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EQUIVALENCE OF WEIGHTED DT-MODULI OF CONVEX FUNCTIONS

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Abstract: This work present a new conclusion for weighted DT-moduli of smoothness (DTMS). Furthermore, the best weighted approximation on a finite closed interval $\mathbb{D} = [-1, 1]$ are computed by DTMS. For any $r \in \mathbb{N}_\circ$, $0 < p \leq \infty$, $1 \leq \eta \leq r$ and $\phi(x) = \sqrt{1 - x^2}$, the equivalences

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \sim \varpi_{i,r}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}$$

$$\sim\ \varpi_{i+1,r-1}^{\phi}\ (f^{(r)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p}$$

and

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \sim \varpi_{i+\eta}^{\phi}\ (f, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{\alpha,\beta,p} \sim \|\theta_{\mathcal{N}}\|^{-\eta}$$

$$\times\, \varpi_{i,2\eta}^{\phi}\ (f^{(2\eta)}, \|\theta_{\mathcal{N}}\|)_{\alpha+\eta,\beta+\eta,p}$$

are valid.

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1. Introduction

Let us begin with the following notation which was introduced by [8]. For $\alpha, \beta \in J_p$, let us denote $J_p = (\frac{-1}{p}, 0)$ if $0 < p < \infty$, and $J_p = [0, \infty)$ if $p = \infty$. The following is definition of $\mathbb{L}_{p,r}^{\alpha,\beta}$ and $\mathbb{L}_{p}^{\alpha,\beta}$ spaces.

Definition 1. ([8]) Let $w_{\alpha,\beta}(x) = (1+x)^\alpha (1-x)^\beta$ be the (classical) Jacobi weight. Define

$$\mathbb{L}_{p,r}^{\alpha,\beta} = \{f : \mathbb{D} \longrightarrow \mathbb{R} : f^{(r-1)} \in AC_{loc}(-1,1), \ \ \|f\|_{w_{\alpha,\beta},p} = \|w_{\alpha,\beta} f^{(r)}\|_p$$
$$< \infty \ , \ \text{ and } \ 1 \le p \le \infty\},$$
$$\mathbb{L}_{p}^{\alpha,\beta} = \{f : \mathbb{D} \longrightarrow \mathbb{R} : \|f\|_{\alpha,\beta,p} = \|w_{\alpha,\beta} f\|_p < \infty, \ \text{ and } \ 0 < p \le \infty\},$$

and for convenience, denote $\mathbb{L}_{p,0}^{\alpha,\beta} = \mathbb{L}_{p}^{\alpha,\beta}$.

Hierarchy foundations of the moduli of smoothness began modern with the work of Ditzian and Totik, 1987 (see [4]). They established better continuous moduli of the function in a norm space. Then Kopotun et al. [5] contributed the function approximated by $\mathbb{L}_{p,r}^{\alpha,\beta}$ and $\mathbb{L}_{p}^{\alpha,\beta}$ spaces. That work would need to discuss the best weighted approximation of convex function. By the same token, our study attempts to fill a gap in the existing literature by construct a new best weighted approximation as an extension of Kopotun's work [6, 7, 8]. Our work here depends on the constructed symmetric difference of f in [2].

2. Methodology for weighted DTMS

Let $\mathbb{D} = [-1, 1]$ be measurable subset of $\mathbb{R}$ and $\mathbf{P} = \{\mathbb{D}_j\}_{j \in \mathbb{N}}$ be a family of finite subsets of $\mathbb{D}$. We have Lebesgue partition $\mathbf{P}$ of $\mathbb{D}$, if $\mathbb{D}_j$ are measurable sets, $\cup_{j \in \mathbb{N}} \mathbb{D}_j = \mathbb{D}$ and $\mathbb{D}_j \cap \mathbb{D}_\iota = \emptyset$, for $j \neq \iota$. Now, the following definition is referred to as Lebesgue Stieltjes integral-i, a term that will be used extensively throughout this paper.

