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STABILITY ANALYSIS OF CHAOTIC NEW HAMILTONIAN SYSTEM BASED ON HÉNON-HEILES MODEL USING ADAPTIVE CONTROLLED HYBRID PROJECTIVE SYNCHRONIZATION

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Abstract: This research article deals with a systematic approach to investigate hybrid projective synchronization among identical new chaotic Hamiltonian systems using adaptive control method. First, nonlinear adaptive controllers are designed to estimate the unknown parameters of the given system and also to attain the stability criteria of the error dynamics of the system. Second, the required hybrid projective synchronization in the considered identical systems via adaptive control method is achieved by using Lyapunov stability theory. Additionally, numerical simulations are conducted using MATLAB software to show the efficient performances of the proposed adaptive controller design. Remarkably, both the analytical as well as computational results are in excellent agreement. Moreover, the considered technique has many applications in the field of secure communication and image encryption.

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1. Introduction

Undoubtedly, the chaos synchronization and chaos control (CSCC) have been active research areas in applied mathematics for the past three decades. More precisely, CSCC focuses on the behaviour exploration of immensely irregular or disordered nonlinear dynamical systems and plays a significant role in several fields, for instance, secure communication [17], robotics [18], neural networks [1], biomedical engineering [21], ecological models [23], finance models [26], oscillations [4], jerk systems [28], encryption [29], etc. Subsequently, CSCC have sought significant attention in various research fields.

A significant characteristic of chaotic systems, described as "Butterfly Effect" is high sensitivity dependence on initial conditions. This property of chaotic systems was first reported by Lorenz [15] in 1963 while analysing a weather prediction model. Most importantly, Pecora and Caroll [19] first introduced in 1990 the notion of chaos synchronization. In chaos synchronization phenomenon, the state trajectories of two or more chaotic/ hyperchaotic systems are regulated to follow the similar dynamics. In recent years, chaos synchronization of chaotic systems using various control techniques has become a fascinating and engaging area of study for the researchers and scientists. Many significant techniques are introduced and studied to control [22, 2, 8] and synchronization [24, 12, 25, 9, 10, 17, 3, 7, 11] of chaos occurring in dynamical systems.

Specifically, Hubler [6] in 1989 firstly introduced adaptive control method (ACM) in chaotic systems. Since then, many researches have been conducted using ACM [13, 8, 10, 11]. Keeping the above discussions in view, our primal aim in this paper is to study hybrid projective synchronization (HPS) among identical newly described Hamiltonian chaotic systems [27] based on Hénon-Heiles model by ACM. Basically, Hénon and Heiles [5] in 1964 first modeled the Hénon-Heiles model which describes the nonlinear motion of a star around a galactic centre with the motion restricted to a plane.

This paper is organized as follows: ection 2 comprises of few essential preliminaries to be used throughout the paper. Section 3 elucidates the basic structured characteristics of the given Hamiltonian chaotic model in detail. Section 4 investigates of the (HPS) method for the given system via ACM. Section 5 consists of the numerical simulations which are displayed graphically using MATLAB. Section 6 concludes the present paper.

2. Preliminaries

The master system and slave system are considered as:

$$\dot{z}_m = f_1(z_m), \tag{1}$$
$$\dot{z}_s = f_2(z_s) + w, \tag{2}$$

where $z_m = (z_{m1}, z_{m2}, \ldots, z_{mn})^T$, $z_s = (z_{s1}, z_{s2}, \ldots, z_{sn})^T$ are the state variables of (1) and (2) respectively, $f_1, f_2 : R^n \to R^n$ are two nonlinear continuous vector functions and $w = (w_1, w_2, \ldots, w_n) \in R^n$ is the properly designed controller.

We define the hybrid projective synchronization (HPS) error as:

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|z_s(t) - \zeta z_m(t)\| = 0, \tag{3}$$

for some $\zeta = \text{diag}(\zeta_1, \zeta_2, \ldots, \zeta_n)$ and $\|\cdot\|$ represents vector norm.

Remark 1. For $\zeta_1 = \zeta_2 = \ldots = \zeta_n = 1$, complete synchronization is achieved.

