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SECOND HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

Jae Ho Choi

Department of Mathematics Education
Daegu National University of Education
219 Jungangdaero, Namgu, Daegu 42411, KOREA

Abstract: The main object of the present paper is to investigate the upper bound of the second Hankel determinant $|a_2a_4 - a_3^2|$ for the analytic functions defined by Ruscheweyh derivative. Furthermore, several basic properties such as inclusion, Hadamard product are also considered.

AMS Subject Classification: 30C45, 30C50
Key Words: analytic functions; Hadamard product; Hankel determinant; Ruscheweyh derivative

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$, $\mathcal{S}^*(\beta)$ $(0 \leq \beta < 1)$ and $\mathcal{K}(\beta)$ $(0 \leq \beta < 1)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are *univalent*, *starlike of order* β and *convex of order* β functions in $\mathbb{U}$ (cf. [6], [20]). In particular, $\mathcal{S}^*(0) = \mathcal{S}$ and $\mathcal{K}(0) = \mathcal{K}$ are the familiar classes of starlike and convex functions in $\mathbb{U}$, respectively.

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In 1976, Noonan and Thomas [17] defined the qth Hankel determinant of $f(z)$ for $q \geq 1$ and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_{n+1} & \cdots & a_{n+q-1} \end{vmatrix}.$$

This Hankel determinant is useful and has also been considered by several authors. For example, Noor in [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions f given by (1) with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence was defined and some of its properties were discussed by Layman [10]. It is well known ([4]) that for $f \in \mathcal{S}$ and given by (1) the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $n = 1$. After that, Fekete-Szegö further generalized the estimate $|a_3 - \mu a_2^2|$ with real μ and $f \in \mathcal{S}$. For results related to the functional, see [3], [7], [9], [14] and [19]. Here we consider the second Hankel determinant in the case of $q = 2$ and $n = 2$, namely,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{2}$$

In particular, sharp bounds on $H_2(2)$ were obtained by several authors of articles [8], [15]-[16], [23] and [25] for different subclasses of univalent functions.

For the functions $f, g \in \mathcal{A}$ and given by the series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (z \in \mathbb{U}),$$

we define the *Hadamard product (or convolution)* of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z) \qquad (z \in \mathbb{U}).$$

Making use of the Hadamard product, Carlson-Shaffer [2] defined the linear operator $\mathcal{L}(a, c) : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{L}(a, c) f(z) = \Phi(a, c; z) * f(z) \qquad (f \in \mathcal{A}), \tag{3}$$

where

$$\Phi(a,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}z^{k+1} \qquad (z\in\mathbb{U};c\notin\mathbb{Z}_0^-:=\{0,-1,-2,\cdots\}) \tag{4}$$

and $(\lambda)_k$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$\begin{aligned}(\lambda)_k &= \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}\\ &= \begin{cases}1 & (k=0)\\ \lambda(\lambda+1)\cdots(\lambda+k-1) & (k\in\mathbb{N}:=\{1,2,3,\cdots\}).\end{cases}\end{aligned}$$

The Carlson-Shaffer operator $\mathcal{L}(a,c)$ maps $\mathcal{A}$ onto itself and $\mathcal{L}(c,a)$ is the inverse of $\mathcal{L}(a,c)$, provided that $a\notin\mathbb{Z}_0^-$ (see also [15], [24]). Moreover, it can be readily verified from (3) and (4) that

$$\mathcal{L}(a,b)\mathcal{L}(c,d)f=\mathcal{L}(c,d)\mathcal{L}(a,b)f \qquad (b,d\notin\mathbb{Z}_0^-;f\in\mathcal{A}) \tag{5}$$

and

$$\mathcal{L}(a,b)\mathcal{L}(b,c)f=\mathcal{L}(a,c)f \qquad (b,c\notin\mathbb{Z}_0^-;f\in\mathcal{A}). \tag{6}$$

In [21] Ruscheweyh introduced the operator $D^\gamma:\mathcal{A}\to\mathcal{A}$ defined by Hadamard product:

$$\begin{aligned}D^\gamma f(z) &= \frac{z}{(1-z)^{\gamma+1}}*f(z)=z+\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)}{\Gamma(\gamma+1)(k-1)!}a_kz^k\\ &\qquad (\gamma\ge -1;z\in\mathbb{U};f\in\mathcal{A}),\end{aligned} \tag{7}$$

which implies that

$$D^nf(z)=\frac{z\left(z^{n-1}f(z)\right)^{(n)}}{n!} \qquad (n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}).$$

