International Journal of Applied Mathematics

Volume 34 No. 2 2021, 297-310
ISSN: 1311-1728 (printed version); ISSN: 1314-8060 (on-line version)
doi: http://dx.doi.org/10.12732/ijam.v34i2.7

TIME-DEPENDENT SOURCE IDENTIFICATION SCHRÖDINGER TYPE PROBLEM

Allaberen Ashyralyev[1,2,3], Mesut Urun[1,4 §]

[1] Department of Mathematics, Near East University
Nicosia, TRNC Mersin – 10, TURKEY
[2] Peoples' Friendship University of Russia
(RUDN University), Ul. Miklukho Maklaya 6
Moscow – 117198, RUSSIA
[3] Institute of Mathematics and Mathematical Modeling
Almaty, KAZAKHSTAN

[4] Maritime Vocational School, Galatasaray University
Istanbul – 34349, TURKEY

Abstract: In this study, the source identification problem with non-local boundary conditions for the time-dependent, one-dimensional Schrödinger equation is studied. Stability estimates are constructed for the solution of source identification problem. A first order of accuracy difference scheme is investigated for the numerical solution of this problem.

AMS Subject Classification: 65J22, 39A14, 35N25
Key Words: identification problem; Schrödinger equation; nonlocal problem; stability

1. Introduction

Source identification problems (SIPs) have significant role in natural science, applied sciences, engineering, quantum mechanics, diffusion equations, heat equations (see, e.g., [1], [2], [3] [4]). The theory and applications of SIPs for partial differential equations(PDEs) were studied in various investigations (see,

Received: December 15, 2020

§Correspondence author

e.g., [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27]). In this paper we investigate one-dimensional time-dependent SIP for the Schrödinger equation (SE)

$$\begin{cases} iz_t - (a(x)\, z_x)_x + \delta z = p(t)\, b(x) + g(t,x), \\ 0 < t < T, x \in (0,l), \\ z(0,x) = \varphi(x), x \in [0,l], \\ z(t,0) = z(t,l), z_x(t,0) = z_x(t,l), \\ \int_0^l z(t,x)\, dx = \zeta(t), \ \ t \in [0,T]. \end{cases} \tag{1}$$

Assume that $0 < a \leq a(x), g(t,x), \zeta(t), \varphi(x), b(x)$ and $a(x)$ are given sufficiently smooth functions with the boundary and integral conditions $b(0) = b(l)$, $b^{'}(0) = b^{'}(l)$ and $\int_0^l b(x)\, dx \neq 0$. In this paper we investigate the stability of differential and difference SIPs. We establish our study in four steps. Section 1 serves as introduction. In Section 2 differential stability of the problem (1) is presented. In Section 3, the stability of the first order of accuracy difference scheme of the problem (1) is presented. Finally, results of numerical experiments are provided.

2. Theorem on Stability of Problem (1)

Assume that H is a Hilbert space and A is a self-adjoint positive-definite operator defined by the formula

$$Az = -\frac{d}{dx}\left(a(x)\frac{dz(x)}{dx}\right) + \delta z(x) \tag{2}$$

with a domain

$$D(A) = \{z : z, z^{''} \in L_2[0,l], \quad z(0) = z(l), \quad z^{'}(0) = z^{'}(l)\}. \tag{3}$$

Let $C(H) = C([0,T], H)$ be the Banach space of all abstract continuous functions $u(t)$ defined on $[0,T]$ with values in H equipped with the norm

$$\|u\|_{C(H)} = \max_{t\in[0,T]} \|u(t)\|_H. \tag{4}$$

Moreover, let $L_2[0,l]$ be the space of all square integrable functions $\psi(x)$ defined on $[0,l]$ equipped with the norm

$$\|\psi\|_{L_2[0,l]} = \left(\int_0^l |\psi(x)|^2\, dx\right)^{\frac{1}{2}} \tag{5}$$

and $W_2^2[0,l] = \{\psi : \psi, \psi_{xx} \in L_2[0,l]\}$.

