z(\tau) - z^1\|_Z < \varepsilon.$$

The main goal of this paper is to prove the following theorem:

Theorem 2. *If the functions f, I_k, h are smooth enough and satisfy the hypotheses (2)-(4), the system (1) is approximately controllable on $[0, \tau]$, for all $\tau > 0$.*

2. Abstract formulation of the problem

In this section, we chose a Hilbert space where system (1) can be written as an abstract differential equation (see [10]); to this end, we consider the following notations: Let us consider the Hilbert space $Z = L^2(\Omega)$ and $0 < \lambda_1 < \lambda_2 < \dots < \lambda_j \rightarrow \infty$ the eigenvalues of operator $A\phi = -\nu\phi_{xx}$. Then, we have the following well known properties:

- a) There exists a complete orthonormal set $\{\phi_j\}$ of eigenvectors of A .
- b) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \langle z, \phi_j \rangle \phi_j = \sum_{j=1}^{\infty} \lambda_j E_j z, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in Z and

$$E_n z = \langle z, \phi_j \rangle \phi_j. \tag{6}$$

So, $\{E_j\}$ is a family of complete orthonormal projections in Z and $z = \sum_{j=1}^{\infty} E_j z, z \in Z$.

- c) $-A$ generates an analytic semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z, \text{ and } \|T(t)\| \leq e^{-\lambda_1 t}, \quad t \geq 0. \tag{7}$$

Consequently, system (1) can be written as an abstract functional differential equations with impulses and nonlocal conditions in Z :

$$\begin{cases} z' = -Az + B_\omega u + f^e(t, z_t, u(t)), & z \in Z, \quad t \geq 0, \\ z(s) + g(z_{\tau_1}, \dots, z_{\tau_q})(s) = \phi(s), & s \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + I_k^e(t_k, z(t_k), u(t_k)), & k = 1, 2, \dots, p, \end{cases} \tag{8}$$

where $u \in L^2(0, \tau; U), U = Z, B_\omega : U \rightarrow Z, B_\omega u = 1_\omega u$ is a bounded linear operator. $z_t \in C([-r, 0]; H_0^1)$ and it is defined by $z_t(s) = z(t + s), -r \leq s \leq 0$ and the functions $I_k^e : [0, \tau] \times Z \times U \rightarrow Z, f^e : [0, \tau] \times C \times U \rightarrow Z$ and $g : C^q([-r, 0]; Z \rightarrow C([-r, 0]; Z)$ are defined by

$$I_k^e(t, z, u)(x) = I_k(t, z(x), u(x)), \quad k = 1, \dots, p, \quad x \in \Omega,$$

$$f^e(t, \phi, u)(x) = \phi(-r, x)\phi_x(-r, x) + f(t, \phi(-r, x)u(x)), \quad x \in \Omega$$

and

$$g(\phi_1, \dots, \phi_q)(s, x) = h(\phi_1(s, x), \dots, \phi_q(s, x)) \quad \text{in } [-r, 0] \times \Omega.$$

Proposition 3. *Under the conditions (2)-(4) the functions f^e and g satisfies for all $z, w \in PC_{t_1..t_P}$:*

$$\|f^e(t, \phi, u)\| \leq \|\phi\|_C^2 + 4 \|a\|_{L_\infty} + 4 \|b\|_{L_\infty} \sqrt{\mu(\Omega)}, \tag{9}$$

$$\|f^e(t, \phi_1, u) - f^e(t, \phi_2, u)\| \leq \{\|\phi_1 - \phi_2\|_C + L\} \|\phi_1 - \phi_2\|_C, \tag{10}$$

$$\|g(z_{\tau_1}, \dots, z_{\tau_q}) - g(w_{\tau_1}, \dots, w_{\tau_q})\| \leq K \|z - w\|_{PC([-r, \tau]; Z)}. \tag{11}$$

