

ON FRAMES AND CONE ADAPTED IRREGULAR SHEARLET COEFFICIENTS

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Abstract: In this paper we study the action of double infinite regular matrix D on horizontal cone-adapted irregular shearlet coefficients. Also, we find the frame bounds by D -transform of function whose shearlet series expansion is known.

AMS Subject Classification: 42C40, 68U10, 47G20

Key Words: shearlets coefficients; double infinite matrices; frames and cone-adapted shearlet transform

1. Introduction

Wavelet frames have important applications, as do various hybrid systems and generalizations such as curvelets and shearlets, which are especially important for analysis in higher dimensions (image or video processing). The shearlets introduced by Labate et al. [10] are frame elements which yield optimally sparse representations [6]. This shearlet system is based on a simple rigorous mathematical framework which not only provides a more flexible theoretical tool for the geometric representation of multidimensional data, but is also more natural for implementations.

We denote \mathbb{Z} the set of integers, \mathbb{R} is the set of real numbers and \mathbb{C} the set of complex numbers. Shearlets are highly anisotropic representation systems, which optimally sparsify $\mathcal{C}^2(\mathbb{R}^2)$ -functions apart from \mathcal{C}^2 -discontinuity curves.

Shearlet system obtained by two procedures: One system coming directly from a group representation of a particular semi-direct product, the so called shearlet group and equipped with a particularly 'nice' mathematical structure, but due to biasedness towards one axis it becomes unattractive for applications point of view, the other system being adapted to a cone-like partitioning of the frequency domain by ensuring an equal treatment of all directions. The main advantages of this system are the ones exhibiting the favorable property of treating the continuum and digital setting uniformly similar to wavelets, and therefore, relevant for applications point of view. The importance of shearlets have been widely acknowledged and since their inception, they have emerged as one of the most effective frameworks for representing multidimensional data ranging over the areas of signal and image processing, remote sensing, data compression and several others (see [1-8], [10], [12], [15]).

The wavelets gave the understanding of many problems in various sciences, engineering and other disciplines. The n -dimensional continuous wavelet transform is able to describe the local regularity of functions and distribution and detect the location of singularity points through its decay at fine scale, it does not provide additional information about the geometry of the set of singularities. The shearlets provide an alternative approach to the curvelets, and exhibit some very distinctive features. Similarly to the curvelets, the shearlets are a multiscale directional system and unlike the curvelets the shearlets form an affine system. That is, they are generated by dilating and translating one single generating function, where the dilation matrix is the product of a parabolic scaling matrix and a shear matrix. The wavelet transform associated with above more general dilation groups is called shearlet transform. Similarly to the theory of affine systems, the continuous shearlets are associated with the whole range of scaling, shear, and translation indices $(a, s, t) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$, whereas the discrete shearlet systems are associated with a sequence in $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$ of discrete scaling, shear and translation indices.

Definition 1. (Cone-Adapted Shearlet Systems). We use the parabolic scaling matrices A_a or \tilde{A}_a , $a > 0$, and shear matrices S_s , $s \in \mathbb{R}$, defined by

$$A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix} \text{ or } \tilde{A}_a = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & a \end{pmatrix} \text{ and } S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

respectively. We partition the frequency plane into the following four cones $\mathcal{C}_1 - \mathcal{C}_4$:

$$\mathcal{C}_i = \left\{ \begin{array}{l} (w_1, w_2) \in \mathbb{R}^2 : |w_1| \geq 1, |w_2| \leq |w_1| : i = 1, \\ (w_1, w_2) \in \mathbb{R}^2 : |w_2| \geq 1, |w_1| \leq |w_2| : i = 2, \\ (w_1, w_2) \in \mathbb{R}^2 : |w_1| \leq -1, |w_2| \leq |w_1| : i = 3, \\ (w_1, w_2) \in \mathbb{R}^2 : |w_2| \leq -1, |w_1| \leq |w_2| : i = 4, \end{array} \right\}$$