Remark 2. For $\zeta_1 = \zeta_2 = \ldots = \zeta_n = -1$, anti-synchronization is attained.

Remark 3. If ζ_i's are not all zeros and $\zeta_i \neq \zeta_j$ for some i and j, then modified projective synchronization is obtained.

3. System Description

Proposed by Vaidyanathan et al. [27], the discussed chaotic system can be written as:

$$\begin{cases} \dot{z}_{m1} = z_{m2} \\ \dot{z}_{m2} = -z_{m1} - 2z_{m1}z_{m3} + az_{m1}^2 \\ \dot{z}_{m3} = z_{m4} \\ \dot{z}_{m4} = -z_{m3} - z_{m1}^2 + z_{m3}^2 + bz_{m3}^4, \end{cases} \tag{4}$$

where $(z_{m1}, z_{m2}, z_{m3}, z_{m4})^T \in R^4$ is the state vector and A and B are parameters. When $A = 1.5$ and $B = -1.9$, the system (4) exhibits chaos. Also, the

Lyapunov exponents of system (4) are $LE_1 = 0.0015$, $LE_2 = 0$, $LE_3 = 0$, $LE_1 = -0.0015$. In addition, Figure 1(a-f) display the phase diagrams of (4). However, the detailed analytic study and numerical results for the system (4) can be found in [27].

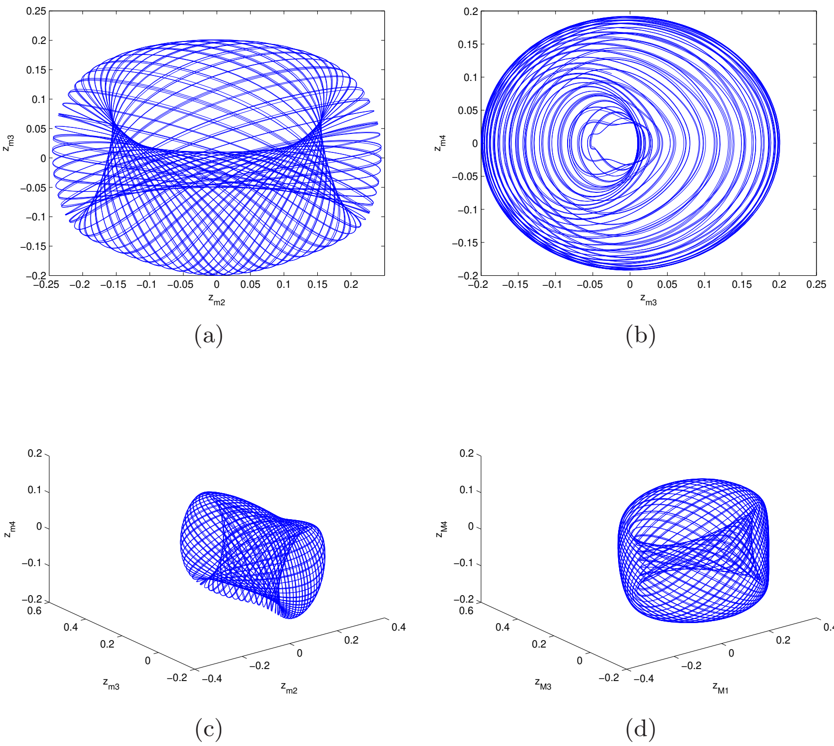

Figure 1: Phase diagrams of Hamiltonian chaotic system in (a) $z_{m2} - z_{m3}$ plane, (b) $z_{m3} - z_{m4}$ plane, (c) $z_{m2} - z_{m3} - z_{m4}$ space, (d) $z_{m1} - z_{m3} - z_{m4}$ space

4. Stability Analysis

In this section, we discuss HPS scheme to design the laws which estimate parameters with adaptive controllers in such a manner that the state vector z_{m1}, z_{m2}, z_{m3} and z_{m4} approaches to equilibrium points as t tends to infinity.

