International Journal of Applied Mathematics

Volume 34 No. 2 2021, 353-361
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v34i2.11

ABOUT UNIFORMLY μ-PARACOMPACT SPACES

Bekbolot E. Kanetov[1 §], Nurila A. Baigazieva[2],
Nurgazy I. Altybaev[3]

[1,2,3] Jusup Balasagyn Kyrgyz National University
Frunze Street, 547, 720033
Bishkek, KYRGYZSTAN

Abstract: In this work we introduce and study uniformly μ-paracompact spaces. In particular, the following problem is solved: What are the uniform spaces which for any finitely-additive open cover λ of power $\leq \mu$ admit a uniformly continuous λ-mapping to some metrizable space?

AMS Subject Classification: 54E15, 54D20
Key Words: finitely-additive cover of power $\leq \mu$; uniform μ-paracompactness; λ-mapping

1. Introduction

One of the most important topological properties of the paracompactness-type is the notion of μ-paracompact spaces. One of the interesting problems of uniform topology is finding uniform analogues of paracompactness-type properties. In this article we show a new approach to the definition of uniform μ-paracompactness of uniform spaces.

Throughout the paper "a uniformity" means a uniformity U defined by using covers (see [3], [4]). For a uniformity U on a set X, τ_U denotes the topology on X induced by U. For a Tychonoff space X by U_X we denote the universal uniformity of X, i.e. the largest uniformity compatible with the topology of X.

We begin with some definitions that we need in the paper.

Received: December 20, 2020

§Correspondence author

Let λ be an open cover of a topological space X and $f : X \to Y$ be a continuous mapping of X to a space Y. The mapping f is called a λ-mapping if every point $y \in Y$ has a neighborhood O_y whose inverse image $f^{-1}O_y$ is contained in at least one element of the cover λ, [1].

For covers α and β of a set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$. The cover consisting of all sets of form $\{A \bigcap B : A \in \alpha, B \in \beta\}$ is said to be the inner intersection of the covers α and β, and is denoted as $\alpha \wedge \beta$. A cover α is finitely-additive if $\alpha^{\angle} = \alpha$, $\alpha^{\angle} = \{\bigcup \beta : \beta \subset \alpha \text{ is finite}\}$.

For a cover α of a set X and $x \in X$, $M \subset X$ we have: $St(x, \alpha) = \{A \in \alpha : x \in A\}, \alpha(x) = \bigcup St(x, \alpha)$, $St(M, \alpha) = \{A \in \alpha : M \bigcap A \neq \varnothing\}, \alpha(M) = \bigcup St(M, \alpha)$.

A topological space X is called μ-paracompact if every open cover α of power $\leq \mu$ has a locally finite open refinement. Marconi [6] investigated uniformly $\mu - M$-paracompact spaces.

A uniform space (X, U) is called:

(i) uniformly $\mu - M$-paracompact if every open cover α of power $\leq \mu$ has a uniformly locally finite open refinement.

(ii) uniformly A-paracompact if every finitely-additive open cover α has a uniform locally finite refinement, [2].

(iii) uniformly B-paracompact, if for each finitely-additive open cover λ of (X, U) there exists a sequence $\{\alpha_n\} \subset U$ such that: for each point $x \in X$ there exists a number $n \in N$ and an element $L \in \lambda$ with the property $\alpha_n(x) \subset L$, [5].

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) onto a uniform space (Y, V). The mapping f is called:

(1) precompact if for each cover $\alpha \in U$ there exist a cover $\beta \in V$ and a finite cover $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$, [3];

(2) uniformly perfect if it is both precompact and perfect, [3];

(3) uniformly open if f maps each open cover $\alpha \in U$ to an open cover $f\alpha \in V$, [3];

(4) strongly uniformly open if for each cover $\alpha \in U$ there exist a cover $\beta \in V$ such that $f(\alpha(x)) \supset \beta(f(x))$ for all $x \in X$, [7].

2. Uniform μ-Paracompactness

Let (X, U) be a uniform space.