3. Controllability of the linear equation

In this section we shall present some characterization of the approximate controllability of the corresponding linear heat equation. To this end, we note that, for all $z_0 \in Z$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = -Az + B_\omega u(t), & z \in Z, \\ z(t_0) = z_0, \end{cases} \tag{12}$$

admits only one mild solution defined, for all $t \in [t_0, \tau]$, with $0 \leq t_0 \leq \tau$, by

$$z(t) = z(t, t_0, z_0, u) = T(t)z_0 + \int_{t_0}^t T(t-s)B_\omega u(s) ds. \tag{13}$$

Definition 4. Corresponding with (12), we define the following operators: The controllability operator $\mathfrak{G}_{\tau\delta} : L^2(\tau - \delta, \tau; U) \rightarrow Z$ defined by

$$\mathfrak{G}_{\tau\delta} u = \int_{\tau-\delta}^{\tau} T(\tau-s)B_\omega u(s) ds, \quad u \in L^2(\tau - \delta, \tau; U), \tag{14}$$

and adjoint of this operator $\mathfrak{G}_{\tau\delta}^* : Z \rightarrow L^2(\tau - \delta, \tau; U)$ is given by

$$(\mathfrak{G}_{\tau\delta}^* z)(t) = B_\omega^* T^*(\tau - t)z, \quad t \in [\tau - \delta, \tau].$$

The Grammian controllability operator is given by:

$$Q_{\tau\delta} = \mathfrak{G}_{\tau\delta} \mathfrak{G}_{\tau\delta}^* = \int_{\tau-\delta}^{\tau} T(\tau-t)B_\omega B_\omega^* T^*(\tau-t) dt. \tag{15}$$

The following lemma holds in general for a linear bounded operator $G : W \rightarrow Z$ between Hilbert spaces W and Z (see [6],[7],[8],[9] and [14]).

Lemma 5. *The following statements are equivalent to the approximate controllability of the linear system (12) on $[\tau - \delta, \tau]$:*

- a) $\overline{\text{Rang}(\mathfrak{G}_{\tau\delta})} = Z$.
- b) $\text{Ker}(\mathfrak{G}_{\tau\delta}^*) = \{0\}$.
- c) $\langle Q_{\tau\delta}z, z \rangle > 0, z \neq 0$ in Z .
- d) $\lim_{\alpha \rightarrow 0^+} \alpha(\alpha I + Q_{\tau\delta})^{-1}z = 0$.
- e) For all $z \in Z$ we have $\mathfrak{G}_{\tau\delta}u_\alpha = z - \alpha(\alpha I + Q_{\tau\delta})^{-1}z$, where

$$u_\alpha = \mathfrak{G}_{\tau\delta}^*(\alpha I + Q_{\tau\delta})^{-1}z, \quad \alpha \in (0, 1].$$

So, $\lim_{\alpha \rightarrow 0^+} \mathfrak{G}_{\tau\delta}u_\alpha = z$ and the error $E_{\tau\delta}z$ of this approximation is given by the formula

$$E_{\tau\delta}z = \alpha(\alpha I + Q_{\tau\delta})^{-1}z, \quad \alpha \in (0, 1].$$

Remark 6. *The foregoing lemma implies that the family of linear operators $\Gamma_{\alpha\tau\delta} : Z \rightarrow W$, defined for $0 < \alpha \leq 1$ by*

$$\Gamma_{\alpha\tau\delta}z = \mathfrak{G}_{\tau\delta}^*(\alpha I + Q_{\tau\delta})^{-1}z, \tag{16}$$

satisfies

$$\lim_{\alpha \rightarrow 0^+} \mathfrak{G}_{\tau\delta}\Gamma_{\alpha\tau\delta} = I \tag{17}$$

in the strong topology.

