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MODIFIED TWO STEPS IMPLICIT MIDPOINT ITERATION METHOD TO APPROXIMATE FIXED POINTS FOR NONEXPANSIVE MAPPINGS

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Abstract: In this paper, we establish the weak convergence of fixed points for nonexpansive mappings in the setting of uniformly convex Banach spaces using modified two steps implicit midpoint rule.

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1. Introduction

Let K be a normed linear space and $F : K \to K$ a given mapping. Let $x_0 \in K$ be arbitrary and $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$. The sequence $\{x_n\}$ is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n F\left(\frac{x_n + x_{n+1}}{2}\right). \tag{1}$$

This implicit midpoint rule was defined by Alghamdi et al. [7] in 2014 to approximate fixed point for the nonexpansive mapping F and for more results refer [6], [3], [9].

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Here we extend equation (1) to two steps iteration scheme as

$$\left.\begin{aligned} y_{n+1} &= (1-\alpha_n)y_n + \alpha_n F\left(\frac{y_n + z_n}{2}\right) \\ z_n &= (1-\beta_n)y_n + \beta_n F\left(\frac{y_n + y_{n+1}}{2}\right),\ n \geq 0 \end{aligned}\right\}, \tag{2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and using this iteration scheme we shall prove the weak convergence fixed point theorem.

2. Preliminaries

Here we give some important results which are needed for our main results.

Lemma 1. ([8]) *Let K be a normed linear space and $t \in [0,1]$. Then for each $x, y \in K$,*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2.$$

Lemma 2. ([2]) *A Banach space K is uniformly convex if and only if there exists a continuous strictly increasing convex mapping $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g\left(\|x-y\|^2\right),$$

for all $x, y \in B_r[0] = \{x \in K : \|x\| \leq r\}$ and $0 \leq t \leq 1$.

Lemma 3. ([4]) *Let $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ be a strictly increasing mapping. If a sequence $\{x_n\}$ in $[0,\infty)$ satisfies $\lim\limits_{n\to\infty} g(x_n) = 0$, then $\lim\limits_{n\to\infty} x_n = 0$.*

Lemma 4. ([5]) *Let $\{x_n\}$ and $\{y_n\}$ are two sequences of non-negative numbers such that $x_{n+1} \leq x_n + y_n$ for all $n \geq 1$.*

If $\sum\limits_{n=1}^{\infty} y_n$ converges, then $\lim\limits_{n\to\infty} x_n$ exists.

3. Main Results

Lemma 5. *Let K be a nonempty bounded closed and convex subset of a uniformly convex Banach space X and let $F : K \to K$ be a nonexpansive mapping such that $Fix(F) := \{x : Fx = x\} \neq \emptyset$. Suppose $\{y_n\}$ is given by (2). Then $\lim_{n\to\infty} \|y_n - p\|$ exists for all $p \in \mathrm{Fix}(F)$.*

Proof. For any $p \in \mathrm{Fix}(F)$, we have

$$\begin{aligned}
\|y_{n+1} - p\|^2 &= \|(1-\alpha_n)y_n + \alpha_n F\left(\frac{y_n + z_n}{2}\right) - p\|^2 \\
&\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n \|F\left(\frac{y_n + z_n}{2}\right) - p\|^2 \\
&\leq (1-\alpha_n)\|y_n - p\|^2 + \alpha_n \|\frac{y_n + z_n}{2} - p\|^2 \\
&\leq (1-\alpha_n)\|y_n - p\|^2 + \frac{\alpha_n}{2}\|y_n - p\|^2 + \frac{\alpha_n}{2}\|z_n - p\|^2 \\
&\leq \left(1 - \frac{\alpha_n}{2}\right)\|y_n - p\|^2 \\
&+ \frac{\alpha_n}{2}\|(1-\beta_n)y_n + \beta_n F\left(\frac{y_n + y_{n+1}}{2}\right) - p\|^2. \\
\|y_{n+1} - p\|^2 &\leq \left(1 - \frac{\alpha_n}{2}\right)\|y_n - p\|^2 + \frac{\alpha_n}{2}(1-\beta_n)\|y_n - p\|^2 \\
&+ \frac{\alpha_n\beta_n}{2}\|F\left(\frac{y_n + y_{n+1}}{2}\right) - p\|^2.
\end{aligned}$$