Definition 2. ([1])Let $\mathbb{D}$ be measurable set, $f : \mathbb{D} \longrightarrow \mathbb{R}$ be a bounded function, and $\mathcal{L}_i : \mathbb{D} \longrightarrow \mathbb{R}$ be nondecreasing function for $i \in \mathbb{N}$. For a Lebesgue partition $\mathbf{P}$ of $\mathbb{D}$, put $\underline{\mathrm{LS}}(f, \ \mathbf{P}, \ \underline{\mathcal{L}}) = \sum_{j=1}^{n} \prod_{i \in \mathbb{N}} \ m_j \ \mathcal{L}_i(\mu(\mathbb{D}_j))$ and $\overline{\mathrm{LS}}(f, \ \mathbf{P}, \ \underline{\mathcal{L}}) = \sum_{j=1}^{n} \prod_{i \in \mathbb{N}} \ M_j \ \mathcal{L}_i(\mu(\mathbb{D}_j))$ where μ is a measure function of $\mathbb{D}$, $m_j = \inf\{f(x) : \ x \in \mathbb{D}_j\}$, $M_j = \sup\{f(x) : \ x \in \mathbb{D}_j\}$, and

$\underline{\mathcal{L}} = \mathcal{L}_1,\ \mathcal{L}_2,\ \cdots$. Also, $\mathcal{L}_i(x_j) - \mathcal{L}_i(x_{j-1}) > 0$, $\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) \leq \overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}})$, $\prod_{i\in\mathbb{N}} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \sup\{\underline{\mathrm{LS}}(f,\ \underline{\mathcal{L}})\}$ and $\prod_{i\in\mathbb{N}} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \inf\{\overline{\mathrm{LS}}(f,\ \underline{\mathcal{L}})\}$ where $\underline{\mathrm{LS}}(f,\ \underline{\mathcal{L}}) = \{\underline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) :\ \mathbf{P}$ part of set $\mathbb{D}\}$ and $\overline{\mathrm{LS}}(f,\ \underline{\mathcal{L}}) = \{\overline{\mathrm{LS}}(f,\ \mathbf{P},\ \underline{\mathcal{L}}) :$ $\mathbf{P}$ part of set $\mathbb{D}\}$. If $\prod_{i\in\mathbb{N}} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\mathbb{N}} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}$ where $\underline{\mathrm{d}\mathcal{L}} = d\mathcal{L}_1\times\ d\mathcal{L}_2\times\ \cdots$. Then f is integral $\int_i$ according to $\mathcal{L}_i$ for $i \in \mathbb{N}$.

The class of all Lebesgue–Stieltjes integrable functions is defined as follows. Let I_f be the class of all functions of Lebesgue–Stieltjes sense whose integral-i of f satisfies Definition 2, i.e.,

$$\begin{aligned} I_f &= \{f : f \text{ is integrable function according to } \mathcal{L}_i,\ i \in \mathbb{N}\} \\ &= \{f : \prod_{i\in\mathbb{N}} \underline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}} = \prod_{i\in\mathbb{N}} \overline{\int}_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}}\}. \end{aligned} \tag{1}$$

Definition 3. ([2]) Let $f \in \Delta^{(2)}(Y_s)$, for $i \in \mathbb{N}$, the symmetric difference of f is denoted by

$$\mho^i_{h\phi}(f,\ x) = \begin{cases} \prod_{i\in\mathbb{N}} \int_i^{\mathbb{D}} f\ \underline{\mathrm{d}\mathcal{L}_\phi}, & \text{if } f \in I_f\ , \\ 0\,, & \text{otherwise.} \end{cases} \tag{2}$$

Definition 4. ([2]) For $\alpha, \beta \in J_p$, $r \in \mathbb{N}_\circ$ and $0 < p \leq \infty$, $\Phi^{p,r}(w_{\alpha,\beta})$ space is defined as

$$\Phi^{p,r}(w_{\alpha,\beta}) = \{f : f \in \mathbb{L}^{\alpha,\beta}_{p,r} \cap I_f \text{ and } \mho^i_{h\phi}(f,x) < \infty\},$$

and $\Phi^{p,0}(w_{\alpha,\beta}) = \Phi^p(w_{\alpha,\beta})$.

The following features of symmetric difference are added:

$$\Phi^{p,r+1}(w_{\alpha,\beta}) = \Phi^{p,r}(w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}}). \tag{3}$$

By virtue of (3), we see the following immediate consequence

$$(w_{\alpha,\beta}) \times \mho^i_{h,\phi}(f^{(r+1)}, x) = (w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}}) \times \mho^{i+1}_{h,\phi}(f^{(r)}, x)\ . \tag{4}$$

Definition 5. ([2]) A weighted DTMS in $\Phi^{p,r}(w_{\alpha,\beta})$ is defined by

$$\varpi^{\phi}_{i,r}\ (f^{(r)},\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}$$

$$= \sup \{\|w_{\alpha,\beta}\ \phi^r \mho^i_{h\phi}(f^{(r)},\ x)\|_p\ ,\ 0 < h \le \|\theta_{\mathcal{N}}\|\},$$

where $\|\theta_{\mathcal{N}}\| < 2(i^{-1})$, $\mathcal{N} \ge 2$.