Definition 1. A uniform space (X, U) is called uniformly μ-paracompact if for each finitely-additive open cover λ of power $\leq \mu$ of (X, U) there exists a sequence $\{\alpha_n\} \subset U$ such that:

$$(U) \quad \text{for each } x \in X \text{ there are } n \in N \text{ and } L \in \lambda \text{ with } \alpha_n(x) \subset L.$$

Proposition 2. *If a space (X, U) is uniformly μ-paracompact, then its topological space (X, τ_U) is μ-paracompact. Conversely, if a Tychonoff space (X, τ) is μ-paracompact, then the uniform space (X, U_X) is uniformly μ-paracompact.*

Proof. Let α be an arbitrary finitely-additive open cover of power $\leq \mu$ of (X, τ_U). Then for α there exists a sequence $\{\beta_n\} \subset U$ satisfying condition (U). Consequently, there exists a pseudometric σ on X such that $\beta_{n+1}(x) \subset \{y : d(x, y) \leq \frac{1}{2^{n+1}}\} \subset \beta_n(x)$ for $n \in N$. The system $\langle\alpha\rangle = \{\langle A\rangle_\sigma : A \in \alpha\}$ is an open cover of the pseudometric space (X, σ), where $\langle A\rangle_\sigma$ is the interior of the set A in the topology τ_σ, induced by the pseudometric σ. By μ-paracompactness of the space (X, τ_σ), the cover $\langle\alpha\rangle = \{\langle A\rangle_\sigma : A \in \alpha\}$ has a locally finite open refinement β. Since $\tau_A \subset \tau_U$, then β is a refinement of α. Consequently, (X, τ_U) is μ-paracompact.

Conversely, let (X, τ) be μ-paracompact. Since the system of all open covers forms the base of the universal uniformity U_X, the space (X, U_X) is uniformly μ-paracompact. □

It follows from this proposition that if X is a μ-paracompact but not paracompact space, and X is endowed with the universal uniformity U_X, then the space (X, U_X) is uniformly μ-paracompact, but not uniformly paracompact.

Proposition 3. *Every uniformly $\mu - M$-paracompact space is uniformly μ-paracompact.*

Proof. The proof follows from the fact that a uniform space is uniformly $\mu - M$-paracompact if and only if each finitely-additive open cover of power $\leq \mu$ is uniform (see [6], page 320, condition 2). □

Corollary 4. *Every countably uniformly M-paracompact space is countably uniformly paracompact.*

Proposition 5. *Every uniformly B-paracompact space is uniformly μ-paracompact.*

Proof. The proof follows directly from Definition 1 and the definition of B-parocompactness of uniform spaces. □

Corollary 6. *Every uniformly B-paracompact space is countably uniformly paracompact.*

The following theorem establishes a characterization of uniformly μ-paracompact spaces by means of λ-mappings.

Theorem 7. *A uniform space (X, U) is uniformly μ-paracompact if and only if for every finitely-additive open cover λ of power $\leq \mu$ of (X, U) there exists a uniformly continuous λ-mapping f of (X, U) onto a metrizable uniform space (Y, V).*

Proof. Necessity. Let λ be an arbitrary finitely-additive open cover of power $\leq \mu$ of (X, U). By virtue of the uniform μ-paracompactness of the space (X, U) for λ there exists a sequence $\{\alpha_n\}$ of uniform covers satisfying the condition (U). Then there exists a pseudometric σ on X such that $\alpha_{n+1}(x) \subset \{y : d(x, y) \leq \frac{1}{2^{n+1}}\} \subset \alpha_n(x)$ for any $x \in X$, $n \in N$. The equivalence relation " $\sim$ "on X is introduced in a standard way: $x_1 \sim x_2$ if and only if $\rho(x_1, x_2) = 0$ for all $x_1, x_2 \in X$. By Y_λ we denote the factor-set of the set X with respect to the equivalence relation " $\sim$ " and by $f : X \to Y_\lambda$ denote the natural mapping of the set X to the set Y_λ. It is easily checked that d is a metric, where $d(y_1, y_2) = \rho(f^{-1}y_1, f^{-1}y_2)$, $y_1, y_2 \in Y$, V_λ is the uniformity on the Y_λ generated by the metric σ and $f : (X, U) \to (Y_\lambda, V_\lambda)$ is a λ-mapping.