and a centered rectangle

$$R = \{(w_1, w_2) \in \mathbb{R}^2 : \|(w_1, w_2)\|_\infty < 1\}.$$

The rectangle R corresponds to the low frequency content of a signal, which is customarily represented by translations of some scaling function. Anisotropy now comes into play when encoding the high frequency content of a signal, which corresponds to the cones $\mathcal{C}_1 - \mathcal{C}_4$ where the cones \mathcal{C}_1 and \mathcal{C}_3 as well as \mathcal{C}_2 and \mathcal{C}_4 are treated separately. Since the low frequency part already has been studied extensively and the horizontal cone $\mathcal{C}_1 \cup \mathcal{C}_3$ and vertical cone $\mathcal{C}_2 \cup \mathcal{C}_4$ are treated similarly, therefore, in this paper our focus will be only on the horizontal cone.

Definition 2. (Cone-Adapted Regular Discrete Shearlet System). For some sampling vector $c = (c_1, c_2) \in (\mathbb{R}^+)^2$, the system generated by a scaling function $\phi \in L^2(\mathbb{R}^2)$ and shearlets $\varphi, \tilde{\varphi} \in L^2(\mathbb{R}^2)$ is defined by

$$SH(c; \phi, \varphi, \tilde{\varphi}) = \Phi(c_1, \phi) \cup \Psi(c, \varphi) \cup \tilde{\Psi}(c, \tilde{\varphi}),$$

where

$$\left\{ \begin{array}{l} \Phi(c_1, \phi) = \{\phi_m = \phi(\cdot - c_1 m) : m \in \mathbb{Z}^2\}, \\ \Psi(c, \varphi) = \left\{ \varphi_{j,k,m} = 2^{\frac{3j}{4}} \varphi(S_{-k} A_{2^j} \cdot - M_c m) : j \geq 0, |k| \leq [2^{\frac{j}{2}}] \right\}, \\ \tilde{\Psi}(c, \tilde{\varphi}) = \left\{ \tilde{\varphi}_{j,k,m} = 2^{\frac{3j}{4}} \tilde{\varphi}(S_{-k}^T \tilde{A}_{2^j} \cdot - \tilde{M}_c m) : j \geq 0, |k| \leq [2^{\frac{j}{2}}] \right\}, \end{array} \right.$$

and

$$M_c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad \tilde{M}_c = \begin{pmatrix} c_2 & 0 \\ 0 & c_1 \end{pmatrix},$$

is known as cone-adapted regular discrete shearlet system.

We set:

$$\Lambda_{cone} = \left\{ (j, k, m) : j \geq 0, |k| \leq [2^{\frac{j}{2}}], m \in \mathbb{Z}^2 \right\}.$$

Definition 3. (Cone-Adapted Regular Discrete Shearlet Transform). Similar to discrete wavelet transform the cone-adapted regular discrete shearlet transform is defined for some function $f \in L^2(\mathbb{R}^2)$ by

$$SH_{\phi, \varphi, \tilde{\varphi}} f(m', (j, k, m), (\tilde{j}, \tilde{k}, \tilde{m})) = (\langle f, \phi_{m'} \rangle, \langle f, \varphi_{j,k,m} \rangle, \langle f, \tilde{\varphi}_{\tilde{j}, \tilde{k}, \tilde{m}} \rangle),$$

where

$$SH_{\phi, \varphi, \tilde{\varphi}} f : \mathbb{Z}^2 \times \Lambda_{cone}^2 \rightarrow \mathbb{C}^3$$

and

$$\langle f, g \rangle = \int_{\mathbb{R}^2} f(x) \overline{g(x)} dx, \|f\|^2 = \langle f, f \rangle^{\frac{1}{2}}.$$