Note that $D^0f(z)=f(z)$, $D^1f(z)=zf'(z)$ and

$$D^\gamma f(z)=\mathcal{L}(\gamma+1,1)f(z) \qquad (z\in\mathbb{U}). \tag{8}$$

A function $f\in A$ is said to be in the class $\mathcal{R}_\gamma(\alpha,\rho)$ $(\gamma\ge -1,|\alpha|<\pi/2,0\le\rho<1)$ if it satisfies the inequality

$$\mathrm{Re}\left\{e^{i\alpha}\frac{D^\gamma f(z)}{z}\right\}>\rho\cos\alpha \qquad (z\in\mathbb{U}). \tag{9}$$

Write

$$\mathcal{R}_\gamma(0,\rho) = \mathcal{R}_\gamma(\rho).$$

As is usually the case, we let $\mathcal{P}$ be the family of all functions p analytic in $\mathbb{U}$ for which $\mathrm{Re}(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \qquad (z \in \mathbb{U}). \tag{10}$$

It follows from (9) that

$$f \in \mathcal{R}_\gamma(\alpha,\rho) \Leftrightarrow e^{i\alpha}\frac{D^\gamma f(z)}{z} = [(1-\rho)p(z)+\rho]\cos\alpha + i\sin\alpha, \tag{11}$$

where α is real, $|\alpha| < \pi/2$ and $p(z) \in \mathcal{P}$.

We note that

$$\mathcal{R}_0(\alpha,\rho) = \left\{ f \in \mathcal{A} \mid \mathrm{Re}\left\{ e^{i\alpha}\frac{f(z)}{z} \right\} > \rho\cos\alpha \right\},$$

$$\mathcal{R}_1(\alpha,\rho) = \left\{ f \in \mathcal{A} \mid \mathrm{Re}\left\{ e^{i\alpha} f'(z) \right\} > \rho\cos\alpha \right\},$$

and the class $\mathcal{R}_1(0,0) = \mathcal{R}$ has been studied in [13].

The object of the present paper is to determine the functional $|a_2a_4 - a_3^2|$ for the function $f \in \mathcal{R}_\gamma(\alpha,\rho)$. We also obtain some basic properties of the class $\mathcal{R}_\gamma(\alpha,\rho)$. Our investigation includes a recent result of Janteng et al. [8].

2. Main results

In order to prove our results, we need the following lammas.

Lemma 1. (see [4]) *Let the function $p \in \mathcal{P}$ and be given by the power series* (10). *Then $|c_k| \le 2$ for each $k \in \mathbb{N}$.*

Lemma 2. (see [11] and [12]) *Let the function $p \in \mathcal{P}$ and be given by the power series* (10). *Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{12}$$

for some x with $|x| \le 1$ and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \tag{13}$$

for some z with $|z| \le 1$.

Lemma 3. (see [22]) *Let f and g be starlike of order 1/2. Then, for each function $F(z)$, satisfying* $\mathrm{Re}(F(z)) > \beta$ $(0 \leq \beta < 1)$, *one has*

$$\mathrm{Re}\left(\frac{f(z) * F(z)g(z)}{f(z) * g(z)}\right) > \beta \qquad (z \in \mathbb{U}).$$

We begin by proving the following theorem.

Theorem 1. *Let $-\pi/2 < \alpha < \pi/2$, $0 \leq \rho < 1$ and $\gamma \geq 0$. Suppose that the function f given by* (1) *be in the class $\mathcal{R}_\gamma(\alpha, \rho)$. Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \begin{cases} \dfrac{16(1-\rho)^2 \cos^2 \alpha}{(\gamma+1)^2(\gamma+2)^2} & (\gamma \leq 6) \\[2ex] \dfrac{(1-\rho)^2(17\gamma^2+36\gamma+36)\cos^2\alpha}{\gamma(\gamma+1)^2(\gamma+2)^2(\gamma+3)} & (\gamma \geq 6). \end{cases} \tag{14}$$

