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THE SOLVABILITY OF BOUNDARY VALUE PROBLEM FOR NONLINEAR ELLIPTIC-PARABOLIC EQUATIONS

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Abstract: The nonlinear elliptic-parabolic equations of nondivergent structure is considered. The solvability of the boundary value problems is investigated.

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1. Introduction

The theory of elliptic-parabolic equations ascends to the classical paper by Keldysh [1] in which the correct statements of the boundary value problems for the equations with one space variable were found. G. Fickera [2] has established a weak solvability of the first boundary value problem for a wide class of the second order equations with the non-negative characteristic form (see also [3]). As to strong solvability of the first boundary value problem for elliptic-parabolic equations in the non-divergent form with smooth coefficients, we shall note in this connection the papers [4, 5, 6]. The similar result for the equations in the case when the coefficients satisfy the Cordes condition is obtained in [7].

The theory of nonlinear elliptic-parabolic equation have many applications. For example, the class of nonlinear operators represent the well-known Richards equation, which serves as a basic model for the filtration of water in unsaturated soils (see [9, 10]), see also [11, 12, 13].

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Let us consider in $Q_T = \Omega \times (0;T)$, where Q_T be the cylinder, Ω is a bounded domain in R^n, $n \geq 2$ with the smooth boundary $\partial\Omega$, following the boundary value problem

$$Lu = \sum_{i,j=1}^{n} a_{ij}\left(x,t,u\right) u_{ij} + \psi\left(x,t\right) u_{tt} - u_t = f\left(x,t\right), \tag{1}$$

$$u|_{\Gamma(Q_T)} = 0. \tag{2}$$

Here

$$u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_i = \frac{\partial u}{\partial x_i}, \quad u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2},$$

$\Gamma(Q_T) = (\partial\Omega \times (0,T)) \cup \{(x,t) : t = 0, x \in \Omega\}$ is parabolic boundary of Q_T and

$$\psi\left(x,t\right) = \omega\left(t\right) \lambda\left(\rho\right) \varphi\left(T-t\right), \tag{3}$$

where $\lambda(\rho) \geq 0$, $\lambda(\rho) \in C^1[0, diam\Omega]$, $\varphi(z) \geq 0$, $|\lambda(\rho)| \leq \alpha\sqrt{\lambda(\rho)}$, $\varphi'(z) \geq 0$, $\varphi(z) \in C^1[0,T]$, $\varphi(0) = \varphi'(0) = 0$, $\varphi(z) \geq \beta z \varphi'(z)$, $\omega(t) \in C^1[0,T]$, $\omega(t) \geq 0$, and α, β are positive constants.

Assume that the coefficients of the equation (1) the following conditions hold: $(a_{ij}(x,t))$ is a real symmetrical matrix with real measurable elements in Q_T for every $(x,t) \in Q_T$, and $\xi \in R^n$ and satisfies the inequalities hold:

$$\gamma\left|\xi\right|^2 \leq \sum_{i,j=1}^{n} a_{ij}\left(x,t,u\right) \xi_i \xi_j \leq \gamma^{-1}\left|\xi\right|^2, \tag{4}$$

where $\gamma \in (0,1]$, is a constant.

The purpose of this paper is to obtain solvability boundary problem for nonlinear equations in an appropriate Sobolev spaces.

Before we obtained some a coersive estimations which be used to proving a unique strong solvability of the first boundary value problem (1)-(2) at every $f(x,t) \in L_2(Q_T)$.

The paper is organized as follows. In Section 2 we present some definitions and preliminary results. In Section 3 we give main results.

2. Definitions and preliminary results

For $R > 0$ and $x^0 \in R^n$ we denote the ball $B_R(x^0) = \{x : |x - x_0| < \; < R\}$ and a cylinder $B_R(x^0) \times (0,T) = Q_T^R(x_0)$. Let $\overline{B}_R(x^0) \subset \Omega$. We say that $u(x,t) \in$

$A\left(Q_T^R(x^0)\right)$ if $u(x,t)\in C^{\infty}\left(\overline{Q}_T^R(x^0)\right)$, $u|_{t=0}=0$ and $\sup p\ u\in\left(\overline{Q}_T^{\rho}(x^0)\right)$ for some $\rho\in(0,R)$.

Let us introduce the Banach space of functions $u(x,t)$ given on Q_T with finite norms

$$\|u\|_{W_2^{1,1}(Q_T)}=\left(\int\limits_{Q_T}\left(u^2+\sum_{i=1}^{n}u_{x_i}^2+u_t^2\right)dxdt\right)^{\frac{1}{2}}<\infty,$$

$$\|u\|_{W_{2,\psi}^{2,2}(Q_T)}$$

$$=\left(\int\limits_{Q_T}\left\{\left(u^2+\sum_{i=1}^{n}u_i^2+\sum_{i,j=1}^{n}u_{i,j}^2\right)+u_t^2+\psi^2(x,t)u_{tt}^2\right\}dxdt\right)^{\frac{1}{2}}.$$

Suppose $\overset{\circ}{W}{}_{2,\psi}^{2,2}(Q_T)$ is a subspace of the space$W_{2,\psi}^{2,2}(Q_T)$ that contains the set of all functions from $C^{\infty}\left(\overline{Q}_T\right)$ vanishing on the parabolic boundary $\Gamma(Q_T)$.

Let us consider the operator L which is arising by problem (1)-(2).

Now we like to get some coercive estimates for strong solutions to the problem (1)-(2). First we give the results for the model operator and applying these estimates we obtain the following.