Theorem 1. *Assume that $g(t,x)$ and $\zeta(t)$ are continuously differentiable functions with $\zeta \in C[0,T]$. Then the problem (1) has a unique solution $z \in C(L_2[0,l])$ and $p \in C[0,T]$ and for the solution of problem (1) the following stability estimates hold:*

$$\|z_t\|_{C(L_2[0,l])} + \|z\|_{C\left(W_2^2[0,l]\right)} + \|p\|_{C[0,T]} \leq M(b,\delta)\Big[\|\varphi\|_{W_2^2[0,l]} + \|g(0,\cdot)\|_{L_2[0,l]} + \|g_t\|_{C(L_2[0,l])} + \|\zeta_t\|_{C[0,T]}\Big]. \tag{6}$$

Here, M denotes a positive constant which may differ in time and thus is not a subject of precision. We use the notation $M(b,c)$ to stress the fact that the constant depends only on b and c.

Proof. Let us denote

$$z(t,x) = w(t,x) - i\eta(t)\, b(x), \tag{7}$$

where

$$\eta(t) = \int_0^t p(s)\,ds, \quad \eta(0) = 0, \tag{8}$$

and let $w(t,x)$ be the solution of the following problem:

$$\begin{cases} iw_t(t,x) - (a(x)\,w_x(t,x))_x + \delta w(t,x) \\ = g(t,x) + \eta(t)\left[(a(x)\,b_x(x))_x - \delta b(x)\right], \\ x \in (0,l),\, t \in (0,T), \\ w(0,x) = \varphi(x),\, x \in [0,l], \\ w(t,0) = w(t,l),\, w_x(t,0) = w_x(t,l),\, t \in [0,T]. \end{cases} \tag{9}$$

Applying the integral condition in (1) and formula (7), we can write

$$\eta(t) = ib\left(\zeta(t) - \int_0^l w(t,x)\,dx\right), \quad b = \frac{1}{\int_0^l b(x)\,dx}. \tag{10}$$

Applying formulas (8) and (10), we get

$$p(t) = ib\left(\zeta'(t) - \int_0^l w_t(t,x)\,dx\right). \tag{11}$$

Using that $\int_0^l b(x)dx \neq 0$, we obtain the estimate

$$|p(t)| \leq K_1(a)\left[|\zeta_t(t)| + \|w_t(t,\cdot)\|_{L_2[0,l]}\right. \tag{12}$$

for each $t, t \in [0,T]$. From (9) it follows that

$$z_t(t,x) = w_t(t,x) - ip(t)a(x) \tag{13}$$

and

$$\|z_t\|_{C(L_2[0,l])} \leq \|w_t\|_{C(L_2[0,l])} + \|p\|_{C[0,T])} \|a\|_{L_2[0,l]} . \tag{14}$$

□

Theorem 2. *Under the assumptions of Theorem 1, problem (9) has a unique solution in* $C(L_2[0,l])$ *and the following stability estimate is satisfied*:

$$\begin{aligned} \|w_t\|_{C(L_2[0,l])} \leq K_2(b,\delta) \Big[\|\varphi\|_{W_2^2[0,l]} + |\zeta(0)| \\ + \|g(0,\cdot)\|_{L_2[0,l]} + \|g_t\|_{C(L_2[0,l])} + \|\zeta_t\|_{C[0,T]} \Big] . \end{aligned} \tag{15}$$

Proof. Problem (9) is equivalent to the problem of solving the integral equation

$$\begin{aligned} w(t,x) = e^{iAt}\varphi(x) \\ + \int_0^t e^{iA(t-p)} \left\{ g(p,x) - b\left(\zeta(p) - \int_0^l w(p,x)\,dx\right) Ab(x) \right\} dp. \end{aligned} \tag{16}$$

Here, e^{iAt} is the operator-function generated by the operator A and defined as the solution of the initial value problem

$$iw_t + Aw(t), \quad t > 0, \ w(0) = \varphi. \tag{17}$$

Taking the derivative and using integration by parts, we get

$$\begin{aligned} w_t(t,x) = iAe^{iAt}\varphi(x) - g(0,x) - b\left(\zeta(0) - \int_0^l w(0,x)dx\right) Ab(x) \\ - \int_0^t e^{iA(t-p)} \left\{ \frac{\partial g(p,x)}{\partial p} - b\left(\zeta_p(p) - \int_0^l \frac{\partial w(p,x)}{\partial p} dx\right) Ab(x) \right\} dp. \end{aligned} \tag{18}$$