The Fourier transform is the unitary operator that maps $f \in L^2(\mathbb{R}^2)$ into the function \hat{f} defined by

$$\hat{f}(w) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i \langle x, w \rangle} dx$$

when $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and by the appropriate limit for the general $f \in L^2(\mathbb{R}^2)$. The function \hat{f} is also square integrable. Indeed, Fourier transform maps $L^2(\mathbb{R}^2)$ one-to-one onto itself and the inverse Fourier transform is defined by

$$\check{f}(x) = \int_{\mathbb{R}^2} f(w) e^{2\pi i \langle x, w \rangle} dw.$$

Notice that the sampling set

$$\left\{ (2^{-j}, k 2^{\frac{-j}{2}}, S_{k 2^{\frac{-j}{2}}} A_{2^{-j}} M_c m) : j \geq 0, |k| \in \left\{ -[2^{\frac{j}{2}}], \dots, [2^{\frac{j}{2}}] \right\} \right\},$$

forces a change in the ordering of parabolic scaling and shearing, bearing in mind this fact, we call new class of irregular parameters.

Now, let us set:

$$S_{cone} = \left\{ (a, s, t) : a \in (0, 1], s \in [-(1 + a^{\frac{1}{2}}), 1 + a^{\frac{1}{2}}], t \in \mathbb{R}^2 \right\}.$$

Definition 4. Let Δ and $\Lambda, \tilde{\Lambda}$ be discrete subsets of \mathbb{R}^2 and S_{cone} respectively, and let $\phi \in L^2(\mathbb{R}^2)$ as well as $\varphi, \tilde{\varphi} \in L^2(\mathbb{R}^2)$. Then the (cone-adapted) irregular discrete shearlet system is defined by

$$SH(\Delta, \Lambda, \tilde{\Lambda}, \phi, \varphi, \tilde{\varphi}) = \Phi(\Delta, \phi) \cup \Psi(\Lambda, \varphi) \cup \tilde{\Psi}(\tilde{\Lambda}, \tilde{\varphi}),$$

where

$$\begin{cases} \Phi(\Delta, \phi) = \{ \phi_t = \phi(\cdot - t) : t \in \Delta \}, \\ \Psi(\Lambda, \varphi) = \left\{ \varphi_{a,s,t} = a^{\frac{-3}{4}} \varphi(A_a^{-1} S_s^{-1} \cdot - t) : (a, s, t) \in \Lambda \right\}, \\ \tilde{\Psi}(\tilde{\Lambda}, \tilde{\varphi}) = \left\{ \tilde{\varphi}_{a,s,t} = a^{\frac{-3}{4}} \tilde{\varphi}(\tilde{A}_a^{-1} S_s^{-T}(\cdot - t)) : (a, s, t) \in \tilde{\Lambda} \right\}. \end{cases}$$

Then the associated (Cone-Adapted) Irregular Discrete Shearlet Transform $SH_{\phi, \varphi, \tilde{\varphi}}(f) : \Delta \times \Lambda \times \tilde{\Lambda} \rightarrow \mathbb{C}^3$ of some function $f \in L^2(\mathbb{R}^2)$ is given by

$$SH_{\phi, \varphi, \tilde{\varphi}} f(t', (a, s, t), (\tilde{a}, \tilde{s}, \tilde{t})) = (\langle f, \phi_{t'} \rangle, \langle f, \varphi_{a, s, t} \rangle, \langle f, \tilde{\varphi}_{\tilde{a}, \tilde{s}, \tilde{t}} \rangle).$$

From now, we restrict our study on horizontal cone $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_3$ and define

$$L^2(\mathcal{C}) = \left\{ f \in L^2(\mathbb{R}^2) : \text{supp } \hat{f} \subseteq \mathcal{C} \right\}.$$

Since one main motivation for considering irregular sets of parameters are stability questions, such a constraint seems very natural.

The feasible set of parameters and feasible shearlet system introduces in [8] is summarizing as:

Let

$$\begin{aligned} & \left\{ (a_j, s_{j,k}, t_{j,k,m}^c) : j \geq 0, k \in K_j, \right\}, K_j \subseteq \mathbb{Z}, c = (c_1, c_2) \in (\mathbb{R}^+)^2, \\ & \{a_j\}_{j \geq 0} \subset \mathbb{R}^+, \{s_{j,k}\}_{j \geq 0, k \in K_j} \subset \mathbb{R}, \{t_{j,k,m}^c\}_{j \geq 0, k \in K_j, m \in \mathbb{Z}^2} \subset \mathbb{R}^2, \end{aligned}$$

be an arbitrary discrete set of parameters in S_{cone} .