The system (4) is selected as the master system and the corresponding slave system may be defined as:

$$\begin{cases} \dot{z}_{s1} = z_{s2} + w_1 \\ \dot{z}_{s2} = -z_{s1} - 2z_{s1}z_{s3} + Az_{s1}^2 + w_2 \\ \dot{z}_{s3} = z_{s4} + w_3 \\ \dot{z}_{s4} = -z_{s3} - z_{s1}^2 + z_{s3}^2 + Bz_{s3}^4 + w_4, \end{cases} \tag{5}$$

where w_1, w_2, w_3 and w_4 are adaptive nonlinear controllers to be constructed so that HPS between two identical Hamiltonian chaotic systems will be attained. Also, Figure 2(a-d) show the phase diagrams of the system (5).

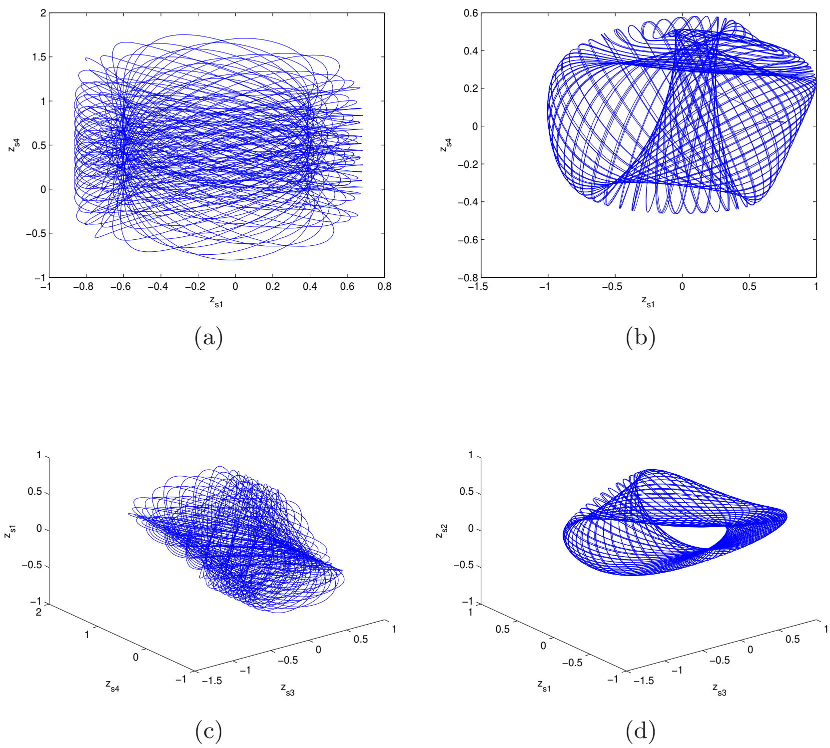

Figure 2: Phase diagrams of Hamiltonian chaotic system in (a) $z_{s1}-z_{s4}$ plane, (b) $z_{s2}-z_{s3}$ plane, (c) $z_{s3}-z_{s4}-z_{s1}$ space, (d) $z_{s3}-z_{s1}-z_{s2}$ space

We formulate the state errors as

$$\begin{cases} e_1 = z_{s1} - \zeta_1 z_{m1} \\ e_2 = z_{s2} - \zeta_2 z_{m2} \\ e_3 = z_{s3} - \zeta_3 z_{m3} \\ e_4 = z_{s4} - \zeta_4 z_{m4}. \end{cases} \tag{6}$$

Our ultimate goal here is to construct controllers w_i, $(i = 1, 2, 3, 4)$ so that the synchronization errors defined in (6) satisfy

$$\lim_{t\to\infty} e_i(t) = 0 \quad \text{for} \quad i = 1, 2, 3, 4.$$

The consequent error dynamics turns out to be

$$\begin{cases} \dot{e}_1 = e_2 + (\zeta_2 - \zeta_1)z_{m2} + w_1 \\ \dot{e}_2 = -e_1 + (\zeta_2 - \zeta_1)z_{m1} - 2(z_{s1}z_{s3} - \zeta_2 z_{m1}z_{m3}) \\ \qquad +A(z_{s1}^2 - \zeta_2 z_{m1}^2) + w_2 \\ \dot{e}_3 = e_4 + (\zeta_4 - \zeta_3)z_{m4} + w_3 \\ \dot{e}_4 = -e_3 + (\zeta_4 - \zeta_3)z_{m4} - (z_{s1}^2 - \zeta_4 z_{m1}^2) \\ \qquad +(z_{s3}^2 - \zeta_4 z_{m3}^2) + B(z_{s3}^4 - \zeta_4 z_{m3}^4) + w_4. \end{cases} \tag{7}$$