Applying formula (18), Hölder's inequality, estimate (13) and the estimate

$$\|e^{iAt}\|_{L_2[0,l] \to L_2[0,l]} \leq e^{-\delta t}, \tag{19}$$

we get

$$\|w_t(t,\cdot)\|_{L_2[0,l]} \leq \|A\varphi\|_{L_2[0,l]} + \|g(0,\cdot)\|_{L_2[0,l]} + K_3(l,a,b)\left(\|\varphi\|_{L_2[0,l]} + |\zeta(0)|\right)$$

$$+T\|g_t\|_{C(L_2[0,l])} + K_4(l,a,b)\|\zeta_t\|_{C[0,T]} + K_5(l,a,b) \int_0^t \|w_p(p,\cdot)\|_{L_2[0,l]}\,dp.$$

Then, applying the integral inequality, we get

$$\|w_t(t,\cdot)\| \leq K_6\,(l,a,b)\, e^{K_5(l,a,b)t} \tag{20}$$

for any $t \in [0,T]$, where

$$K_6\,(l,a,b) = \|A\varphi\|_{L_2[0,l]} + \|g(0,\cdot)\|_{L_2[0,l]} + K_3(l,a,b)\left(\|\varphi\|_{L_2[0,l]} + |\zeta(0)|\right)$$

$$+ T\,\|g_t\|_{C(L_2[0,l])} + K_4\,(l,a,b)\,\|\zeta_t\|_{C[0,T]}\,.$$

□

3. Stability of Difference Problem

Let $C_\tau\,(H) = C\left([0,T]_\tau\,,H\right)$ be the normed space of all mesh functions $\theta^\tau = \{\theta_k\}_{k=0}^N$ defined on

$$[0,T]_\tau = \{t_k = k\tau,\ 0 \leq k \leq N,\ N\tau = T\}$$

with values in H equipped with the norm

$$\|\theta^\tau\|_{C_\tau(H)} = \max_{0\leq k\leq N} \|\theta_k\|_H\,. \tag{21}$$

Moreover, $L_{2h} = L_2\,[0,l]_h$ and $W_{2h}^2 = W_2^2\,[0,l]_h$ are normed spaces of all mesh functions $\gamma^h\,(x) = \{\gamma_n\}_{n=0}^M$ defined on

$$[0,l]_h = \{x_n = nh,\quad 0 \leq n \leq M,\ Mh = l\}$$

equipped with the norms

$$\left\|\gamma^h\right\|_{L_{2h}} = \left\{\sum_{i=0}^M |\gamma_i|^2\,h\right\}^{\frac{1}{2}},$$

$$\left\|\gamma^h\right\|_{W_{2h}^2} = \left\{\sum_{i=0}^M |\gamma_i|^2\,h + \sum_{i=1}^{M-1}\left|\frac{\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}}{h^2}\right|^2 h\right\}^{\frac{1}{2}}.$$

Moreover, we introduce the difference operator A_h defined by the formula

$$A_h u^h\,(x) = \left\{-\left(a_{n+1}\frac{u_{n+1} - u_n}{h^2} - a_n\frac{u_n - u_{n-1}}{h^2}\right) + \delta u_n\right\}_{n=1}^{M-1} \tag{22}$$

acting in the space of mesh functions $u^h(x) = \{u_n\}_{n=0}^M$ defined on $[0,l]_h$ satisfying the conditions $u_M = u_0, u_M - u_{M-1} = u_1 - u_0$. For the numerical solution $\{z_k^h(x)\}_{k=0}^N$ of problem (1), we present difference scheme (DS) of the first order of approximation