Definition 5. (Feasible Shearlet System). Let

$\Lambda = \left\{ (a_j, s_{j,k}, t_{j,k,m}^c) : j \geq 0, k \in K_j, m \in \mathbb{Z}^2 \right\}$ be a feasible set of parameters and let $\varphi \in L^2(\mathbb{R}^2)$ be a feasible shearlet. Then we call

$$SH(\Lambda, \varphi) = \left\{ \varphi_{j,k,m} = a_j^{-\frac{3}{4}} \varphi(S_{-s_k} A_{a_j^{-1}} \cdot - M_c m) : j \geq 0, k \in K_j \right\},$$

a feasible shearlet system.

2. Shearlet frame

Like the Fourier transform, the wavelet transform has a discrete and a continuous version. For the continuous wavelet transform one has weaker conditions, especially the orthogonality not necessary for an invertible continuous wavelet transform. So, it is convenient to focus our attention on Parseval frame (PF) wavelets rather than on orthonormal wavelets. Similar to the wavelets the shearlets form a Parseval frame. The Mercedes frame $\{w_1, w_2, w_3\}$ is a simple example of a tight frame and after rescaling $\{u_1, u_2, u_3\}$, $u_i = c w_i$, $c = (\frac{2}{3})^{\frac{1}{2}}$ is

a Parseval frame for \mathbb{R}^2 . Several constructions of discrete shearlet frames are already known to date (see [3],[9],[13],[14],[18]).

Like the continuous shearlets, the discrete shearlets constitute a Parseval frame of the finite Euclidean space $L^2(\mathcal{C})$. Recall that for a Hilbert space H a sequence $\{\varphi_{j,k,m} : j \geq 0, k \in K_j, m \in \mathbb{Z}^2\}$ is a frame if and only if there exist constants $0 < A \leq B < \infty$ that

$$A\|f\|_H^2 \leq \sum_{j \geq 0} \sum_{k \in K_j} \sum_{m \in \mathbb{Z}^2} |\langle f, \varphi_{j,k,m} \rangle|^2 \leq B\|f\|_H^2$$

for all $f \in H$. The frame is called tight if $A = B$ and a Parseval frame if $A = B = 1$. Thus for Parseval frame we have that

$$\|f\|_H^2 = \sum_{j \geq 0} \sum_{k \in K_j} \sum_{m \in \mathbb{Z}^2} |\langle f, \varphi_{j,k,m} \rangle|^2$$

for all $f \in H$, which is equivalent to the reconstruction formula

$$f = \sum_{j \geq 0} \sum_{k \in K_j} \sum_{m \in \mathbb{Z}^2} c(h, j, k, m) \varphi_{j,k,m}, \quad (1)$$

where $c(h, j, k, m) = \langle f, \varphi_{j,k,m} \rangle$. The series representation (1) of f is called shearlet series and analogous to the notion of wavelet coefficients, the $c(h, j, k, m)$ are the shearlet coefficients.

Moricz and Rhoades [19] gave the following definition for matrix transform of a sequence.

Definition 6. Let $D = (d_{iljk})$ be a double infinite matrix of real numbers. Then D -transform of a double sequence $x = \{x_{jk}\}$ is

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_{iljk} x_{jk}, \quad (2)$$

which is called D -means or D -transform of the sequence $x = \{x_{jk}\}$.