$$\begin{aligned}
\|y_{n+1} - p\|^2 &\leq \left(1 - \frac{\alpha_n\beta_n}{2}\right)\|y_n - p\|^2 + \frac{\alpha_n\beta_n}{2}\|\frac{y_n + y_{n+1}}{2} - p\|^2 \\
&\leq \left(1 - \frac{\alpha_n\beta_n}{2}\right)\|y_n - p\|^2 + \frac{\alpha_n\beta_n}{4}\|y_n - p\|^2 \\
&+ \frac{\alpha_n\beta_n}{4}\|y_{n+1} - p\|^2. \\
\|y_{n+1} - p\|^2 &\leq \left(1 - \frac{\alpha_n\beta_n}{4}\right)\|y_n - p\|^2 + \frac{\alpha_n\beta_n}{4}\|y_{n+1} - p\|^2.
\end{aligned}$$

That is,

$$\left(1 - \frac{\alpha_n\beta_n}{4}\right)\|y_{n+1} - p\|^2 \leq \left(1 - \frac{\alpha_n\beta_n}{4}\right)\|y_n - p\|^2.$$

This implies

$$\|y_{n+1}-p\|^2 \le \|y_n-p\|^2, \text{ for all } n \ge 0.$$

Thus, the sequence $\{\|y_n-p\|\}$ is non-increasing and bounded.

Hence, $\lim_{n\to\infty} \|y_n-p\|$ exists. □

Theorem 6. *Let K be a nonempty bounded closed and convex subset of a uniformly convex Banach space X and let $F : K \to K$ be a nonexpansive mapping with $Fix(F) \neq \emptyset$. Assume $\{y_n\}$ is given by (2), where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy either:*

(C) $0 < \epsilon \le \alpha_n \le 1$ *and* $0 < \epsilon \le \beta_n(1-\beta_n)$ *(or)*

(D) $0 < \epsilon \le \alpha_n(1-\alpha_n)$.

Then $\lim_{n\to\infty} \|y_n - Fy_n\| = 0$.

Proof. Let $p \in Fix(T)$. Then we have

$$\begin{aligned}
\|y_{n+1}-p\|^2 &= \left\|(1-\alpha_n)y_n + \alpha_n F\left(\frac{y_n+z_n}{2}\right) - p\right\|^2 \\
&\le (1-\alpha_n)\|y_n-p\|^2 + \alpha_n \left\|F\left(\frac{y_n+z_n}{2}\right) - p\right\|^2 \\
&- \alpha_n(1-\alpha_n) g\left(\left\|y_n - F\left(\frac{y_n+z_n}{2}\right)\right\|\right).
\end{aligned}$$

$$\begin{aligned}
\|y_{n+1}-p\|^2 &\le (1-\alpha_n)\|y_n-p\|^2 + \alpha_n \left\|\frac{y_n+z_n}{2} - p\right\|^2 \\
&- \alpha_n(1-\alpha_n) g\left(\left\|y_n - F\left(\frac{y_n+z_n}{2}\right)\right\|\right) \\
&\le \left(1-\frac{\alpha_n}{2}\right)\|y_n-p\|^2 + \frac{\alpha_n}{2}\|z_n-p\|^2 \\
&- \alpha_n(1-\alpha_n) g\left(\left\|y_n - F\left(\frac{y_n+z_n}{2}\right)\right\|\right) \\
&\le \left(1-\frac{\alpha_n}{2}\right)\|y_n-p\|^2 \\
&+ \frac{\alpha_n}{2}\left\|(1-\beta_n)y_n + \beta_n F\left(\frac{y_n+y_{n+1}}{2}\right) - p\right\|^2 \\
&- \alpha_n(1-\alpha_n) g\left(\left\|y_n - F\left(\frac{y_n+z_n}{2}\right)\right\|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-\frac{\alpha_n}{2}\right)\|y_n-p\|^2+\frac{\alpha_n(1-\beta_n)}{2}\|y_n-p\|^2 \\
&+ \frac{\alpha_n\beta_n}{2}\left\|F\left(\frac{y_n+y_{n+1}}{2}\right)-p\right\|^2 \\
&- \frac{\alpha_n\beta_n(1-\beta_n)}{2}g\left(\left\|y_n-F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \\
&- \alpha_n(1-\alpha_n)g\left(\left\|y_n-F\left(\frac{y_n+z_n}{2}\right)\right\|\right).
\end{aligned}
$$