$$\begin{cases} i\frac{z_n^k - z_n^{k-1}}{\tau} - \left(a_{n+1}\frac{z_{n+1}^k - z_n^k}{h^2} - a_n\frac{z_n^k - z_{n-1}^k}{h^2}\right) + \delta z_n^k = p_k a_n + g_n^k, \\ g_n^k = g(t_k, x_n), t_k \in [0,T]_\tau, x_n \in [0,l]_h, k \in \overline{1,N}, \\ n \in \overline{1,M-1}, N\tau = T, z_n^0 = \varphi_n, \varphi_n = \varphi(x_n), n \in \overline{0,M}, \\ z_M^k = z_0^k, z_M^k - z_{M-1}^k = z_1^k - z_0^k, \sum_{i=1}^M z_i^k h = \zeta(t_k), \ \ k \in \overline{0,N}. \end{cases} \quad (23)$$

Here, it is assumed that $a_M = a_0, a_M - a_{M-1} = a_1 - a_0$, and $\sum_{m=1}^M a_m h \neq 0$. Let us give the following result on the stability of DS (23).

Theorem 3. *For the solution of DS (23) the stability estimates are satisfied:*

$$\left\| \left\{ \frac{1}{\tau}\left(z_k^h - z_{k-1}^h\right) \right\}_{k=1}^N \right\|_{C_\tau(L_{2h})} + \left\| \left\{ z_k^h \right\}_{k=1}^N \right\|_{C_\tau(W_{2h}^2)} + \left\| \{p_k\}_{k=1}^N \right\|_{C[0,T]_\tau}$$

$$\leq K(q) \left[\left\|\varphi^h\right\|_{W_{2h}^2} + \left\|g_1^h\right\|_{L_{2h}} + |\zeta_0| \right.$$

$$\left. + \left\| \left\{ \frac{1}{\tau}\left(g_k^h - g_{k-1}^h\right) \right\}_{k=2}^N \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{1}{\tau}\left(\zeta_k - \zeta_{k-1}\right) \right\}_{k=1}^N \right\|_{C[0,T]_\tau} \right].$$

Proof. Let use denote

$$z_n^k = w_n^k - i\eta_k b_n, \quad (24)$$

where

$$b_n = b(x_n), \eta_k = \sum_{m=1}^k p_m \tau. \quad (25)$$

It is easy to show that $\left\{w_k^h(x)\right\}_{k=0}^N$ is the solution of the DS

$$\begin{cases} i\frac{w_n^k-w_n^{k-1}}{\tau}-\left(a_{n+1}\frac{w_{n+1}^k-w_n^k}{h^2}-a_n\frac{w_n^k-w_{n-1}^k}{h^2}\right)+\delta w_n^k \\ \;=g_n^k+\left[-\frac{1}{h}[a_{n+1}\frac{b_{n+1}-b_n}{h}-a_n\frac{b_n-b_{n-1}}{h}]+\delta a_n\right]i\eta_k, \\ k\in\overline{1,N},\quad n\in\overline{1,M-1}, \\ w_n^0=\varphi_n,\quad n\in\overline{0,M}, \\ w_M^k=w_0^k, w_M^k-w_{M-1}^k=w_1^k-w_0^k,\quad k\in\overline{0,N}. \end{cases} \tag{26}$$

Now we estimate $|p_k|$. Using the discrete analogy of the integral condition in (23) and using (28), we obtain

$$\eta_k=ib_1\left(\zeta_k-\sum_{m=1}^M w_m^k h\right), b_1=\frac{1}{\sum_{m=1}^M b_m h}. \tag{27}$$

Then,

$$p_k=\frac{ib_1}{\tau}\left(\zeta_k-\zeta_{k-1}-\sum_{m=1}^M(w_m^k-w_m^{k-1})h\right). \tag{28}$$

Applying the Cauchy-Schwartz inequality, we get

$$|p_k|\leq K(b)\left[\left|\frac{\zeta_k-\zeta_{k-1}}{\tau}\right|+\left\|\frac{w_k^h-w_{k-1}^h}{\tau}\right\|_{L_{2h}}\right] \tag{29}$$

for all $1\leq k\leq N$ and

$$\left\|\{p_k\}_{k=1}^N\right\|_{C[0,T]_\tau} \tag{30}$$

$$\leq K(b)\left[\left\|\left\{\frac{\zeta_k-\zeta_{k-1}}{\tau}\right\}_{k=1}^N\right\|_{C[0,T]_\tau}+\left\|\left\{\frac{w_k^h-w_{k-1}^h}{\tau}\right\}_{k=1}^N\right\|_{C_\tau(L_{2h})}\right].$$