The estimate (14) *is sharp.*

Proof. Let $f \in \mathcal{R}_\gamma(\alpha, \rho)$. Then, from (11) we have

$$e^{i\alpha}\frac{D^\gamma f(z)}{z} = [(1-\rho)p(z) + \rho]\cos\alpha + \sin\alpha \qquad (z \in \mathbb{U}), \tag{15}$$

where $p \in \mathcal{P}$ and is given by (10). By using the series expansion of $D^\gamma f(z)$ and $p(z)$ as in (7) and (10), equating the coefficients in (15) yields

$$\begin{aligned} e^{i\alpha}(\gamma+1)a_2 &= (1-\rho)c_1\cos\alpha, \\ e^{i\alpha}\frac{(\gamma+1)(\gamma+2)}{2}a_3 &= (1-\rho)c_2\cos\alpha, \\ e^{i\alpha}\frac{(\gamma+1)(\gamma+2)(\gamma+3)}{6}a_4 &= (1-\rho)c_3\cos\alpha. \end{aligned} \tag{16}$$

Therefore, from (16) we have

$$\left|a_2a_4 - a_3^2\right| = \frac{2(1-\rho)^2\cos^2\alpha}{(\gamma+1)^2(\gamma+2)}\left|\frac{3c_1c_3}{\gamma+3} - \frac{2c_2^2}{\gamma+2}\right|. \tag{17}$$

Since the function $p(z)$ and $p(e^{i\theta}z)$ $(\theta \in \mathbb{R})$ are members of the class $\mathcal{P}$ simultaneously, we assume that without loss of generality that $c_1 > 0$. For convenience of notation, we take $c_1 = c$ $(c \in [0, 2])$.

Using (12) and (13) in (17), we obtain

$$\begin{aligned} &\left|a_2a_4 - a_3^2\right| \\ = \; &\frac{(1-\rho)^2\cos^2\alpha}{2(\gamma+1)^2(\gamma+2)}\left|\frac{3c}{\gamma+3}\,\{c^3 + 2(4-c^2)cx - c(4-c^2)x^2\right. \end{aligned}$$

$$
\begin{aligned}
&+ 2(4-c^2)(1-|x|^2)z\} - \frac{2\{c^4+2c^2(4-c^2)x+x^2(4-c^2)^2\}}{\gamma+2}\Bigg| \\
= \; & \frac{(1-\rho)^2\cos^2\alpha}{2(\gamma+1)^2(\gamma+2)} \Bigg| \left(\frac{3}{\gamma+3}-\frac{2}{\gamma+2}\right)c^4 + \left(\frac{6c(4-c^2)c^2}{\gamma+3}\right. \\
&\left. - \frac{4(4-c^2)c^2}{\gamma+2}\right)x - \left(\frac{3c^2(4-c^2)}{\gamma+3}+\frac{2(4-c^2)^2}{\gamma+2}\right)x^2 \\
&+ \frac{6c(4-c^2)(1-|x|^2)z}{\gamma+3}\Bigg| \\
= \; & \frac{(1-\rho)^2\cos^2\alpha}{2(\gamma+1)^2(\gamma+2)} \Bigg| \frac{\gamma c^4}{(\gamma+2)(\gamma+3)} + \frac{2\gamma(4-c^2)c^2}{(\gamma+2)(\gamma+3)}x \\
&- \frac{(4-c^2)(\gamma c^2+8\gamma+24)}{(\gamma+2)(\gamma+3)}x^2 + \frac{6c(4-c^2)(1-|x|^2)z}{\gamma+3}\Bigg|.
\end{aligned}
$$

An application of triangle inequality and replacement of $|x|$ by μ give

$$
\begin{aligned}
&|a_2a_4-a_3^2| \\
\le \; & \frac{(1-\rho)^2\cos^2\alpha}{2(\gamma+1)^2(\gamma+2)}\left[\frac{\gamma c^4}{(\gamma+2)(\gamma+3)}+\frac{2\gamma c^2(4-c^2)}{(\gamma+2)(\gamma+3)}\mu\right. \\
&\left.+\frac{(4-c^2)(\gamma c^2+8\gamma+24)}{(\gamma+2)(\gamma+3)}\mu^2+\frac{6c(4-c^2)}{\gamma+3}-\frac{6c(4-c^2)}{\gamma+3}\mu^2\right] \\
= \; & \frac{(1-\rho)^2\cos^2\alpha}{2(\gamma+1)^2(\gamma+2)}\left[\frac{\gamma c^4}{(\gamma+2)(\gamma+3)}+\frac{6c(4-c^2)}{\gamma+3}\right. \\
&\left.+\frac{2\gamma c^2(4-c^2)}{(\gamma+2)(\gamma+3)}\mu+\frac{(4-c^2)(c-2)(\gamma c-4\gamma-12)}{(\gamma+2)(\gamma+3)}\mu^2\right] \qquad (18) \\
:= \; & F(c,\mu),
\end{aligned}
$$

where $0 \le c \le 2$ and $0 \le \mu \le 1$.