$$
\begin{aligned}
\|y_{n+1}-p\|^2 &\leq \left(1-\frac{\alpha_n\beta_n}{4}\right)\|y_n-p\|^2+\frac{\alpha_n\beta_n}{4}\|y_{n+1}-p\|^2 \\
&- \frac{\alpha_n\beta_n(1-\beta_n)}{2}g\left(\left\|y_n-F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \\
&- \alpha_n(1-\alpha_n)g\left(\left\|y_n-F\left(\frac{y_n+z_n}{2}\right)\right\|\right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
\left(1-\frac{\alpha_n\beta_n}{4}\right) &\times \|y_{n+1}-p\|^2 \\
&\leq \left(1-\frac{\alpha_n\beta_n}{4}\right)\|y_n-p\|^2 \\
&- \frac{\alpha_n\beta_n(1-\beta_n)}{2}g\left(\left\|y_n-F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \\
&- \alpha_n(1-\alpha_n)g\left(\left\|y_n-F\left(\frac{y_n+z_n}{2}\right)\right\|\right).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|y_{n+1}-p\|^2 &\leq \|y_n-p\|^2 \\
&- \frac{2\alpha_n\beta_n(1-\beta_n)}{4-\alpha_n\beta_n}g\left(\left\|y_n-F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \\
&- \frac{4\alpha_n(1-\alpha_n)}{4-\alpha_n\beta_n}g\left(\left\|y_n-F\left(\frac{y_n+z_n}{2}\right)\right\|\right).
\end{aligned}
$$

Now we rewrite the above equation into the following two forms:

$$
\frac{2\alpha_n\beta_n(1-\beta_n)}{4-\alpha_n\beta_n}g\left(\left\|y_n-F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \leq \|y_n-p\|^2-\|y_{n+1}-p\|^2, \quad (3)
$$

$$
\frac{4\alpha_n(1-\alpha_n)}{4-\alpha_n\beta_n}g\left(\left\|y_n-F\left(\frac{y_n+z_n}{2}\right)\right\|\right) \leq \|y_n-p\|^2-\|y_{n+1}-p\|^2. \quad (4)
$$

Case (i) Assume condition (A) holds.
From equation (3),

$$\begin{aligned} \frac{\alpha_n\beta_n(1-\beta_n)}{2} &\times g\left(\left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \\ &\leq \frac{2\alpha_n\beta_n(1-\beta_n)}{4-\alpha_n\beta_n} g\left(\left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \\ &\leq \|y_n - p\|^2 - \|y_{n+1}-p\|^2. \end{aligned}$$

By condition (A), $\alpha_n\beta_n(1-\beta_n) \geq \epsilon^2$. Then the above equation becomes

$$\frac{\epsilon^2}{2} g\left(\left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \leq \|y_n - p\|^2 - \|y_{n+1}-p\|^2.$$

Taking sum on the first m terms of the above equation, we have get

$$\frac{\epsilon^2}{2}\sum_{n=0}^{m} g\left(\left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) \leq \|y_0 - p\|^2 - \|y_{m+1}-p\|^2.$$

It is evident that, as $m \to \infty$,

$$\sum_{n=0}^{\infty} g\left(\left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) < \infty, \tag{5}$$

which gives that

$$\lim_{n\to\infty} g\left(\left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|\right) = 0.$$

Then by Lemma 3,

$$\lim_{n\to\infty} \left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\| = 0.$$