Let $\hat{T}_\eta = \{t_j\}_{j=0}^{\eta}$, be a partition and $\eta > 1$. The set $t_j = t_{j,\eta} = -\cos(\frac{j\Pi}{\eta})$, $j = 0, \cdots, \eta$, is called the Chebyshev partition of $[-1, 1]$.

The following theorem covers all weighted DTMS equivalents in this paper.

Theorem 6. ([2]) *Assume that $s, r \in \mathbb{N}_\circ$, $\alpha, \beta \in J_p$, $0 < p \le \infty$ and $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$. If $\mathbf{P}$ is Lebesgue partition of $\mathbb{D}$, and $\hat{T}_\eta$ is Chebyshev partition with $\mathbf{P} \cap \hat{T}_\eta \ne \emptyset$. Then, for any constant c may be depend on η and $J_{j,\eta}$ and may be depend on $|\mathbb{D}| \le \delta_\circ$ for some $\delta_\circ \in \mathbb{R}^+$, we have*

$$\varpi^{\phi}_{i+1,r}\ (f^{(r)},\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p} \sim c(\delta_\circ)\varpi^{\phi}_{i,r+1}\ (f^{(r+1)},\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p} \tag{5}$$

$$\sim c(\delta_\circ) \times \varpi^{\phi}_{i+1,r}\ (f^{(r)},\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p} \sim \|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p \tag{6}$$

$$\sim c(\eta, J_{j,\eta})\{\varpi^{\phi}_{i+2\eta,i+\eta}\ (f^{(i+\eta)},\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}\ : \textit{if}\ |\mathbb{D}| \le c(\eta, J_{j,\eta})\} \tag{7}$$

and

$$\|\theta_{\mathcal{N}}\|^\eta \times \varpi^{\phi}_{i+\eta}\ (f,\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}$$

$$\sim c(\eta, J_{j,\eta})\varpi^{\phi}_{i,2\eta}\ (f^{(2\eta)},\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\eta,\beta+\eta},p} \tag{8}$$

$$\sim \|w_{\alpha,\beta}\phi^\eta f^{(\eta)}\|_p \sim c(\eta, J_{j,\eta}) \times \{\varpi^{\phi}_{i,i+2\eta}\ (f^{(i+2\eta)},\ \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\frac{\eta}{2},\beta+\frac{\eta}{2}},p}$$

$$: \textit{if}\ |\mathbb{D}| > c(\eta, J_{j,\eta})\}\ . \tag{9}$$

Lemma 7. *If $f \in \Delta^{(2)} \cap \Phi^{p,r}(w_{\alpha,\beta})$ is such that $f^{(r)}(x) = p_n^{(r)}(x)$, where $p_n \in \pi_n \cap \Delta^{(2)}$, $\mathcal{N} \ge k \ge 2$ and $s \in \mathbb{S}(\hat{T}_n, r+2) \cap \Delta^{(2)} \cap \Phi^{p,r}(w_{\alpha,\beta})$. Then there exists a constant $c_1 = c_1(k)$ such that*

$$\|f - s\|_{w_{\alpha,\beta},p} \le c_1\ \varpi^{\phi}_{i,r}(f, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}\ . \tag{10}$$

Proof. By virtue of ([3], proof of Theorem 3.3), then (10) is valid. □

3. Main result

Our contributions to discussion of constrained approximation are continuing with this section. The outcomes that are related directly to the proves from Section 2 are demonstrated as well. Now, we ready to position the following definition.

Definition 8. For $\alpha, \beta \in J_p$ and $f \in I_f$, we set

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p$$

$$= \inf\{\|f - p_n\|_{\alpha,\beta,p}\ , p_n \in \pi_n \cap \Delta^{(2)} \cap I_f\ , f \in \Delta^{(2)} \cap \Phi^p(w_{\alpha,\beta})\},$$

to denote the degree of best convex polynomial approximation of f.