Applying (24), we obtain

$$\frac{z_n^k-z_n^{k-1}}{\tau}=\frac{w_n^k-w_n^{k-1}}{\tau}-ip_kb_n$$

and

$$\left\|\left\{\frac{1}{\tau}\left(z_k^h-z_{k-1}^h\right)\right\}_{k=1}^N\right\|_{C_\tau(L_{2h})} \tag{31}$$

$$\leq \left\| \left\{ \frac{1}{\tau} \left(w_k^h - w_{k-1}^h \right) \right\}_{k=1}^N \right\|_{C_\tau(L_{2h})} + \left\| \{p_k\}_{k=1}^N \right\|_{C[0,T]_\tau} \left\| \{b_n\}_{n=1}^M \right\|_{L_{2h}}.$$

Therefore, the statements complete the proof of Theorem 3. □

Theorem 4. *For the solution of DS (26) the stability estimate is satisfied:*

$$\left\| \left\{ \frac{1}{\tau} \left(w_k^h - w_{k-1}^h \right) \right\}_{k=1}^N \right\|_{C_\tau(L_{2h})} \leq K_3(a) \left[\left\| \varphi^h \right\|_{W_{2h}^2} + |\zeta_0| \right. \tag{32}$$

$$\left. + \left\| g_1^h \right\|_{L_{2h}} + \left\| \left\{ \frac{1}{\tau} \left(g_k^h - g_{k-1}^h \right) \right\}_{k=2}^N \right\|_{C_\tau(L_{2h})} + \left\| \left\{ \frac{\zeta_k - \zeta_{k-1}}{\tau} \right\}_{k=1}^N \right\|_{C[0,T]_\tau} \right].$$

Proof. Problem (26) is equivalent to the following abstract problem

$$\begin{cases} i\frac{w_k^h - w_{k-1}^h}{\tau} + A^h w_k^h = g_k^h + iA^h b^h \eta_k \\ t_k = k\tau, 1 \leq k \leq N, N\tau = T, w_0^h = \varphi^h. \end{cases} \tag{33}$$

Therefore,

$$w_k^h = R^k \varphi^h - i \sum_{j=1}^k R^{k-j+1} \tau \left(g_j^h + iA^h a^h \eta_j \right), \tag{34}$$

where $R = (I + i\tau A^h)^{-1}$. Taking the difference derivative, we get

$$\frac{w_k^h - w_{k-1}^h}{\tau} = -iA^h R^k \varphi^h - iR^k (g_1^h + iA^h a^h \eta_1) \tag{35}$$

$$-i \sum_{j=2}^k R^{k-j+1} (g_j^h - g_{j-1}^h + iA^h a^h (\eta_j - \eta_{j-1})).$$

Applying formula (35) and estimate (29), and the following estimate

$$\|R\|_{H \longrightarrow H} \leq \frac{1}{1 + \delta\tau},$$

we get

$$\left\| \frac{w_k^h - w_{k-1}^h}{\tau} \right\|_{L_{2h}} \leq \left\| A_h^x \varphi^h \right\|_{L_{2h}} + \left\| g_1^h \right\|_{L_{2h}}$$

$$+\sum_{j=2}^{k}\left\|g_j^h-g_{j-1}^h\right\|_{L_{2h}}+K_4(a)\sum_{j=1}^{k}\tau\left[\left|\frac{\zeta_j-\zeta_{j-1}}{\tau}\right|+\left\|\frac{w_j^h-w_{j-1}^h}{\tau}\right\|_{L_{2h}}\right].$$