Now

$$\begin{aligned} \|y_{n+1}-y_n\|^2 &= \left\|(1-\alpha_n)y_n + \alpha_n F\left(\frac{y_n+z_n}{2}\right) - y_n\right\|^2 \\ &\leq \alpha_n \left\|y_n - F\left(\frac{y_n+z_n}{2}\right)\right\|^2. \\ \|y_{n+1}-y_n\|^2 &\leq \alpha_n \left\|y_n - F\left(\frac{y_n+y_{n+1}}{2}\right)\right\|^2 \end{aligned}$$

$$+ \quad \alpha_n \left\| F\left(\frac{y_n + y_{n+1}}{2}\right) - F\left(\frac{y_n + z_n}{2}\right) \right\|^2.$$

$$\begin{aligned}
\|y_{n+1} - y_n\|^2 &\leq \alpha_n \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2 + \frac{\alpha_n}{2} \|y_{n+1} - z_n\|^2 \\
&\leq \alpha_n \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2 \\
&+ \frac{\alpha_n}{2} \left\| y_{n+1} - (1-\beta_n)y_n - \beta_n F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2. \\
\|y_{n+1} - y_n\|^2 &\leq \alpha_n \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2 + \frac{\alpha_n}{2} \|y_{n+1} - y_n\|^2 \\
&+ \frac{\alpha_n \beta_n}{2} \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
\left(1 - \frac{\alpha_n}{2}\right) \|y_{n+1} - y_n\|^2 &\leq \alpha_n \left(1 + \frac{\beta_n}{2}\right) \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2 \\
\|y_{n+1} - y_n\|^2 &\leq \frac{\alpha_n(2+\beta_n)}{2-\alpha_n} \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2 \\
&\leq \alpha \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right) \right\|^2,
\end{aligned}$$

where $\alpha = \min\left\{\frac{3}{2}, \frac{\alpha_n(2+\beta_n)}{2-\alpha_n}\right\}$.

This implies that

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$$

Case (ii) Suppose condition (B) holds.

Then from equation (4),

$$\begin{aligned}
\alpha_n(1-\alpha_n) &\times g\left(\left\| y_n - F\left(\frac{y_n + z_n}{2}\right) \right\|\right) \\
&\leq \frac{4\alpha_n(1-\alpha_n)}{4 - \alpha_n\beta_n} \\
&\times g\left(\left\| y_n - F\left(\frac{y_n + z_n}{2}\right) \right\|\right)
\end{aligned}$$

$$\leq \quad \|y_n - p\|^2 - \|y_{n+1} - p\|^2.$$

By condition (B), $\alpha_n(1-\alpha_n) \geq \epsilon$. Then the above equation becomes

$$\epsilon g\left(\left\|y_n - F\left(\frac{y_n + z_n}{2}\right)\right\|\right) \leq \|y_n - p\|^2 - \|y_{n+1} - p\|^2.$$

Taking sum on the first m terms of the above equation, we have get

$$\epsilon \sum_{n=0}^{m} g\left(\left\|y_n - F\left(\frac{y_n + z_n}{2}\right)\right\|\right) \leq \|y_0 - p\|^2 - \|y_{m+1} - p\|^2.$$

It is evident that, as $m \to \infty$,

$$\sum_{n=0}^{\infty} g\left(\left\|y_n - F\left(\frac{y_n + z_n}{2}\right)\right\|\right) < \infty \tag{6}$$

which gives that

$$\lim_{n\to\infty} g\left(\left\|y_n - F\left(\frac{y_n + z_n}{2}\right)\right\|\right) = 0.$$

Then by Lemma 3,

$$\lim_{n\to\infty} \left\|y_n - F\left(\frac{y_n + z_n}{2}\right)\right\| = 0.$$

Now,

$$\begin{aligned}
\|y_{n+1} - y_n\|^2 &= \left\|(1-\alpha_n)y_n + \alpha_n F\left(\frac{y_n + z_n}{2}\right) - y_n\right\|^2 \\
&\leq \alpha_n \left\|y_n - F\left(\frac{y_n + z_n}{2}\right)\right\|^2. \\
\|y_{n+1} - y_n\|^2 &\leq \left\|y_n - F\left(\frac{y_n + z_n}{2}\right)\right\|^2.
\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0.$$