Theorem 9. *Suppose that $\alpha, \beta \in J_p$, $1 \le p \le \infty$ and $f \in \Delta^{(2)} \cap \Phi^p(w_{\alpha,\beta})$. Then,*

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \le c\varpi_i^{\phi}(f, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{\alpha,\beta,p} \tag{11}$$

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \le c\varpi_{i,1}^{\phi}(f', \|\theta_{\mathcal{N}}\|, \mathbb{D})_{\alpha,\beta,p} \tag{12}$$

and

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \le c\varpi_i^{\phi}(f, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{\alpha+\delta,\beta+\delta,p}, \tag{13}$$

where c is a constant may depend on α, β or, may depend on δ.

Proof. Let $f \in \Delta^{(2)} \cap \Phi^p(w_{\alpha,\beta})$, $\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p$ be degree of best convex polynomial approximation of f, that defined based on Definition 8. Then the degree of best convex polynomial approximation of f less than or equal $\|f - p_n\|_{\alpha,\beta,p}$ and the polynomial p_n in $\pi_n \cap \Delta^{(2)} \cap I_f$. Then by Lemma 7, there is a constant c, and degree of best convex polynomial approximation of f less than or equal $c\|\phi^r \mho_{h\phi}^i(f, x)\|_{\alpha,\beta,p}$. Therefore, (11) is proven. Next, the proof (12) is similar to proof of equivalent (5). Finally, by (3), then $\Phi^{p,r}(w_{\alpha-\delta,\beta-\delta}) \subset \Phi^{p,r}(w_{\alpha,\beta})$ where $\delta < 1$. Therefore $\Phi^{p,r}(w_{\alpha,\beta}) \subset \Phi^{p,r}(w_{\alpha+\delta,\beta+\delta})$, thus by (11), (13) is implied. □

The reworked outcomes by the following theorem. In fact, this theorem is the most important $\Phi^p(w_{\alpha,\beta})$ space characterization.

Theorem 10. *For $r \in \mathbb{N}_\circ$, $\alpha, \beta \in J_p$ and $p < \infty$. If $f \in \Delta^{(2)} \cap \Phi^p(w_{\alpha,\beta})$, $\mathbf{P}$ is Lebesgue partition of $\mathbb{D}$, and $\hat{T}_\eta$ is Chebyshev partition with $\mathbf{P} \cap \hat{T}_\eta \neq \emptyset$. Then,*

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c\varpi_{i,r}^{\phi}(f^{(r)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p} \tag{14}$$

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c(\delta_\circ)\varpi_{i+1,r-1}^{\phi}(f^{(r)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p} \tag{15}$$

and

$$\varpi_{i,r}^{\phi}(f^{(r)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p} \leq c^{-1}\|w_{\alpha-\frac{\eta}{2},\beta-\frac{\eta}{2}}\phi^\eta f^{(\eta)}\|_p, \tag{16}$$

where $1 \leq \eta \leq r$, $\frac{|\mathbb{D}|}{c} > 1$ and c is a constant depend on η and $J_{j,\eta}$.

Proof. Suppose that $f \in \Delta^{(2)} \cap \Phi^p(w_{\alpha,\beta})$ and $\mathbf{P}, \hat{T}_\eta$ are Lebesgue and Chebyshev partitions of $\mathbb{D}$ with $\mathbf{P} \cap \hat{T}_\eta \neq \emptyset$. The proof (14) is given by Theorem 6, Lemma 7 and (4). Thus, by (14) and equivalent of (5), (6), then (15) is inferred. Next, let c be a constant defined based on (7), (9), and $\frac{|\mathbb{D}|}{c} > 1$. Then assume from (8), $r = i + 2\eta$ and $r \geq 3$. Hence, (16) is also attained. □

Theorem 11. *For $r \in \mathbb{N}_\circ$ and $\alpha, \beta \in J_p$, there is a constant c may be depend on $r, \alpha, \beta, p, \varpi_{1,r}^{\phi}$ and may be depend on $r, \alpha, \beta, p, \varpi_{1,r}^{\phi}$, η and $J_{j,\eta}$ such that $f \in \Delta^{(2)} \cap \Phi^{p,r}(w_{\alpha,\beta})$, $J_{j,\eta} = [u_{j-(\eta+i)}, u_{j-(\eta+i)+1}]$ and $1 \leq \eta \leq r$. Then,*