Next, the adaptive nonlinear controllers are designed by

$$\begin{cases} w_1 = -e_2 - (\zeta_2 - \zeta_1)z_{m2} - L_1 e_1 \\ w_2 = e_1 - (\zeta_2 - \zeta_1)z_{m1} + 2(z_{s1}z_{s3} - \zeta_2 z_{m1}z_{m3}) \\ \qquad -\hat{A}(z_{s1}^2 - \zeta_2 z_{m1}^2) - L_2 e_2 \\ w_3 = -e_4 - (\zeta_4 - \zeta_3)z_{m4} - L_3 e_3 \\ w_4 = e_3 - (\zeta_4 - \zeta_3)z_{m4} + (z_{s1}^2 - \zeta_4 z_{m1}^2) - (z_{s3}^2 - \zeta_4 z_{m3}^2) \\ \qquad -\hat{B}(z_{s3}^4 - \zeta_4 z_{m3}^4) - L_4 e_4, \end{cases} \tag{8}$$

where $L_1 > 0$, $L_2 > 0$, $L_3 > 0$, $L_4 > 0$ are gain constants.

On substituting the controllers (8) in error dynamics (7), we get

$$\begin{cases} \dot{e}_1 = -L_1 e_1 \\ \dot{e}_2 = (A - \hat{A})(z_{s1}^2 - \zeta_2 z_{m1}^2) - L_2 e_2 \\ \dot{e}_3 = -L_3 e_3 \\ \dot{e}_4 = (B - \hat{B})(z_{s3}^4 - \zeta_4 z_{m3}^4) - L_4 e_4, \end{cases} \tag{9}$$

where $\hat{A}$, $\hat{B}$ are estimated quantities of unknown parameter A, B respectively.

Defining the parameter estimation error as:

$$\tilde{A} = A - \hat{A}, \quad \tilde{B} = B - \hat{B}. \tag{10}$$

Using (10), the error dynamics (9) is written as:

$$\begin{cases} \dot{e}_1 = -L_1 e_1 \\ \dot{e}_2 = \tilde{A}(x_{s1}^2 - \zeta_2 x_{m1}^2) - L_2 e_2 \\ \dot{e}_3 = -L_3 e_3 \\ \dot{e}_4 = \tilde{B}(x_{s3}^4 - \zeta_4 x_{m3}^4) - L_4 e_4. \end{cases} \tag{11}$$

On differentiating parameter estimation error (10), one finds that

$$\dot{\tilde{A}} = -\dot{\hat{A}}, \quad \dot{\tilde{B}} = -\dot{\hat{B}}. \tag{12}$$

Constructing the classic Lyapunov function by the rule:

$$V = \frac{1}{2}[e_1^2 + e_2^2 + e_3^2 + e_4^2 + \tilde{A}^2 + \tilde{B}^2], \tag{13}$$

which implying that V is positive definite.

The derivative of Lyapunov function V may be written as:

$$\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4 - \tilde{A}\dot{\hat{A}} - \tilde{B}\dot{\hat{B}}. \tag{14}$$

Keeping (14) in mind, we are describing the parameter estimates laws as:

$$\begin{cases} \dot{\hat{A}} = (z_{s1}^2 - \zeta_2 z_{m1}^2)e_2 + L_5\tilde{A} \\ \dot{\hat{B}} = (z_{s3}^4 - \zeta_4 z_{m3}^4)e_4 + L_6\tilde{B}, \end{cases} \tag{15}$$

where $L_5 > 0$ and $L_6 > 0$ are gain constants.