We now maximize the function $F(c,\mu)$ on the closed square [0,2]×[0,1]. Since

$$
\begin{aligned}
\frac{\partial F}{\partial \mu} = \; & \frac{(1-\rho)^2\cos^2\alpha}{2(\gamma+1)^2(\gamma+2)}\left[\frac{2\gamma c^2(4-c^2)}{(\gamma+2)(\gamma+3)}\right. \\
&\left.+\frac{2(4-c^2)(c-2)(\gamma c-4\gamma-12)}{(\gamma+2)(\gamma+3)}\mu\right],
\end{aligned}
$$

$c-2<0$ and $\gamma c-4\gamma-12<0$, we have $\partial F/\partial\mu > 0$ for $0<c<2$ and $0<\mu<1$. Thus $F(c,\mu)$ cannot have a maximum in the interior of the closed square [0,2]×[0,1]. Moreover, for fixed $c \in [0,2]$,

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c).$$

One can obtain that

$$\left|a_2a_4 - a_3^2\right| \le G(c),$$

where

$$\begin{aligned} G(c) &= \frac{(1-\rho)^2\cos^2\alpha}{2(\gamma+1)^2(\gamma+2)}\left[\frac{\gamma c^4}{(\gamma+2)(\gamma+3)} + \frac{6c(4-c^2)}{\gamma+3}\right. \\ &\quad \left. + \frac{2\gamma c^2(4-c^2)}{(\gamma+2)(\gamma+3)} + \frac{(4-c^2)(c-2)(\gamma c - 4\gamma - 12)}{(\gamma+2)(\gamma+3)}\right]. \end{aligned}$$

Since

$$G'(c) = \frac{-4c(1-\rho)^2\cos^2\alpha}{(\gamma+1)^2(\gamma+2)^2(\gamma+3)}\left\{\gamma c^2 - \gamma + 6\right\},$$

we have to consider following two cases:

Case 1. For $\gamma \le 6$, $G'(c) < 0$ for $0 < c < 2$ and has real critical point at $c = 0$. Also $G(c) > G(2)$. Therefore, $\max_{0\le c\le 2}$ occurs at $c = 0$. Thus the upper bound of (18) corresponds to $\mu = 1$ and $c = 0$. Hence, we get

$$\left|a_2a_4 - a_3^2\right| \le \frac{16(1-\rho)^2\cos^2\alpha}{(\gamma+1)^2(\gamma+2)^2}.$$

Case 2. Let $\gamma \ge 6$. After necessary calculations, it is obtain that

$$G'(0) = 0 \qquad \text{and} \qquad G'\left(\sqrt{1-6/\gamma}\right) = 0.$$

Since

$$G''(0) > 0 \qquad \text{and} \qquad G''\left(\sqrt{1-6/\gamma}\right) < 0,$$

$G(c)$ has a maximum at $c = \sqrt{1-6/\gamma}$. Hence, we have

$$\left|a_2a_4 - a_3^2\right| \le \frac{(1-\rho)^2(17\gamma^2+36\gamma+36)\cos^2\alpha}{\gamma(\gamma+1)^2(\gamma+2)^2(\gamma+3)}.$$

Equality holds for the function

$$f(z) = \Phi(1,\gamma+1;z) * e^{-i\alpha}\left[z\left(\frac{1+(1-2\rho)z^2}{1-z^2}\cos\alpha + i\sin\alpha\right)\right].$$

This completes the proof of Theorem 1. □

Putting $\alpha = 0$ in Theorem 1, we get the following consequence.