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c\|\theta_{\mathcal{N}}\|^\eta \varpi_{i+\eta}^{\phi}(f, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}$$

and

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c(\eta, J_{j,\eta})\varpi_{i,2\eta}^{\phi}(f^{(2\eta)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\eta,\beta+\eta},p}\ .$$

Proof. Since, by (5) and (14), then $\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c_\circ\varpi_{i,r}^{\phi}(f^{(r)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}$, where $c_\circ = c_\circ(\delta_\circ)$ for some $\delta_\circ \in \mathbb{R}^+$. Let $r = i + 2\eta$, $1 \leq \eta \leq r$, then, by virtue of (9), there exists a constant $c = c(\eta, J_{j,\eta})$, and $|\mathbb{D}| > c$. Then,

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c\varpi_{i,i+2\eta}^{\phi}(f^{(i+2\eta)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\frac{\eta}{2},\beta+\frac{\eta}{2}},p}\ .$$

Hence by (9) and (8), we obtain

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c\|\theta_{\mathcal{N}}\|^\eta \varpi_{i+\eta}^{\phi}(f, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}$$

and

$$\mathcal{E}_n^{(2)}(f, w_{\alpha,\beta})_p \leq c\varpi_{i,2\eta}^{\phi}(f^{(2\eta)}, \|\theta_{\mathcal{N}}\|, \mathbb{D})_{w_{\alpha+\eta,\beta+\eta},p}\ .$$

□

4. Applications

The function $f(x) = x^4 - x$ is defined as a convex function at the interval $[0, 1]$. The value of moduli of smoothness in (14) of Theorem 10 is verified here. From Definition 3, $\mho^i_{h\phi}(f^{(r)}, x) = \prod_{i\in\mathbb{N}} \int_i^{\mathbb{D}} f^{(r)} \underline{\mathrm{d}\mathcal{L}_\phi}$.

If $r = 0$. Let $f(x) = x^4 - x$. Let $\mathcal{L}_i : [0, 1] \to \mathbb{R}$, $i = 1, 2$, and $\mathcal{L}_1(x) = \sin(x)$, $\mathcal{L}_2(x) = \sqrt{x}$ be nondecreasing functions. Let $\mathbf{P} = \{\mathbb{D}_1, \mathbb{D}_2\}$ be Lebesgue partition such that $\mathbb{D}_1 = [0, 0.62)$ and $\mathbb{D}_2 = [0.62, 1]$. Then, $\mu(\mathbb{D}_1) = 0.62$ and $\mu(\mathbb{D}_2) = 0.38$. So, $h \le \|\theta_\mathcal{N}\|, \|\theta_\mathcal{N}\| = 0.62 < 2(2^{-1}) = 1$. Hence,

$$\varpi^\phi_2(f, \|\theta_\mathcal{N}\|, [0, 1])^2_{w_{\alpha,\beta},2} = \sup_{0<h\le\|\theta_\mathcal{N}\|} \|\mho^2_{h\phi}(f, x)\|^2_{\alpha,\beta,2} \tag{17}$$

and

$$\|\mho^2_{h\phi}(f, x)\|^2_{\alpha,\beta,2} = \|w_{\alpha,\beta}\mho^2_{h\phi}(f, x)\|^2_2.$$

Then,

$$\mho^2_{h\phi}(f, x) = \prod_{i=1}^{2} \int_i^{\mathbb{D}} f \underline{\mathrm{d}\mathcal{L}_\phi} = \inf \underline{\mathrm{LS}}(f, \underline{\mathcal{L}_\phi}) = \sum_{j=1}^{2}\prod_{i=1}^{2} m_j \times \mathcal{L}_i(\mu(\mathbb{D}_j))$$

$$= m_1 \times \mathcal{L}_1(\mu(\mathbb{D}_1)) \times \mathcal{L}_2(\mu(\mathbb{D}_1)) + m_2 \times \mathcal{L}_1(\mu(\mathbb{D}_2)) \times \mathcal{L}_2(\mu(\mathbb{D}_2)),$$

where

$$m_1 = \inf_{x\in\mathbb{D}_1} |f(x - h\sqrt{1 - x^2}) - 2f(x) + f(x + h\sqrt{1 - x^2})|,$$

and

$$m_2 = \inf_{x\in\mathbb{D}_2} |f(x - h\sqrt{1 - x^2}) - 2f(x) + f(x + h\sqrt{1 - x^2})|.$$