Sufficiency. Let λ be an arbitrary finitely-additive open cover of power $\leq \mu$ of (X, U) and $f : (X, U) \to (Y_\lambda, V_\lambda)$ be a λ-mapping of the uniform space (X, U) onto a metrizable uniform space (Y_λ, V_λ). Let $\{\beta_n\}$ be a countable base of uniformity V_λ. Then $\{\alpha_n\} \subset U$, $\alpha_n = f^{-1}\beta_n$. We show that the sequence $\{\alpha_n\}$ satisfies condition (U). Let $x \in X$ be an arbitrary point. Then there exists a neighborhood $\beta_n(y)$ of a point $y \in Y_\lambda$ such that $f^{-1}\beta_n(y) \subset L$, for some $L \in \lambda$. Therefore, $\alpha_n(x) \subset L$. So, the space (X, U) is uniformly μ-paracompact. □

Corollary 8. *A uniform space (X, U) is countably uniformly paracompact if and only if for every finitely-additive countably open cover λ of (X, U) there exists a uniformly continuous λ-mapping f of (X, U) onto a metrizable uniform space (Y, V).*

Proposition 9. *The product $(X,U)\times(Y,V)$ of a uniformly μ-paracompact space (X,U) and a compact space (Y,V) is uniformly μ-paracompact.*

Proof. Let (X,U) be uniformly μ-paracompact and (Y,V) a compact space. Then according to Example 2.3.3., [4, p. 95] the projection $\pi_X : (X,U) \times (Y,V) \to (X,U)$ is uniformly perfect, i.e. it is an ω-mapping of the product $(X,U) \times (Y,V)$ onto uniformly μ-paracompact space (Y,V) for any finitely-additive open cover ω of power $\leq \mu$ of the product $(X,U) \times (Y,V)$. According to Theorem 1, the product $(X,U) \times (Y,V)$ is uniformly μ-paracompact. □

Corollary 10. *The product $(X,U) \times (Y,V)$ of a countably uniformly paracompact space (X,U) and a compact space (Y,V) is countably uniformly paracompact.*

Theorem 11. *Let $f : (X,U) \to (Y,V)$ be a uniformly perfect mapping from a uniform space (X,U) to a uniform space (Y,V). Then uniform $\mu - M$ - paracompactness is preserved both in the image and the preimage direction.*

Proof. Let (X,U) be uniformly $\mu - M$-paracompact and α be an arbitrary finitely-additive open cover of power $\leq \mu$ of (Y,V). Then $f^{-1}\alpha \in U$. Since the mapping f is precompact, there exist a cover $\beta \in V$ and a finite cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \succ f^{-1}\alpha$. It is clear that $(f^{-1}\beta \wedge \gamma)^{\angle} = (f^{-1}\beta)^{\angle} \succ (f^{-1}\alpha)^{\angle}$. Therefore, $\beta^{\angle} \succ \alpha$, i.e. $\alpha \in V$. Consequently, (Y,V) is uniformly $M - \mu$-paracompact.

Conversely, let (Y,V) be uniformly $M - \mu$-paracompact and α be an arbitrary finitely-additive open cover of power $\leq \mu$ of (X,U). Then $\{f^{-1}y : y \in Y\} \succ \alpha$. Since f is a closed mapping, it follows that $\beta = \{f^{\#}A : A \in \alpha\}$ is an open cover of power $\leq \mu$ of (Y,V), $f^{\#}A = Y\backslash f(X\backslash A)$. Then $\beta^{\angle} \in V$ and $f^{-1}\beta^{\angle} \succ \alpha$. Consequently, (X,U) is uniformly $M - \mu$-paracompact. □