Putting $\left\|\frac{w_k^h-w_{k-1}^h}{\tau}\right\|_{L_{2h}}=Z_k$, we obtain the discrete analogy of integral inequality

$$Z_k\leq\left\|A_h^x\varphi^h\right\|_{L_{2h}}+\left\|g_1^h\right\|_{L_{2h}}+\sum_{j=2}^{N}\left\|g_j^h-g_{j-1}^h\right\|_{L_{2h}}+K_4(a)\sum_{j=1}^{N}|\zeta_j-\zeta_{j-1}|$$

$$+K_4(a)\sum_{j=1}^{k}\tau Z_j.$$

From that it follows

$$Z_k\leq\left[\left\|A_h^x\varphi^h\right\|_{L_{2h}}+\left\|g_1^h\right\|_{L_{2h}}+\sum_{j=2}^{N}\left\|g_j^h-g_{j-1}^h\right\|_{L_{2h}}\right.$$

$$\left.+K_4(a)\sum_{j=1}^{N}|\zeta_j-\zeta_{j-1}|\right]\times\frac{1}{(1-K_4(a)\tau)^k}.$$

Since

$$\frac{1}{1-K_4(a)\tau}=1+\frac{K_4(a)\tau}{1-K_4(a)\tau},$$

we have

$$\frac{1}{(1-K_4(a)\tau)^k}\leq e^{\frac{K_4(a)T}{1-K_4(a)\tau}}.$$

$$Z_k\leq\left[\left\|A_h^x\varphi^h\right\|_{L_{2h}}+\left\|g_1^h\right\|_{L_{2h}}+\sum_{j=2}^{N}\left\|g_j^h-g_{j-1}^h\right\|_{L_{2h}}\right.$$

$$\left.+K_4(a)\sum_{j=1}^{N}|\zeta_j-\zeta_{j-1}|\right]\times e^{\frac{K_4(a)T}{1-K_4(a)\tau}}$$

for any k. From that it follows (32). □

4. Numerical Experiment

For the numerical experiment we consider the SIP

$$\begin{cases} iz_t - z_{xx} + z = p\left(t\right)\left(1+\sin 2x\right) + e^{it}\left(-1+3\sin(2x)\right), \\ x\in(0,\pi), \quad t\in(0,1), \\ z\left(0,x\right) = 1+\sin 2x, \quad x\in[0,\pi], \\ z\left(t,0\right) = z\left(t,\pi\right), z_x\left(t,0\right) = z_x\left(t,\pi\right), \int_0^\pi z\left(t,x\right)dx = \pi e^{it}, t\in[0,1], \end{cases} \tag{36}$$

for a SE. The exact solution of this problem is $(z,p) = \left((1+\sin 2x)e^{it}, e^{it}\right)$. Applying difference scheme (23) for problem (36), we get

$$\begin{cases} \frac{i}{\tau}\left(z_n^k - z_n^{k-1}\right) - \frac{1}{h^2}\left(z_{n+1}^k - 2z_n^k + z_{n-1}^k\right) + z_n^k \\ \quad = p_k\left(1+\sin 2x_n\right) + \left(-1+3\sin(2x_n)\right)e^{it_k}, \\ t_k = k\tau, x_n = nh, k\in\overline{1,N}, \quad n\in\overline{1,M-1}, \\ z_n^0 = 1+\sin 2x_n, n\in\overline{0,M}, \quad Mh=\pi, \quad N\tau=1, \\ z_M^k = z_0^k, z_M^k - z_{M-1}^k = z_1^k - z_0^k, \sum_{m=1}^M z_m^k h = \pi e^{it_k}, \quad k\in\overline{0,N}, \end{cases} \tag{37}$$

The algorithm for obtaining the solution $\left\{\left\{z_n^k\right\}_{k=0}^N\right\}_{n=0}^M$ and $\{p_k\}_{k=1}^N$ of DS (37) contains three steps. We introduce η_k by the formula

$$\eta_k = \sum_{m=1}^k p_m\tau, \qquad k\in\overline{1,N}, \quad \eta_0 = 0. \tag{38}$$

