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A HARMONIC MEAN INEQUALITY FOR THE EXPONENTIAL INTEGRAL FUNCTION

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Abstract: By using purely analytical techniques, we establish a harmonic mean inequality for the classical exponential integral function.

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1. Introduction

The classical exponential integral function is defined as [1, p. 228]

$$\begin{aligned} E(s) &= \int_s^\infty \frac{e^{-t}}{t} dt \\ &= \int_1^\infty \frac{e^{-st}}{t} dt \qquad (1) \\ &= \Gamma(0, s) \end{aligned}$$

for $s \in (0, \infty)$ where $\Gamma(u, s)$ is the upper incomplete gamma function defined as

$$\Gamma(u, s) = \int_s^\infty t^{u-1} e^{-t} dt.$$

It satisfies the following properties among others.

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$$E'(s) = -\frac{e^{-s}}{s}, \tag{2}$$

$$E''(s) = \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} = -\left(1 + \frac{1}{s}\right) E'(s). \tag{3}$$

For more properties of this special function, one may refer to [1], [8] and [13]. It is often applied in astrophysics, neutron physics, quantum chemistry as well as other applied sciences. Due to its importance, it has been studied in diverse ways. See for instance [3], [5], [12], [14], [15], [16] and [17].

In the present work, we continue the investigation. Specifically, our objective is to establish a harmonic mean inequality for the function. For harmonic mean inequalities involving other special functions, the interested reader may refer to [2], [4], [6], [9], [10], [11], [18], [19].

2. Results

We begin with the following auxiliary results.

Lemma 1. *For $s \in (0, \infty)$, the inequality*

$$E(s) + E(1/s) \geq 2\,\Gamma(0, 1) \tag{4}$$

is satisfied, with equality when $s = 1$.

Proof. The case for $s = 1$ is self-evident. Hence let $\mathcal{K}(s) = E(s) + E(1/s)$ for $s \in (0, 1) \cup (1, \infty)$. Then

$$\begin{aligned} s\,\mathcal{K}'(s) &= sE'(s) - \frac{1}{s}E'(1/s) \\ &= e^{-\frac{1}{s}} - e^{-s} \\ &= h(s). \end{aligned}$$

Then $h(s) < 0$ if $s \in (0, 1)$ and $h(s) > 0$ if $s \in (1, \infty)$. Hence $\mathcal{K}(s)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$. In either case, we have

$$\mathcal{K}(s) > \lim_{s \to 1} \mathcal{K}(s) = 2\Gamma(0, 1)$$

which completes the proof of the lemma. □

Lemma 2. *For $s \in (0, \infty)$, the inequality*

$$E(s)E(1/s) \leq \Gamma^2(0,1) \tag{5}$$

is satisfied, with equality when $s = 1$.

Proof. The case for $s = 1$ is self-evident. Hence let $\mathcal{L}(s) = E(s)E(1/s)$ for $s \in (0,1) \cup (1, \infty)$. Then

$$\begin{aligned} s\, e^s e^{1/s}\, \mathcal{L}'(s) &= e^s E(s) - e^{1/s} \frac{1}{s} E(1/s) \\ &= v(s). \end{aligned}$$

Now let $\alpha(s) = e^s E(s)$. Then by using the fact that $E(s) < e^{-s}/s$ for all positive s (see [7]), we conclude that

$$e^{-s} \alpha'(s) = E(s) - \frac{e^{-s}}{s} < 0.$$

Hence $\alpha(s)$ is decreasing on $(0, \infty)$. Consequently, $v(s) > 0$ if $s \in (0,1)$ and $v(s) < 0$ if $s \in (1, \infty)$. Thus, $\mathcal{L}(s)$ is increasing on $(0,1)$ and decreasing on $(1, \infty)$. In either case, we have

$$\mathcal{L}(s) < \lim_{s \to 1} \mathcal{L}(s) = \Gamma^2(0,1)$$

which completes the proof of the lemma. □

We now state the main results of this paper in the following theorem.

Theorem 3. *For $s \in (0, \infty)$, the inequality*

$$\frac{2E(s)E(1/s)}{E(s) + E(1/s)} \leq \Gamma(0,1) \tag{6}$$

is satisfied, with equality when $s = 1$.

First Proof. By applying Lemma 1 and Lemma 2, we obtain

$$\frac{2E(s)E(1/s)}{E(s) + E(1/s)} \leq \frac{2E(s)E(1/s)}{2\Gamma(0,1)} \leq \frac{\Gamma^2(0,1)}{\Gamma(0,1)} = \Gamma(0,1)$$

which completes the proof. □

Second Proof. The case for $s = 1$ is self-evident. Hence let $\mathcal{M}(s) = \frac{2E(s)E(1/s)}{E(s)+E(1/s)}$ and $\lambda(s) = \ln \mathcal{M}(s)$ for $s \in (0,1) \cup (1,\infty)$. Then

$$\lambda'(s) = \frac{E'(s)}{E(s)} - \frac{1}{s^2}\frac{E'(1/s)}{E(1/s)} - \frac{E'(s) - \frac{1}{s^2}E'(1/s)}{E(s) + E(1/s)}$$

which implies that

$$s\left[E(s) + E(1/s)\right]\lambda'(s) = s\frac{E'(s)}{E(s)}E(1/s) - \frac{1}{s}\frac{E'(1/s)}{E(1/s)}E(s).$$

This further implies that

$$\begin{aligned} s\left[\frac{1}{E(s)} + \frac{1}{E(1/s)}\right]\lambda'(s) &= s\frac{E'(s)}{E(s)} - \frac{1}{s}\frac{E'(1/s)}{E(1/s)} \\ &:= \Delta(s). \end{aligned}$$

Now let $p(s) = \frac{sE'(s)}{E^2(s)}$ for $s \in (0,\infty)$. Then by using (2) and (3), we obtain

$$\begin{aligned} E^3(s)p'(s) &= E(s)E'(s) + sE(s)E''(s) - 2(E'(s))^2 \\ &= e^{-s}E(s) + 2e^{-s}E'(s). \end{aligned}$$

Furthermore, by using (1), we obtain

$$\begin{aligned} \frac{E^3(s)}{e^{-s}}p'(s) &= E(s) + 2E'(s) \\ &= \int_1^\infty \frac{e^{-st}}{t}dt - 2\int_1^\infty e^{-st}dt \\ &= \int_1^\infty \left[\frac{1}{t} - 2\right]e^{-st}dt \\ &< 0. \end{aligned}$$

Hence, $p'(s) < 0$ which shows that $p(s)$ is decreasing for all $s \in (0,\infty)$. By the decreasing property of $p(s)$, we arrive at the conclusion that $\Delta(s) > 0$ if $s \in (0,1)$ and $\Delta(s) < 0$ if $s \in (1,\infty)$. Thus, $\lambda'(z) > 0$ if $s \in (0,1)$ and $\lambda'(s) < 0$ if $s \in (1,\infty)$. These mean that, $\mathcal{M}(s)$ is increasing on $(0,1)$ and decreasing on $(1,\infty)$. In either case, we have

$$\mathcal{M}(s) < \lim_{s\to 1}\mathcal{M}(s) = E(1) = \Gamma(0,1)$$

which completes the proof. □

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