Lemma 1. *Let condition* (3)-(4) *for coefficients and weight be fulfilled. Then for any function* $u(x,t)\in A\left(Q_T^R(x^0)\right)$, *there exists* $T_1(\psi(x,t),n)$ *such that for* $T\leq T_1$ *and following estimate holds:*

$$\int\limits_{Q_T^R(x^0)}\left[\left(\sum_{i,j=1}^{n}u_{ij}^2+u_i^2\right)+\psi^2(x,t)u_{tt}^2+u_t^2+\psi(x,t)\sum_{i=1}^{n}u_{it}^2\right]dxdt$$

$$\leq(1+D(T)S)\int\limits_{Q_T^R(x^0)}(Lu)^2dxdt, \tag{5}$$

where $S=S(\psi,n)$ *is some constant,* $D(T)=\sup\limits_{[0,T]}\psi'(t)+\sup\limits_{[0,T]}\varphi'(t)$.

Proof. For proof, we calculate $\int\limits_{Q_T}(Lu)^2dxdt$. We have

$$\int\limits_{Q_T}(Lu)^2dxdt=i_1+i_2+i_3+i_4+i_5.$$

Later we will consider each addend separately. Applying integration by parts with respect to variables x_i, x_j and taking into account $\left.\frac{\partial u}{\partial x_j}\right|_{\partial B_R} = 0$ we obtain estimate of integrals i_1. Also we calculate integrals i_2, i_3, i_4, i_5. □

Let $\delta = \sup\limits_{Q_T} \left(\sum\limits_{i,j=1}^{n} (a_{ij}(x,t,u) - \delta_{ij})^2 \right)^{\frac{1}{2}}$, where δ_{ij} are the Kronecker symbols.

Lemma 2. *Let the coefficients of the operators L satisfy conditions* (3)-(4)*. Then there exists T_2 such that for every $T \leq T_2$ and $\varepsilon > 0$ the estimate holds*:

$$\|u\|_{W_{2,\psi}^{2,2}(Q_T)} \leq C(\psi,\delta,n,\Omega)\,\|Lu\|_{L_2(Q_T)} + \varepsilon\|u\|_{W_{2,\psi}^{2,2}(Q_T)} + C(\psi,\delta,n,\Omega)\,\|u\|_{L_2(Q_T)}$$

for any function $u(x,t) \in C^\infty(\bar{Q}_T(x_0))$, $u|_{t=0} = 0$.

Lemma 3. *Let the conditions of Lemma 2 be satisfied. Then at $T \leq T_2$ for any function $u(x,t) \in \overset{\circ}{W}{}_{2,\varphi_\varepsilon}^{2,2}(Q_T)$ it holds the estimate*

$$\|u\|_{W_{2,\psi}^{2,2}(Q_T)} \leq C(\psi,\delta,n,\Omega)\,\|Lu\|_{L_2(Q_T)} + C(\psi,\delta,n,\Omega)\,\|u\|_{L_2(Q_T)}\,.$$

Similarly to Lemma 1, we can proof of Lemmas 2 and 3 with using Friedrich's inequality.

Now we give the following coercive estimate for solution boundary problem (1)-(2).

Theorem 1. *Let for strong solutions of the problem* (1)-(2) *the conditions* (3),(4) *be fulfilled. Then there exists $T_0(\psi,\delta,n\Omega)$ such that for every $T \leq T_0$ the estimate*

$$\|u\|_{W_{2,\psi}^{2,2}(Q_T)} \leq C(\psi,\delta,n,\Omega)\,\|Lu\|_{L_2(Q_T)}\,. \tag{6}$$

holds for any functions $u(x,t) \in W_{2,\psi}^{2,2}(Q_T)$.

Proof. It is enough to prove the estimate (6) for smooth functions from

$\overset{\circ}{W}{}^{2,2}_{2,\psi}(Q_T)$. We have for any $t \in (0,T)$ and any $x \in \Omega$

$$u(x,t) = \int_0^t u_t(x,\tau)d\tau.$$

Using the Cauchy-Bunyakovsky inequality, we write

$$u^2(x,t) = T\int_0^T u_t^2(x,\tau)\,d\tau.$$

Then

$$\int_{Q_R^T} u^2(x,t)\,dxdt = T^2\int_{Q_R^T} u_t^2(x,t)\,dxdt.$$

Thus,

$$\|u\|_{L_2(Q_R^T)} \le T\,\|u_t\|_{L_2(Q_R^T)} \le T\,\|u\|_{W^{2,2}_{2,\psi}(Q_T)}\,.$$

Let $T_0 = \min\left\{T_2, \frac{1}{2C}\right\}$. Then at $T \le T_0$ we obtain estimate (6). The theorem is proved. □

Theorem 2. *Let for strong solutions of the problem* (1)-(2) *the conditions* (3),(4) *be fulfilled. Then there exists* $T_0(\psi,\delta,n\Omega)$ *such that for every* $T \le T_0$ *problem* (1)-(2) *is solvable in space* $W^{2,2}_{2,\psi}(Q_T)$ *for any* $f(x,t) \in L_2(Q_T)$ *and the estimate holds*

$$\|u\|_{W^{2,2}_{2,\psi}(Q_T)} \le C(\psi,\delta,n,\Omega)\,\|f\|_{L_2(Q_T)}\,. \tag{7}$$

Proof. The estimate (7) and the uniqueness of the solution follow from the coercive estimates. The existence of a solution is proved by considering the family of operators. □

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