Lemma 7. *$Q_{\tau\delta} > 0$ if, and only if, the linear system (12) is controllable on $[\tau - \delta, \tau]$. Moreover, given an initial state y_0 and a final state z^1 we can find a sequence of controls $\{u_\alpha^\delta\}_{0 < \alpha \leq 1} \subset L^2(\tau - \delta, \tau; U)$*

$$u_\alpha = u_{\alpha,\delta} = \mathfrak{G}_{\tau\delta}^*(\alpha I + \mathfrak{G}_{\tau\delta}\mathfrak{G}_{\tau\delta}^*)^{-1}(z^1 - T(\tau)y_0), \quad \alpha \in (0, 1],$$

such that the solutions $y(t) = y(t, \tau - \delta, y_0, u_\alpha)$ of the initial value problem

$$\begin{cases} y' = Ay + B_\omega u_\alpha(t), & y \in Z, \quad t > 0, \\ y(\tau - \delta) = y_0, \end{cases} \tag{18}$$

satisfies

$$\lim_{\alpha \rightarrow 0^+} y(\tau, \tau - \delta, y_0, u_\alpha) = z^1.$$

Equivalently,

$$\lim_{\alpha \rightarrow 0^+} y(\tau) = \lim_{\alpha \rightarrow 0^+} \left\{ T(\delta)y_0 + \int_{\tau-\delta}^\tau T(\tau - s)B_\omega u_\alpha(s) ds \right\} = z^1.$$

4. The system with impulses, delay and nonlocal conditions

In this section, we shall prove the main result of this paper, the interior approximate controllability of the Burguers equation with impulses, delay and nonlocal conditions given by (1), which is equivalent to prove the approximate controllability of the system (8). In this regard, according to [1, 12, 15, 16, 17], for all $\phi \in C$ and $u \in L^2(0, \tau; U)$ the nonlocal Cauchy problem (8) admits only one mild solution $z \in PC_{t_1..t_P}([-r, \tau]; Z)$ given by

$$\begin{aligned}
 z(t) &= T(t)\phi(0) - T(t)[g(z_{\tau_1}, \dots, z_{\tau_q})(0)] + \int_0^\tau T(t-s)B_\omega u(s) ds \\
 &+ \int_0^t T(t-s)f^e(s, z_s, u(t)) ds \\
 &+ \sum_{0 < t_k < t} T(t-t_k)I_k^e(t_k, z(t_k)u(t_k)), \quad t \in [0, \tau], \\
 z(t) &+ (g(z_{\tau_1}, \dots, z_{\tau_q}))(t) = \phi(t), \quad t \in [-r, 0].
 \end{aligned}
 \tag{19}$$

Now, we are ready to prove the main result of this paper, which is the interior approximate controllability of Burgers equation with impulses, delays and nonlocal conditions.

Proof of Theorem 2.

Given $\phi \in C$, a final state z^1 and $\varepsilon > 0$, we want to find a control $u_\alpha^\delta \in L^2(0, \tau; U)$ steering the system from $\phi(0)$ to an ε -neighborhood of z^1 on time τ . Precisely, for $\alpha > 0$ and $0 < \delta < \min\{\tau - t_p, r\}$ small enough, there exists control $u_\alpha^\delta \in L^2(0, \tau; U)$ such that corresponding of solutions $z^{\delta, \alpha}$ of (8) satisfies;

$$\left\| z^{\delta, \alpha}(\tau) - z^1 \right\| \leq \varepsilon.$$

In fact, we consider any $u \in L^2(0, \tau; U)$ and the corresponding solution $z(t) = z(t, 0, \phi, u)$ of the Cauchy problem (8). For $\alpha \in (0, 1]$, we define the control $u_\alpha^\delta \in L^2(0, \tau; U)$ as follows

$$u_\alpha^\delta(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq \tau - \delta, \\ u_\alpha(t), & \text{if } \tau - \delta < t \leq \tau, \end{cases}$$

where

$$u_\alpha(t) = B_\omega^* T^*(\tau - t)(\alpha I + \mathfrak{G}_{\tau\delta} \mathfrak{G}_{\tau\delta}^*)^{-1} (z^1 - T(\delta)z(\tau - \delta)), \quad \tau - \delta < t \leq \tau.$$