In 1926 Robinson [20] defined the double regular matrix as:

A double matrix $D = (d_{iljk})$ is said to be regular if the following conditions hold:

- (i). $\lim_{i,l \rightarrow \infty} \sum_{j,k=0}^{\infty} d_{iljk} = 1$,
- (ii). $\lim_{i,l \rightarrow \infty} \sum_{j,k=0}^{\infty} |d_{iljk}| = 0$, ($k = 0, 1, 2, \dots$),
- (iii). $\lim_{i,l \rightarrow \infty} \sum_{j,k=0}^{\infty} |d_{iljk}| = 0$, ($j = 0, 1, 2, \dots$),
- (iv). $\|D\| = \sup_{i,l > 0} \sum_{j,k=0}^{\infty} |d_{iljk}| < \infty$.

Either of condition (ii) and (iii) implies that $\lim_{i,l \rightarrow \infty} d_{iljk} = 0$.

In this paper our main aim is to study the action of double infinite regular matrix D on $f \in L^2(\mathcal{C})$ and on its horizontal cone-adapted irregular shearlet

coefficients. We find the frame bounds for D -transform of $f \in L^2(\mathcal{C})$ whose shearlet series expansion is given by (1).

Now we have the following

Theorem 7. *Let $D = (d_{iljk})$ be a double nonnegative regular matrix and $SH(\Lambda, \varphi)$ be a feasible shearlet system. If*

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in K_j} \sum_{m \in \mathbb{Z}^2} \langle f, \varphi_{j,k,m} \rangle \varphi_{j,k,m}$$

is a horizontal cone-adapted shearlet expansion of $f \in L^2(\mathcal{C})$ with shearlet coefficients

$$c(h, j, k, m) = \langle f, \varphi_{j,k,m} \rangle_{L^2(\mathcal{C})} \equiv \langle f, \varphi_{j,k,m} \rangle = \int_{\mathcal{C}} f(x) \overline{\varphi_{j,k,m}(x)} dx,$$

where $\left\{ \varphi_{j,k,m} = a_j^{-\frac{3}{4}} \varphi(S_{-s_k} A_{a_j^{-1}} \cdot - M_c m) : j \geq 0, k \in K_j, m \in \mathbb{Z}^2 \right\}$, then the shearlet frame bounds for D -transform of $f \in L^2(\mathcal{C})$ is

$$\delta^* \|f\|^2 \leq \sum_{j \in \mathbb{Z}} \sum_{k \in K_j} \sum_{m \in \mathbb{Z}^2} |\langle Df, \varphi_{j,k,m} \rangle|^2 \leq \delta_1 \|f\|^2$$

where $0 \leq \delta^, \delta_1 < \infty$.*

Proof. The feasible set of parameters of feasible shearlet $\{\varphi_{j,k,m}\}$ characterize a function $f \in L^2(\mathcal{C})$ by means of shearlet coefficients $\langle f, \varphi_{j,k,m} \rangle$ if

$$\langle f, \varphi_{j,k,m} \rangle = \langle g, \varphi_{j,k,m} \rangle \Leftrightarrow f = g,$$

or equivalently,

$$\langle f, \varphi_{j,k,m} \rangle = 0 \Leftrightarrow f = 0,$$

this characterization is numerically stable if small perturbations in the shearlets coefficients $\langle f, \varphi_{j,k,m} \rangle$ of f correspond to small perturbations of the function f in the $L^2(\mathcal{C})$ -norm that is the feasible shearlet coefficient sequence of two functions are close in $l^2(\mathbb{Z}^2)$, the functions themselves are close in $L^2(\mathcal{C})$. It leads to that if

$$\sum_{j \geq 0} \sum_{k \in K_j} \sum_{m \in \mathbb{Z}^2} |\langle Df, \varphi_{j,k,m} \rangle|^2$$

is small, then $\|Df\|^2$ is small.