Theorem 4. *The chaotic systems* (4)-(5) *are asymptotically hybrid projective synchronized for all initial states* $(z_{m1}(0), z_{m2}(0), z_{m3}(0), z_{m4}(0)) \in R^4$ *by the designed adaptive controller* (8) *and the parameter update law* (15).

Proof. The Lyapunov function V as considered in (13) is positive definite function. On simplification, Eqns. (11), (14) and (15) reduces to

$$\dot{V} - L_1 e_1^2 - L_2 e_2^2 - L_3 e_3^2 - L_4 e_4^2 - L_5\tilde{A}^2 - L_6\tilde{B}^2 < 0,$$

ensuring that $\dot{V}$ is negative definite.

Thus, using LST [20], we conclude that synchronization error $e(t) \to 0$ exponentially as $t \to \infty$ for all initial conditions $e(0) \in R^4$. This finishes the proof. □

5. Numerical Simulation and Discussion

This section performs some numerical simulations to illustrate effectively the proposed HPS technique via ACM. $\zeta = diag(\zeta_1, \zeta_2, \ldots, \zeta_n)$. The initial states of master (4) and slave systems (5) are $(z_{m1}(0) = 0.2, z_{m2}(0) = 0, z_{m3}(0) = -0.2, z_{m4}(0) = 0)$ and $(z_{s1}(0) = 0.2, z_{s2}(0) = 0.2, z_{s3}(0) = -0.2, z_{s4}(0) = 0)$ respectively. Also, the scaling matrix ζ is selected as $\zeta_1 = 4$, $\zeta_2 = -3$, $\zeta_3 = 2$, and $\zeta_4 = -5$. The control gains are chosen as $L_i = 6$ for $i = 1, 2, \ldots, 6$. In addition, simulation results concerning the hybrid projective synchronized trajectories of systems (4) and (5) are shown in Figure 3(a-d). Moreover, Figure 4(a-e) show that the synchronization error $(e_1, e_2, e_3, e_4) = (0.4, 0.2, 0, 0)$ converging to zero as t tending to infinity. In Figure 5(a-b), it is noted that the estimated quantities $(\hat{A}, \hat{B})$ of unknown parameters converging to their original values asymptotical with time. Hence, the proposed HPS synchronization strategy in master and slave systems is achieved computationally.

6. Conclusion

In this paper, hybrid projective synchronization in newly designed Hamiltonian chaotic systems has been investigated using ACM. Keeping Lyapunov stability theory in view, adaptive controllers have been described to attain asymptotic stability of the error dynamics of the given system. Further, numerical simulations through MATLAB are presented to validate the efficiency of the proposed methodology. Remarkably, the theoretical results completely agree with the computational results. Such scheme may be utilised to control the nonlinear motion of a star around a galactic centre with motion restricted to a plane. In addition, the proposed strategy may find important applications in the areas of image encryption and secure communication.