Since $0 < \delta < \tau - t_p$, then $\tau - \delta > t_p$ and the corresponding solution $z^{\delta,\alpha}(t) = z(t, 0, \phi, u_\alpha^\delta)$ of the nonlocal Cauchy problem (8) at time τ can be written as follows:

$$\begin{aligned} z^{\delta,\alpha}(\tau) &= T(\tau)\phi(0) - T(\tau)[g(z_{\tau_1}, \dots, z_{\tau_q})(0)] + \int_0^\tau T(\tau - s)B_\omega(s) ds \\ &+ \int_0^\tau T(\tau - s)f^e(s, z_s^{\delta,\alpha}, u_\alpha^\delta(s)) ds \\ &+ \sum_{0 < t_k < \tau} T(\tau - t_k)I_k^e(t_k, z(t_k), u_\alpha^\delta(t_k)) \\ &= T(\delta) \left\{ T(\tau - \delta)\phi(0) - T(\tau - \delta)[g(z_{\tau_1}, \dots, z_{\tau_1})(0)] \right. \\ &+ \int_0^{\tau - \delta} T(\tau - \delta - s)B_\omega(s)u_\alpha^\delta(s) ds \\ &+ \int_0^{\tau - \delta} T(\tau - \delta - s)f^e(s, z_s^{\delta,\alpha}, u_\alpha^\delta(s)) ds \\ &+ \left. \sum_{0 < t_k < \tau - \delta} T(\tau - \delta - t_k)I_k^e(t_k, z^{\delta,\alpha}(t_k), u_\alpha^\delta(t_k)) \right\} \\ &+ \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_\alpha^\delta(s) ds + \int_{\tau - \delta}^\tau T(\tau - s)f^e(s, z_s^{\delta,\alpha}, u_\alpha^\delta(s)) ds \\ &= T(\delta)z(\tau - \delta) + \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_\alpha(s) ds \\ &+ \int_{\tau - \delta}^\tau T(\tau - s)f^e(s, z_s^{\delta,\alpha}, u_\alpha(s)) ds. \end{aligned}$$

So,

$$\begin{aligned} z^{\delta,\alpha}(\tau) &= T(\delta)z(\tau - \delta) + \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_\alpha(s) ds \\ &+ \int_{\tau - \delta}^\tau T(\tau - s)f^e(s, z_s^{\delta,\alpha}, u_\alpha(s)) ds. \end{aligned}$$

The corresponding solution $y^{\delta,\alpha}(t) = y(t, \tau - \delta, z(\tau - \delta), u_\alpha)$ of the initial value problem (18) at time τ is given by:

$$y^{\delta,\alpha}(\tau) = T(\delta)z(\tau - \delta) + \int_{\tau - \delta}^\tau T(\tau - s)B_\omega u_\alpha(s) ds.$$

On the other hand, from Lemma 7 there exists $\alpha > 0$ such that

$$\|y_\alpha^\delta(\tau) - z^1\| \leq \varepsilon/2.$$

Therefore,

$$\left\| z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau) \right\| \leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \left\| f^e(s, z_s^{\delta,\alpha}, u_\alpha^\delta(s)) \right\| ds.$$

Now, since $0 < \delta < r$ and $\tau - \delta \leq s \leq \tau$, then $s - r \leq \tau - r < \tau - \delta$ and

$$z^{\delta,\alpha}(s-r) = z(s-r).$$

Hence, there exists δ small enough such that $0 < \delta < \min\{r, \tau - t_p\}$ and

$$\begin{aligned} \left\| z^{\delta,\alpha}(\tau) - y^{\delta,\alpha}(\tau) \right\| &\leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \left\| f^e(s, z_s, u_\alpha^\delta(s)) \right\| ds \\ &\leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \{ \|z_s\|^2 + 4 \|a\|_{L^\infty} \|z_s\| + 4 \|b\|_{L^\infty} \sqrt{\mu(\Omega)} \} ds < \varepsilon/2. \end{aligned}$$

So,

$$\begin{aligned} \left\| z^{\delta,\alpha}(\tau) - z^1 \right\| &\leq \int_{\tau-\delta}^{\tau} \|T(\tau-s)\| \left\| f^e(s, z_s, u_\alpha^\delta(s)) \right\| ds \\ &+ \left\| y^{\delta,\alpha}(\tau) - z^1 \right\| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This completes the proof of the theorem. \square

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