Corollary 1. *If the function $f(z)$ given by* (1) *be in the class $\mathcal{R}_\gamma(\rho)$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \dfrac{16(1-\rho)^2}{(\gamma+1)^2(\gamma+2)^2} & (\gamma \leq 6) \\[2ex] \dfrac{(1-\rho)^2(17\gamma^2+36\gamma+36)}{\gamma(\gamma+1)^2(\gamma+2)^2(\gamma+3)} & (\gamma \geq 6). \end{cases}$$

Equality holds for the function

$$f(z) = \Phi(1, \gamma+1; z) * \frac{z\left(1+(1-2\rho)z^2\right)}{1-z^2}.$$

Remark. Taking $\gamma = 1$, $\alpha = 0$ and $\rho = 0$ in Theorem 1, we get a recent result due to Janteng et al. [8].

Theorem 2. *Suppose that $-\pi/2 < \alpha < \pi/2$, $0 \leq \rho < 1$ and $\gamma \geq 0$. Then*

$$\mathcal{R}_{\gamma+1}(\alpha, \rho) \subset \mathcal{R}_{\gamma}(\alpha, \rho). \tag{19}$$

Proof. Let $f \in \mathcal{R}_{\gamma+1}(\alpha, \rho)$. By using (5), (6) and (8), we have

$$\begin{aligned} D^{\gamma} f(z) &= \mathcal{L}(\gamma+1, 1) f(z) \\ &= \mathcal{L}(\gamma+2, 1)\mathcal{L}(1, \gamma+2)\mathcal{L}(\gamma+1, 1) f(z) \\ &= \mathcal{L}(\gamma+1, \gamma+2) D^{\gamma+1} f(z) \\ &= \Phi(\gamma+1, \gamma+2; z) * D^{\gamma+1} f(z), \end{aligned}$$

where the function Φ is defined by (4). Therefore,

$$\begin{aligned} e^{i\alpha}\frac{D^{\gamma} f(z)}{z} &= \frac{\Phi(\gamma+1, \gamma+2; z) * \left(e^{i\alpha} D^{\gamma+1} f(z)/z\right) \cdot z}{\Phi(\gamma+1, \gamma+2; z) * z} \\ &= \frac{f(z) * F(z) g(z)}{f(z) * g(z)}, \end{aligned}$$

where $f(z) = \Phi(\gamma+1, \gamma+2; z)$, $g(z) = z$ and $F(z) = e^{i\alpha} D^{\gamma+1} f(z)/z$. We note that $g \in \mathcal{S}^*(1/2)$ and $\mathrm{Re}(F(z)) > \rho\cos\alpha$. Moreover, by using a result of Bernardi [1], we observe that $\Phi(\gamma+1, \gamma+2; z) \in \mathcal{S}^*(1/2)$. Therefore, by applying Lemma 3,

$$\mathrm{Re}\left(e^{i\alpha}\frac{D^{\gamma} f(z)}{z}\right) > \rho\cos\alpha \qquad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}; 0 \leq \rho < 1; z \in \mathbb{U}\right).$$

Hence, $f \in \mathcal{R}_\gamma(\alpha, \rho)$, which completes the proof of Theorem 2. □

Theorem 3. *Let $-\pi/2 < \alpha < \pi/2$, $0 \leq \rho < 1$ and $\gamma \geq 0$. Suppose that $f \in \mathcal{S}^*(1/2)$ and $g \in \mathcal{R}_\gamma(\alpha, \rho)$, then the Hadamard product*

$$(f * g)(z) \in \mathcal{R}_\gamma(\alpha, \rho) \qquad (z \in \mathbb{U}). \tag{20}$$

Proof. Since the Hadamard product is associative and commutative, we obtain

$$D^\gamma(f * g)(z) = f(z) * D^\gamma g(z) \qquad (z \in \mathbb{U}).$$

Therefore, we get

$$e^{i\alpha}\frac{D^\gamma(f * g)(z)}{z} = \frac{f(z) * \left(e^{i\alpha} D^\gamma g(z)/z\right) \cdot z}{f(z) * z}.$$

Thus, by applying Lemma 3, we observe that

$$\operatorname{Re}\left(e^{i\alpha}\frac{D^\gamma(f * g)(z)}{z}\right) > \rho\cos\alpha.$$

Hence, $(f * g)(z) \in \mathcal{R}_\gamma(\alpha, \rho)$, and the proof of Theorem 3 is complete. □

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