Then,

$$p_k = \frac{\eta_k - \eta_{k-1}}{\tau}, \quad \overline{1,N}, \tag{39}$$

$$z_n^k = w_n^k - i\eta_k, \quad k\in\overline{0,N}, \quad n\in[0,M]. \tag{40}$$

Here w_n^k is the solution of the DS

$$\begin{cases} i\frac{w_n^k - w_n^{k-1}}{\tau} - \frac{w_{n+1}^k - 2w_n^k + w_{n-1}^k}{h^2} + w_n^k + \frac{\pi e^{it_k} - \sum_{m=1}^M w_m^k h}{\pi} \\ \quad = e^{it_k}(-1+4\sin 2x), \quad k\in\overline{1,N}, \quad n\in\overline{1,M-1}, \\ w_n^0 = 1+\sin 2x_n, \quad n\in\overline{0,M}, \\ w_0^k = w_M^k, \quad w_M^k - w_{M-1} = w_1^k - w_0^k, k\in\overline{0,N}. \end{cases} \tag{41}$$

Using the discrete analogy of integral condition in (41), we get

$$\eta_k = \frac{\sum_{m=1}^M w_m^k h - \pi e^{it_k}}{i\pi}, \qquad k\in\overline{1,N}. \tag{42}$$

Step 1: According to DS (41), we obtain $\left\{\{w_n^k\}_{k=0}^N\right\}_{n=0}^M$.
Step 2: We will find $\{\eta_k\}_{k=0}^N$, $\{p_k\}_{k=1}^N$ by formulas (42) and (39).
Step 3: We will find $\left\{\{z_n^k\}_{k=0}^N\right\}_{n=0}^M$ by formulas (38) and (40).
The errors are computed by

$$E_z = \max_{k\in\overline{0,N}} \left(\sum_{n=0}^{M} \left| z\left(t_k, x_n\right) - z_n^k \right|^2 h \right)^{\frac{1}{2}}, \tag{43}$$

$$E_p = \max_{k\in\overline{1,N}} \left| p\left(t_k\right) - p_k \right|. \tag{44}$$

Numerical solutions of $z\left(t,x\right)$ at $\left(t_k, x_n\right)$ is z_n^k and of $p(t)$ at t_k is p_k.

The results of numerical experiments for problem (36) are provided in Table 4.1.

Error	$M=N=40$	$M=N=80$	$M=N=160$
E_p	0.0496	0.0248	0.0124
E_z	0.0067	0.0031	0.0015

Table 4.1: Error analysis

As it is seen in Table 4.1, if M and N are multiplied by 2, the value of errors decreases approximately 1/2 for the DS. This shows that it has the first order of accuracy.

5. Conclusion

In this article, the SIP with non-local boundary conditions for the one-dimensional SE is studied. Stability estimates are established for SIP differential and difference problems. Results of the numerical experiments presented.

References

[1] A.I. Prilepko, D.G. Orlovsky, I.A. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, Marcel Dekker (1987).

[2] S.I. Kabanikhin, Methods for solving dynamic inverse problems for hyperbolic equations, *J. of Inverse and Ill-Posed Problems*, **12** (2014), 493-517.

[3] Y.Y. Belov, *Inverse Problems for Partial Differential Equations*, In: *Inverse and Ill-Posed Problems Series*, 32, De Grayter (2002).

[4] Y.A. Gryazin, M.V. Klibanov, T.R. Lucas, Imaging the diffusion coefficient in a parabolic inverse problem in optical tomography, *Inverse Problems*, **15**, No 2 (1999), 373-397.

[5] C. Ashyralyyev, High order of accuracy difference schemes for the inverse elliptic problem with Dirichlet condition, *Boundary Value Problems*, **2014**, No 5 (2014); doi: 10.1186/1687-2770.

[6] C. Ashyralyyev, High order approximation of the inverse elliptic problem with Dirichlet-Neumann conditions, *Filomat*, **28**, No 5 (2014), 947-962.

[7] A. Ashyralyev, C. Ashyralyyev, On the problem of determining the parameter of an elliptic equation in a Banach space, *Nonlinear Analysis: Modelling and Control*, **19**, No 3 (2014), 350-366.

[8] A. Ashyralyev, M. Urun, Determination of a control parameter for the difference Schrödinger equation, *Abstract and Applied Analysis*, **2013** (2013), ID 548201.