In particular, there exists $\alpha < \infty$ such that

$$|\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle|^2 \leq 1 \Rightarrow \|Df\|^2 < \alpha.$$

For $f \in L^2(\mathcal{C})$, define

$$\tilde{f} = f[\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle]^{\frac{-1}{2}}.$$

Then,

$$\langle D\tilde{f}, \varphi_{j,k,m} \rangle = [\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle]^{\frac{-1}{2}} \langle Df, \varphi_{j,k,m} \rangle$$

or

$$\begin{aligned} & |\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle D\tilde{f}, \varphi_{j,k,m} \rangle|^2 \\ &= [\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle]^2 [\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle]^{-1} \\ &= [\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle]^2. \end{aligned}$$

It gives

$$|\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle D\tilde{f}, \varphi_{j,k,m} \rangle|^2 \leq 1, \|D\tilde{f}\|^2 \leq \alpha,$$

or

$$|\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle|^2 \geq \alpha^{-1} \|Df\|^2.$$

Since D is regular matrix, it gives

$$\delta^* \|f\|^2 = \alpha^{-1} \|D\|^2 \|f\|^2 \leq |\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle Df, \varphi_{j,k,m} \rangle|^2.$$

Now let $f \in L^2(\mathcal{C})$. Then

$$\begin{aligned} & |\Sigma_{j \in \mathbb{Z}} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle f, \varphi_{j,k,m} \rangle|^2 \\ &= |\Sigma_{j \in \mathbb{Z}} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} \langle \hat{f}, \hat{\varphi}_{j,k,m} \rangle|^2 \\ &= |\Sigma_{j \geq 0} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2} a_j^{\frac{3}{2}} \int_{\mathcal{C}} \hat{f}(w) \overline{\hat{\varphi}(S_{s_k}^T A_{a_j} w)} e^{2\pi i \langle w, A_{a_j} S_{s_k} c m \rangle} dw|^2. \end{aligned} \tag{3}$$

Now first we consider the sum over $m \in \mathbb{Z}^2$. For this, set $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$.

Then by change of variables, we have

$$\begin{aligned}
& \Sigma_{m \in \mathbb{Z}^2} a_j^{\frac{3}{2}} \left| \int_{\mathcal{C}} \hat{f}(w) \overline{\hat{\varphi}(S_{s_k}^T A_{a_j} w)} e^{2\pi i \langle w, A_{a_j} S_{s_k} c m \rangle} dw \right|^2 \\
&= \Sigma_{m \in \mathbb{Z}^2} \frac{a_j^{\frac{-3}{2}}}{|\det(M_c)|} \left| \int_{\mathbb{R}^2} \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \chi_{\mathcal{C}}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \right. \\
&\quad \times \left. \overline{\hat{\varphi}(M_c^{-1} w)} e^{2\pi i \langle w, m \rangle} dw \right|^2 \\
&= \Sigma_{m \in \mathbb{Z}^2} \frac{a_j^{\frac{-3}{2}}}{|\det(M_c)|} \left| \Sigma_{s \in \mathbb{Z}^2} \int_{\Omega+s} \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \chi_{\mathcal{C}}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \right. \\
&\quad \times \left. \overline{\hat{\varphi}(M_c^{-1} w)} e^{2\pi i \langle w, m \rangle} dw \right|^2 \\
&= \Sigma_{m \in \mathbb{Z}^2} \frac{a_j^{\frac{-3}{2}}}{|\det(M_c)|} \left| \int_{\Omega} \Omega \Sigma_{s \in \mathbb{Z}^2} \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + s)) \right. \\
&\quad \times \left. \chi_{\mathcal{C}}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + s)) \overline{\hat{\varphi}(M_c^{-1} (w + s))} e^{2\pi i \langle w, m \rangle} dw \right|^2.
\end{aligned}$$