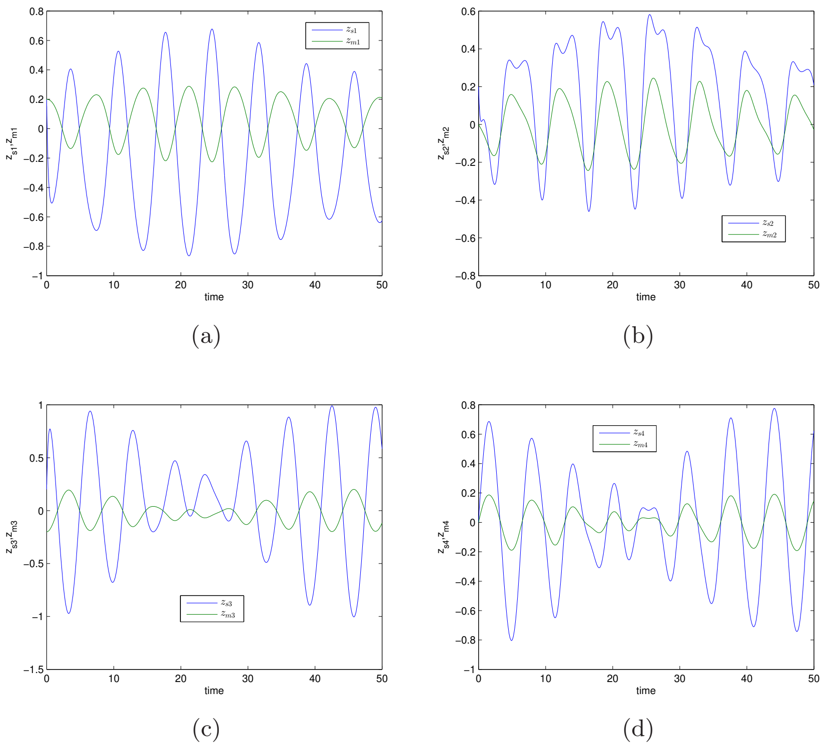

Figure 3: Hybrid projective synchronization of Hamiltonian chaotic systems (a) between $x_{m1}(t) - x_{s1}(t)$, (b) between $x_{m2}(t) - x_{s2}(t)$, (c) between $x_{m3}(t) - x_{s3}(t)$, (d) between $x_{m4}(t) - x_{s4}(t)$

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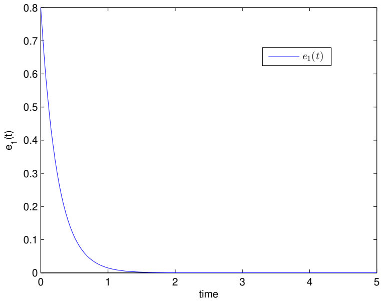
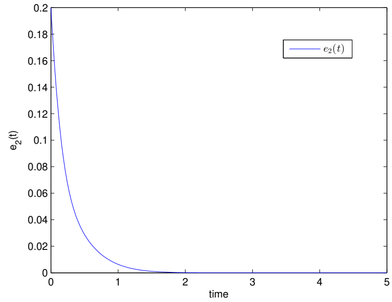
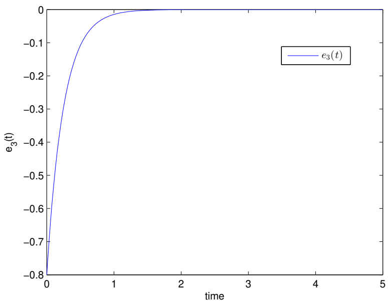


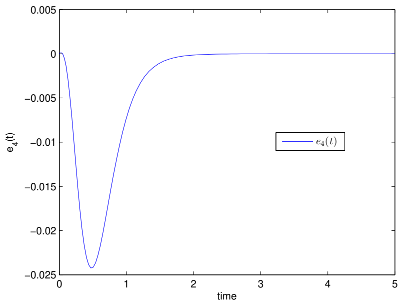
(a)



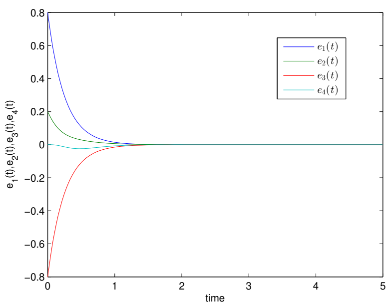
(b)



(c)



(d)



(e)

Figure 4: Dynamics in synchronization error states (a) $(t, e_1(t))$, (b) $(t, e_2(t))$, (c) $(t, e_3(t))$, (d) $(t, e_4(t))$, (e) $(t, e_1(t), e_2(t), e_3(t), e_4(t))$

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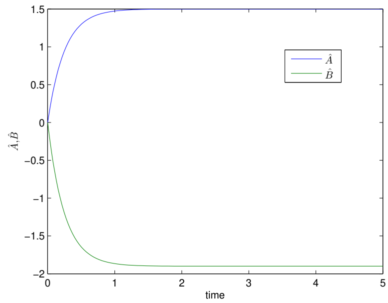

Figure 5: Time history of parameter estimates of Hamiltonian chaotic system

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