[9] A.B. Kostin, The inverse problem of recovering the source in a parabolic equation under a condition of nonlocal observation, *Matematicheskii Sbornik*, **204**, No 10 (2013), 1391-1434 (In Russian).

[10] A. Ashyralyev, On a problem of determining the parameter of a parabolic equation, *Ukrainian Mathematical J.*, **62**, No 9 (2010), 1200-1210.

[11] M. Choulli, M. Yamamoto, Generic well-posedness of a linear inverse parabolic problem with diffusion parameter, *J. of Inverse and III-posed Problems*, **7**, No 3 (1999), 241-254.

[12] A. Ashyralyev, F. Emharab, Source identification problems for hyperbolic differential and difference equations. *J. of Inverse and Ill-posed Problems*, **27**, No 3 (2019), 301-315.

[13] A. Ashyralyev A.U. Sazaklioglu, Investigation of a time-dependent source identification inverse problem with integral overdetermination, *Numerical Functional Analysis and Optimization*, **38**, No 10 (2017), 1276-1294.

[14] S. Saitoh, V.K. Tuan, M. Yamamoto, Reverse convolution inequalities and applications to inverse heat source problems, *J. of Inequalities in Pure and Applied Mathematics*, **3**, No 5 (2002), Art. 80.

[15] N.I. Ivanchov, On the determination of unknown source in the heat equation with nonlocal boundary conditions, *Ukrainian Mathematical J.*, **47**, No 10 (1995), 1647-1652.

[16] V.T. Borukhov, P.N. Vabishchevich, Numerical solution of the inverse problem of reconstructing a distributed right-hand side of a parabolic equation, *Computer Physics Communications*, **126** (2000), 32-36.

[17] A. Ashyralyev, A.S. Erdogan, Well-posedness of the right-hand side identification problem for a parabolic equation, *Ukrainian Mathematical J.*, **66**, No 2 (2014), 165-177.

[18] A.A. Samarskii, P.N. Vabishchevich, *Numerical Methods for Solving Inverse Problems of Mathematical Physics*, Inverse and Ill-posed Problems Series, Walter de Gruyter, Berlin-New York (2007).

[19] G. Di Blasio, A. Lorenzi, Identification problems for parabolic delay differential equations with measurement on the boundary, *J. of Inverse and Ill-Posed Problems*, **15**, No 7 (2007), 709-734.

[20] A. Ashyralyev, D. Agirseven, On source identification problem for a delay parabolic equation, *Nonlinear Analysis: Modelling and Control*, **19**, No 3 (2014), 335-349.

[21] I. Orazov and M.A. Sadybekov, On a class of problems of determining the temperature and density of heat sources given initial and final temperature, *Siberian Mathematical J.*, **53** (2012), 146-151.

[22] A. Ashyralyev, A.S. Erdogan, A.U. Sazaklioglu, Numerical solution of a source identification problem, *Journal of Inverse and Ill-posed Problems*, **27**, No 4 (2019), 457-468.

[23] F.A. Aliev, N.A. Ismailov, A.A. Namazov, M.F. Rajabov, Algorithm for calculating the parameters of formation of gas-liquid mixture in the shoe of gas lift well, *Applied and Computational Mathematics*, **15**, No 3 (2016), 370-376.

[24] B. Hicdurmaz, Finite difference schemes for time-fractional Schrödinger equations via fractional linear multi step method, *International J. of Computer Mathematics* (2020); doi:10.1080/00207160/1834088.

[25] A. Ashyralyev, M. Urun, Time-dependent source identification problem for the Schrödinger equation with nonlocal boundary conditions, In: *AIP*

Conf. Proc., **2183** (Third Int. Conf. of Math. Sci., Maltepe Univ, Istanbul, Sep. 04-08, 2019) (Ed. by H. Cakalli and others), Amer. Inst. Phys. (2019), Art. 070016.

[26] B. Hicdurmaz, *Initial Boundary Value Problems for Fractional Schrödinger Differential Equations*, PhD Thesis, Gebze Technical University, Kocaeli (2015).

[27] A. Sirma, *Nonlocal Boundary Value Problems for Schrödinger Equation*, PhD Thesis, Gebze Institute of Technology, Kocaeli (2007).