Using the Plancherel theorem, we get

$$\begin{aligned}
& \Sigma_{m \in \mathbb{Z}^2} a_j^{\frac{3}{2}} \left| \int_{\mathcal{C}} \hat{f}(w) \overline{\hat{\varphi}(S_{s_k}^T A_{a_j} w)} e^{2\pi i \langle w, A_{a_j} S_{s_k} c m \rangle} dw \right|^2 \\
&= \frac{a_j^{\frac{-3}{2}}}{|\det(M_c)|} \int_{\Omega} |\Sigma_{s \in \mathbb{Z}^2} \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + s)) \times \\
&\quad \chi_{\mathcal{C}}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + s)) \overline{\hat{\varphi}(M_c^{-1} (w + s))}|^2 dw \\
&= \frac{a_j^{\frac{-3}{2}}}{|\det(M_c)|} \int_{\Omega} \Sigma_{m, s \in \mathbb{Z}^2} \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + s)) \times \\
&\quad \chi_{\mathcal{C}}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + s)) \overline{\hat{\varphi}(M_c^{-1} (w + s))} \times \\
&\quad \overline{\hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + m)) \hat{\varphi}(M_c^{-1} (w + m))} dw \\
&= \frac{a_j^{\frac{-3}{2}}}{|\det(M_c)|} \Sigma_{s \in \mathbb{Z}^2} \int_{\Omega+s} \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \times \\
&\quad \chi_{\mathcal{C}}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \overline{\hat{\varphi}(M_c^{-1} w)} \Sigma_{s \in \mathbb{Z}^2} \overline{\hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + m - s))} \times
\end{aligned}$$

$$\begin{aligned}
& \hat{\varphi}(M_c^{-1}(w + m - s))dw \\
&= \frac{a_j^{\frac{-3}{2}}}{|det(M_c)|} \int_{\mathbb{R}^2} \Sigma_{s \in \mathbb{Z}^2} \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \times \\
& \quad \overline{\chi_C(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} w) \hat{\varphi}(M_c^{-1} w) \hat{f}(A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} (w + m))} \times \\
& \quad \overline{\hat{\varphi}(M_c^{-1} w) \hat{\varphi}(M_c^{-1} (w + m))} dw.
\end{aligned}$$

Substituting in (4), we get

$$\begin{aligned}
& |\Sigma_{j \in \mathbb{Z}} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2}| < \hat{f}, \hat{\varphi}_{j,k,m} >|^2 \\
&= \frac{1}{|det(M_c)|} \Sigma_{j \in \mathbb{Z}} \Sigma_{k \in K_j} \int_C |\hat{f}(w)|^2 |\hat{\varphi}(S_{s_k}^{-T} A_{a_j} w)|^2 dw \\
&+ \frac{1}{|det(M_c)|} \Sigma_{j \geq 0} \Sigma_{k \in K_j} \int_C \Sigma_{m \in \mathbb{Z}^2 \setminus \{0\}} \hat{f}(w) \times \\
& \quad \overline{\hat{f}(w + A_{a_j}^{-1} S_{s_k}^{-T} M_c^{-1} m) \hat{\varphi}(S_{s_k}^T A_{a_j} w) \times} \\
& \quad \hat{\varphi}(S_{s_k}^T A_{a_j} w + M_c^{-1} m) dw = S_1 + S_2.
\end{aligned} \tag{4}$$

To find a bound on second summation, we apply Cauchy-Schwartz inequality,

$$|S_2| \leq \frac{1}{det|M_c|} \|\hat{f}\|^2 \Sigma_{m \in \mathbb{Z}^2 \setminus \{0\}} [\beta(M_c^{-1} m) \beta(-M_c^{-1} m)]^{\frac{1}{2}},$$

where $\beta : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$\beta(\xi) = \text{ess sup} \Sigma_{j \geq 0} \Sigma_{k \in K_j} |\hat{\varphi}(S_{s_k}^T A_{a_j} w)| |\hat{\varphi}(S_{s_k}^T A_{a_j} w + \xi)|.$$