Finally,

$$\begin{aligned}
\|y_n - Fy_n\| &\leq \left\|y_n - F\left(\frac{y_n + y_{n+1}}{2}\right)\right\| \\
&+ \left\|F\left(\frac{y_n + y_{n+1}}{2}\right) - Fy_n\right\|
\end{aligned}$$

$$\begin{aligned} &\leq \left\| y_n - F\left(\frac{y_n + y_{n+1}}{2}\right)\right\| + \frac{1}{2}\left\| y_{n+1} - y_n \right\|. \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence the theorem is proven. □

Theorem 7. *Let K be a nonempty bounded closed and convex subset of a uniformly convex Banach space X and let $F : K \to K$ be a nonexpansive mapping with $Fix(F) \neq \emptyset$. Assume $\{y_n\}$ is given by (2), where $\{\alpha_n\}$, $\{\beta_n\}$ satisfy either:*
(C) $0 < \epsilon \leq \alpha_n \leq 1$ *and* $0 < \epsilon \leq \beta_n(1 - \beta_n)$ *(or)*
(D) $0 < \epsilon \leq \alpha_n(1 - \alpha_n)$.
Then $\{y_n\}$ converges weakly to some fixed point of F.

Proof. From Lemma(5), it is clear that the sequence $\{y_n\}$ is bounded. Suppose there is a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup p$ as $i \to \infty$, where $p \in Fix(F)$. Then

$$\frac{y_{n_i} + y_{n_i+1}}{2} \rightharpoonup p \quad \text{as} \quad i \to \infty.$$

By the definition of continuity of F,

$$F\left(\frac{y_{n_i} + y_{n_i+1}}{2}\right) \rightharpoonup p \quad \text{as} \quad i \to \infty.$$

From equation (2),

$$z_{n_i} = (1 - \beta_{n_i})y_{n_i} + \beta_{n_i} F\left(\frac{y_{n_i} + y_{n_i+1}}{2}\right) \rightharpoonup p \quad \text{as} \quad i \to \infty,$$

and

$$y_{n_i} = (1 - \alpha_{n_i})y_{n_i} + \alpha_{n_i} F\left(\frac{z_{n_i} + y_{n_i}}{2}\right) \rightharpoonup p \quad \text{as} \quad i \to \infty.$$

Continuing the above process, by induction we have

$$y_{n_i+m} \rightharpoonup p, \quad \text{for any} \quad m \geq 0.$$

The remaining part of the proof immediately follows from Theorem 2.3 of [1]. □

Example 8. Let K be the closed unit ball in the Hilbert space R and let $F : K \to K$ defined by $Fx = \frac{x}{2}$. Then clearly F is nonexpansive mapping and K is bounded convex subset of R.

By Theorem 7, the sequence $\{y_n\}$ converges weakly to the fixed point of F. By choosing $\alpha_n = \dfrac{n}{n+1}$ and $\beta_n = \dfrac{n}{2n+1}$, condition (B) is satisfied. Equation (2) can be equivalently written as

$$y_{n+1} = \left(\frac{16 - 8\alpha_n - 3\alpha_n\beta_n}{16 - \alpha_n\beta_n}\right) y_n, \quad n \geq 0.$$

If we start from $y_0 = 0.5$, we obtain $y_1 = 1, y_2 = 0.363, y_3 = 0.228,$
$y_4 = 0.131, y_5 = 0.071, y_6 = 0.037, y_7 = 0.019, y_8 = 0.009,$
$y_9 = 0.004, y_{10} = 0.002, y_{11} = 0.001$ and $y_{12} = 0$.

Hence the sequence $\{y_n\}$ converges weakly to 0, which is clearly a fixed point of F with the parameter sequences $\alpha_n = \dfrac{n}{n+1}$ and $\beta_n = \dfrac{n}{2n+1}$.

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