Then,

$$\mho^2_{h\phi}(f, x) = \inf_{x\in\mathbb{D}_1} |(x - h\sqrt{1 - x^2})^4 - (x - h\sqrt{1 - x^2}) - 2(x^4 - x)$$

$$+(x + h\sqrt{1 - x^2})^4 - (x + h\sqrt{1 - x^2})| \times \mathcal{L}_1(0.62)\mathcal{L}_2(0.62)$$

$$+ \inf_{x\in\mathbb{D}_2} |(x - h\sqrt{1 - x^2})^4 - (x - h\sqrt{1 - x^2}) - 2(x^4 - x)$$

$$+(x + h\sqrt{1 - x^2})^4 - (x + h\sqrt{1 - x^2})| \times \mathcal{L}_1(0.38)\mathcal{L}_2(0.38)$$

$$= \inf_{x\in\mathbb{D}_1} |2x^4h^4 - 12x^4h^2 - 4x^2h^4 + 12x^2h^2 + 2h^4| \times \sin(0.62) \times \sqrt{(0.62)}$$

$$+\inf_{x\in\mathbb{D}_2}|2x^4h^4-12x^4h^2-4x^2h^4+12x^2h^2+2h^4|\times\sin(0.38)\times\sqrt{(0.38)}$$

$$=\inf_{x\in\mathbb{D}_1}|-(12h^2-2h^4)x^4+(12h^2-4h^4)x^2+2h^4|\times(0.008)$$

$$+\inf_{x\in\mathbb{D}_2}|-(12h^2-2h^4)x^4+(12h^2-4h^4)x^2+2h^4|\times(0.004).$$

Let $f_1(x)=-(12h^2-2h^4)x^4+(12h^2-4h^4)x^2+2h^4$ be a function of $x\in\mathbb{D}_1$. Let $A=-(12h^2-2h^4), B=(12h^2-4h^4)$ and $C=2h^4$ be a factors variables of the function f_1. Then, the range of f_1 is $R_1=[2h^4,6h^2(4-h^2)]$. Also, if $x\in\mathbb{D}_2$ then the range of f_1 is $R_2=[0,6h^2(4-h^2)]$. Then, $\mho^2_{h\phi}(f,h)=(2h^4)\times(0.008)+(0)\times(0.004)$.

Therefore, the symmetric difference is defined as below:

Table 1: Shows symmetric difference values $\mho^2_{h\phi}(f,h)$ of f.

$f(x)$	r	$\mho^2_{h\phi}(f,h)$
x^4-x	0	$(0.016\ h^4)$

Then, from (17), we get

$$\|(1-x)^\alpha(1+x)^\beta\times(\mho^2_{h\phi}(f,x))\|_2^2=\|(1-x)^\alpha(1+x)^\beta\times(0.016\ h^4)\|_2^2$$

$$=(\int_0^1|(1-x)^\alpha(1+x)^\beta\times(0.016\ h^4)|^2dx). \tag{18}$$

From (18), we have

Case I. Let $\alpha=\beta=1$, then

$$\|(1-x)^\alpha(1+x)^\beta\times(\mho^2_{h\phi}(f,x))\|_2^2=\|(1-x)(1+x)\times(\mho^2_{h\phi}(f,x))\|_2^2$$

$$=(\int_0^1|(1-x^2)\times(0.016h^4)|^2dx)=(0.016h^4)^2\times(\int_0^1|(1-x^2)|^2dx)$$

$$=(0.016h^4)^2\times(\int_0^1(x^4-2x^2+1)dx)$$

$$=(0.016h^4)^2\times[(\int_0^1x^4dx)-(2\int_0^1x^2dx)+(\int_0^1dx)]=(0.016h^4)^2\times(\frac{8}{15}).$$

From (17) and (18), therefore, the weighted DTMS of order 2 was found as follows on the basis of Table 1:

$$\varpi_2^{\phi}(f, \|\theta_{\mathcal{N}}\|, [0,1])_{w_{1,1},2} = \sup_{0<h\leq\|\theta_{\mathcal{N}}\|} \|(1-x^2)\times(\mho_{h\phi}^2(f,x))\|_2$$

$$= (0.016\|\theta_{\mathcal{N}}\|^4)\times\sqrt{(\frac{8}{15})} = (0.011\|\theta_{\mathcal{N}}\|^4).$$