Corollary 12. *Let $f : (X,U) \to (Y,V)$ be a uniformly perfect mapping from a uniform space (X,U) to a uniform space (Y,V). Then countable uniform M-paracompactness is preserved both in the image and the preimage direction.*

Proposition 13. *Let $f : (X,U) \to (Y,V)$ be a strongly uniformly open mapping of a uniformly μ-paracompact space (X,U) to a uniform space (Y,V). Then the space (Y,V) is uniformly μ-paracompact.*

Proof. Let (X, U) be uniformly μ-paracompactum and β be an arbitrarily finitely-additive open cover of power $\leq \mu$ of (Y, V). Since f is a uniformly continuous mapping, it follows that $f^{-1}\beta$ is a finitely-additive open cover of power $\leq \mu$ of (X, U). Then there exists a sequence of uniform covers $\{\alpha_n\}$ satisfying condition (U). Since f is strongly uniformly open, for each $\alpha_n \in U$ there exists a cover $\beta \in V$ such that $f(\alpha_n(x)) \supset \beta_n(f(x))$ for each point $x \in X$. Therefore, for any point $y \in Y$, there exist $n \in N$ and $B \in \beta$ such that $\beta_n(y) \subset B$. Therefore, (Y, V) is uniformly μ-paracompact. □

Corollary 14. *Let $f : (X, U) \to (Y, V)$ be a strongly uniformly open mapping of a countably uniformly paracompact space (X, U) to a uniform space (Y, V). Then space (Y, V) is countably uniformly paracompact.*

Proposition 15. *Let $f : (X, U) \to (Y, V)$ be a uniformly open mapping of a uniformly $\mu - M$-paracompact space (X, U) to a uniform space (Y, V). Then the space (Y, V) is uniformly $\mu - M$-paracompact.*

Proof. Let (X, U) be an uniformly $\mu - M$-paracompactum and β an arbitrary finitely-additive open cover of power $\leq \mu$ of (Y, V). Since f is a uniformly continuous mapping, it follows that family $f^{-1}\beta$ is a finitely-additive open cover of power $\leq \mu$ of (X, U) and $f^{-1}\beta \in U$. Then the uniform openness of f implies that $\beta \in V$. Therefore, (Y, V) is uniformly $\mu - M$-paracompact. □

Corollary 16. *Let $f : (X, U) \to (Y, V)$ be a uniformly open mapping of a countably uniformly M-paracompact space (X, U) to a uniform space (Y, V). Then the space (Y, V) is countably uniformly M-paracompact.*

Proposition 17. *Let (X, U) be a uniformly $\mu - M$- paracompact space and the index of compactness of (X, τ_U) is $\leq \mu$. Then the space (X, U) is complete.*

Proof. Suppose that a Cauchy filter F in (X, U) does not converge at any point of the uniform space (X, U). Then every point $x \in X$ has a neighborhood O_x and $E_x \in F$ such that $O_x \bigcap E_x = \varnothing$. Consider the open cover $\lambda = \{O_x : x \in X\}$ of power $\leq \mu$. Then $\lambda^{\angle} \in U$, therefore $\lambda^{\angle} \bigcap F \neq \varnothing$, i.e. there are $O_{x_k} \in \lambda$, $k = 1, 2, ..., n$, such that $\bigcup\limits_{k=1}^{n} O_x \in \lambda \bigcap F$. Hence,

$$\left(\bigcap_{k=1}^{n} E_{x_k}\right) \bigcap \left(\bigcup_{k=1}^{n} O_{x_k}\right) \neq \varnothing \text{ and } \bigcap_{k=1}^{n} E_{x_k} \in F,\ \bigcup_{k=1}^{n} O_{x_k} \in F.$$

A contradiction. So, (X, U) is complete. $\square$

Corollary 18. *Let (X, U) be a countably uniformly M-para-compact space such that (X, τ_U) is a Lindelöf space. Then the uniform space (X, U) is complete.*

A uniform space (X, U) is called uniformly locally μ-compact if the uniformity of U contains a uniform cover whose closure of each element is μ-compact.