Consequently, if we denote

$$\gamma = \frac{1}{|det(M_c)|} \left\{ \text{ess sup}_{w \in C} \Sigma_{j \geq 0} \Sigma_{k \in K_j} |\hat{\varphi}(S_{s_k}^T A_{a_j} w)|^2 - \beta(mm) \right\} \tag{5}$$

and

$$\delta = \frac{1}{|det(M_c)|} \left\{ \text{ess inf}_{w \in C} \Sigma_{j \geq 0} \Sigma_{k \in K_j} |\hat{\varphi}(S_{s_k}^T A_{a_j} w)|^2 + \beta(mm) \right\}, \tag{6}$$

where $\beta(mm) = \Sigma_{m \in \mathbb{Z}^2 \setminus \{0\}} [\beta(M_c^{-1} m) \beta(-M_c^{-1} m)]^{\frac{1}{2}}$, then we obtain

$$|\Sigma_{j \in \mathbb{Z}} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2}| < \hat{f}, \hat{\varphi}_{j,k,m} >|^2 \leq \delta \|\hat{f}\|^2,$$

or

$$|\Sigma_{j \in \mathbb{Z}} \Sigma_{k \in K_j} \Sigma_{m \in \mathbb{Z}^2}| < Df, \varphi_{j,k,m} >|^2 \leq \delta \|D\|^2 \|f\|^2 = \delta_1 \|f\|^2. \tag{7}$$

Since D is regular matrix. Combining (3) and (8) the proof is complete. \square

3. Action of double infinite regular matrix on horizontal cone-adapted irregular shearlet coefficients

Theorem 8. If $c(h, j, k, m)$ are the horizontal cone-adapted irregular shearlet coefficients of $f \in L^2(\mathcal{C})$, that is $c(h, j, k, m) = \langle f, \varphi_{j,k,m} \rangle$, then the $c^*(h, l, n, m)$ are the mentioned shearlet coefficients of Df , where $c^*(h, l, n, m)$ is defined as the D -transform of $c(h, l, n, m)$ by

$$c^*(h, l, n, m) = \sum_{l \geq 0} \sum_{n \in K_j} \sum_{m \in \mathbb{Z}^2} c(h, l, n, m) \|\varphi_{l,n,m}\|^2.$$

Proof. We have

$$\begin{aligned} \langle Df, \varphi_{l,n,m} \rangle &= \int_{\mathcal{C}} Df(x) \overline{\varphi_{l,n,m}(x)} dx \\ &= \int_{\mathcal{C}} \sum_{l \geq 0} \sum_{n \in K_j} \sum_{m \in \mathbb{Z}^2} d_{lnjk} c(h, j, k, m) \varphi_{j,k,m}(x) \overline{\varphi_{l,n,m}(x)} dx. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{l \geq 0} \sum_{n \in K_j} \sum_{m \in \mathbb{Z}^2} \langle Df, \varphi_{l,n,m} \rangle \varphi_{l,n,m} \\ = \sum_{l \geq 0} \sum_{n \in K_j} \sum_{m \in \mathbb{Z}^2} \int_{\mathcal{C}} \sum_{j \geq 0} \sum_{k \in K_j} \sum_{m \in \mathbb{Z}^2} d_{lnjk} c(h, j, k, m) \times \\ \varphi_{j,k,m}(x) \varphi_{l,n,m}(x) \overline{\varphi_{l,n,m}(x)} dx, \end{aligned}$$

if and only if

$$\begin{aligned} \gamma \|\varphi_{j,k,m}\|^4 \|\hat{f}\|^2 &\leq \sum_{l \geq 0} \sum_{n \in K_j} \sum_{m \in \mathbb{Z}^2} |c(h, l, n, m)|^2 \\ &\leq \delta \|\varphi_{j,k,m}\|^4 \|\hat{f}\|^2, \end{aligned}$$

or

$$c'_1 \|f\|^2 \leq \sum_{l \geq 0} \sum_{n \in K_j} \sum_{m \in \mathbb{Z}^2} |c(h, l, n, m)|^2 \leq c'_2 \|f\|^2,$$

where $0 \leq c_1, c'_1, c_2, c'_2 < \infty$. This completes the proof. \square

4. Conclusions and applications

In this paper we consider a generalized wavelet transform, namely discrete shearlet transform and we study the action of double infinite regular matrix on horizontal cone-adapted irregular shearlet coefficients. The frame bounds by matrix transform of function whose shearlet series expansion is known also have been studied. The Cone-Adapted Shearlet System has been used for giving equal treatment of all directions.

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