Case II. If $\alpha \neq \beta$. Let $\alpha = 3$ and $\beta = 0.25$.

$$\|(1-x)^{\alpha}(1+x)^{\beta}\times(\mho_{h\phi}^2(f,x))\|_2^2 = (\int_0^1 |(1-x)^3(1+x)^{0.25}\times(0.016h^4)|^2dx)$$

$$= (\int_0^1((1-x)^6(1+x)^{0.5}\times(0.016h^4)^2)dx)$$

$$= (0.016h^4)^2\times(\int_0^1((1-x)^6(1+x)^{0.5})dx)$$

$$= (0.016h^4)^2\times(\int_0^1((x^6-6x^5+15x^4-20x^3+15x^2-6x+1)\times\sqrt{1+x})dx)$$

$$= (0.016h^4)^2\times(\int_0^1(x^6\times\sqrt{1+x})dx - 6\int_0^1(x^5\times\sqrt{1+x})dx$$

$$+15\int_0^1(x^4\times\sqrt{1+x})dx - 20\int_0^1(x^3\times\sqrt{1+x})dx + 15\int_0^1(x^2\times\sqrt{1+x})dx$$

$$-6\int_0^1(x\times\sqrt{1+x})dx + \int_0^1(\sqrt{1+x})dx)$$

$$= (0.016h^4)^2\times(\frac{2}{15}x^6(1+x)^{\frac{3}{2}} - \frac{12}{15}[\frac{2}{13}(1+x)^{\frac{13}{2}} - \frac{10}{11}(1+x)^{\frac{11}{2}}$$

$$+\frac{20}{9}(1+x)^{\frac{9}{2}} - \frac{20}{7}(1+x)^{\frac{7}{2}} + 2(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}}] - 6(\frac{2}{13}(1+x)^{\frac{13}{2}}$$

$$-\frac{10}{11}(1+x)^{\frac{11}{2}} + \frac{20}{9}(1+x)^{\frac{9}{2}} - \frac{20}{7}(1+x)^{\frac{7}{2}} + 2(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}})$$

$$+15[2(\frac{1}{11}(1+x)^{\frac{11}{2}} - \frac{4}{9}(1+x)^{\frac{9}{2}} + \frac{6}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{1}{3}(1+x)^{\frac{3}{2}})]$$

$$-20(\frac{2}{9}(1+x)^{\frac{9}{2}} - \frac{6}{7}(1+x)^{\frac{7}{2}} + \frac{6}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}}) + 15[\frac{2}{7}(1+x)^{\frac{7}{2}}$$

$$-\frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}}] - 6(\frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}}) + [\frac{2}{3}(1+x)^{\frac{3}{2}}])$$

$$= (0.016h^4)^2 \times (0.1956 - 6 \times (0.2269) + 15 \times (0.2705) - 20 \times (0.335) + 15 \times (0.4402) - 6 \times (0.6437) + 1.2189) = (0.016h^4)^2 \times (0.1514).$$

From (17) and (18), therefore, the weighted DTMS of order 2 was found as follows on the basis of Table 1:

$$\varpi_2^{\phi}(f, \|\theta_{\mathcal{N}}\|, [0,1])_{w_{3,0.25},2} = \sup_{0<h\leq\|\theta_{\mathcal{N}}\|} \|(1-x)^3(1+x)^{0.25} \times (\mho_{h\phi}^2(f,x))\|_2$$

$$= (0.016\|\theta_{\mathcal{N}}\|^4) \times \sqrt{(0.1514)} = (0.0062\|\theta_{\mathcal{N}}\|^4).$$

5. Conclusions

We have described the behaviors of the weighted DTMS equivalence obtained in this study are constructed with that of the values α and β in this paper. More specifically, we investigated the nature of the best weighted approximation to f due to the above mentioned DTMS behaviors, if f is convex. However, based on our key findings, we obtain the following application:

Table 2: The weighted DTMS of $f(x) = x^4 - x$.

Method	α	β	p	Order DTMS	Value DTMS	Result
weighted DTMS	1	1	2	$\varpi_2^{\phi}(\cdot)_{w_{1,1},2}$	$0.011\|\theta_{\mathcal{N}}\|^4$	0.0016
weighted DTMS	3	0.25	2	$\varpi_2^{\phi}(\cdot)_{w_{3,0.25},2}$	$0.0062\|\theta_{\mathcal{N}}\|^4$	0.00091

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