Proposition 19. *Any uniformly locally μ-compact space is uniformly $\mu - M$-paracompact.*

Proof. Let (X, U) be uniformly locally μ-paracompact and α be a finitely-additive open cover of power $\leq \mu$. Then there exists such a uniform cover β the closure of each element of which is μ-paracompact. Then it is easy to see that β is a refinement of α. Consequently, (X, U) is uniformly $\mu - M$-paracompact. $\square$

Corollary 20. *Any uniformly locally countably compact space is countably uniformly M-paracompact.*

Corollary 21. *Any locally μ-compact topological group is $\mu - M$-paracompact.*

Let us prove the following simple topological lemma.

Lemma 22. *A topological space X is μ-paracompact if and only if every finitely-additive open cover α of power $\leq \mu$ has a locally finite open refinement.*

Proof. Necessity. The necessity is obvious.

Sufficiency. Let α be an arbitrary open cover of power $\leq \mu$ of X. Then for the finitely-additive open cover $\alpha^{\angle}$ of power $\leq \mu$ of (X, U) there exists a locally finite open cover β which is a refinement of it. For each $B \in \beta$, choose $A_B \in \alpha^{\angle}$ such that $B \subset A_B^{\angle}$, where $A_B^{\angle} = \bigcup\limits_{i=1}^{n} A_i$, $A_i \in \alpha$, $i = 1, 2, ..., n$. Let $\alpha_0 = \bigcup\{\alpha_B : B \in \beta\}$, $\alpha_B = \{B \bigcap A_i : i = 1, 2, ..., n\}$. Then α_0 is an open locally finite cover of the space X. It is clear that $\alpha_0 \succ \alpha$. Consequently, X is μ-paracompact. $\square$

A uniform space (X, U) is called uniformly $\mu - A$-paracompact if every finitely-additive open cover α of power $\leq \mu$ has a uniform locally finite refinement.

If a space (X, U) is uniformly $\mu - A$-paracompact, then its topological space (X, τ_U) is μ-paracompact. Conversely, if a space (X, τ) is μ-paracompact, then the uniform space (X, U_X) is uniformly $\mu - A$-paracompact.

Any uniformly A-paracompact space is uniformly $\mu - A$-paracompact.

Proposition 23. *Any uniformly $\mu - A$-compact space is uniformly $\mu - M$-paracompact.*

A uniform space (X, U) is called strongly uniformly locally μ-compact if the uniformity of U contains a locally finite uniform cover whose closure of each element is μ-compact.

Proposition 24. *Any strongly uniformly locally μ-compact space is uniformly $\mu - A$-paracompact.*

Proof. Let (X, U) be strongly uniformly locally μ-paracompact and α is a finitely-additive open cover of power $\leq \mu$. Then there exists such a locally finite uniform cover β the closure of each element of which is μ-paracompact. Then it is easy to see that β is a refinement of the α. Consequently, (X, U) is uniformly $\mu - A$-paracompact. □

References

[1] P.S. Aleksandrov, *Introduction to the Theory of Sets and General Topology*, Nauka, Moscow (1977) (in Russian).

[2] L.V. Aparina, About the uniformity and proximity of Lindelöf analogues, *Uspehi Math. Nauk*, **37** (1982), 141-142 (in Russian).

[3] A.A. Borubaev, *Uniform Spaces and Uniformly Continuous Mappings*, Ilim, Frunze (1990) (in Russian).

[4] R. Engelking, *Genereal Topology*, Revised and completed Ed., Heldermann Verlag, Berlin (1989).

[5] B.E. Kanetov, *Some Classes of Uniform Spaces and Uniformly Continuous Mappings*, Bishkek (2013) (in Russian).

[6] U. Marconi, On the uniform paracompactness, *Rend. Sem. Mat. Univ. Padova*, **72** (1984), 319-328.

[7] A.S. Mishchenko, About uniformly closed mappings, *Fund. Math.*, **124** (1981), 219-228.